Physics through Computational Thinking

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Lecture 22

Damped Harmonic Oscillator: Spring-mass System with Friction

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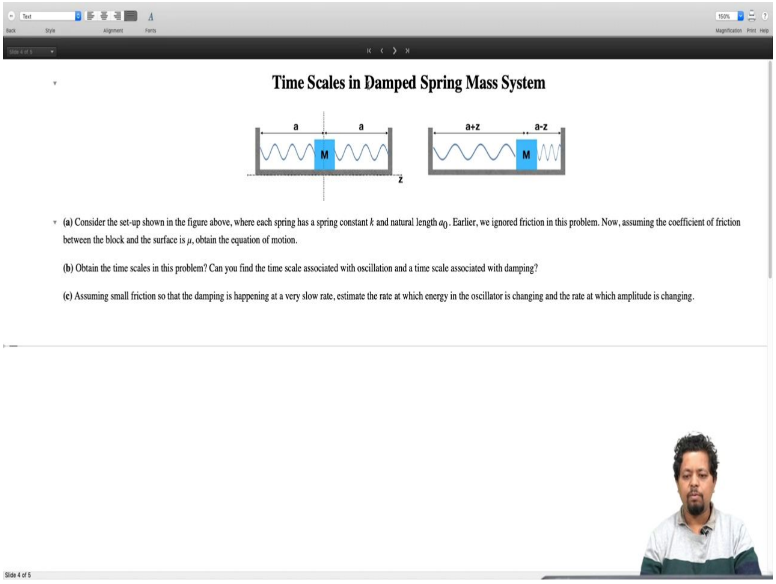

Welcome back, let us take another example from the damped harmonic oscillator and here we are going to take the example of the spring-mass system, So, I have got a coupled spring-mass system like this, we discuss about this in one of the previous videos, we got our spring attached to two ends of a wall and the other end of the spring is attached to a mass which is initially in the centre.

We displace it out to a corner and release it and it goes back and forth and oscillates. Last time, we did not consider friction and as a consequence it was a simple harmonic oscillator. Oscillating with a uniform frequency and uniform amplitude, but this time we will consider that there is some friction in this problem and as a consequence, as the mass oscillates back and forth, it is eventually going to slow down and come to a stop.

So, we are going to analyse this problem, let us look at this problem statement. First, we are going to do is find the equation of motion assuming that the coefficient of friction is μ and spring constant is k, the natural length of the spring is a_0. So, let us go ahead and work this out and after that, we are going to look at the time scales in the problem.

The two-time scales in the problem, there are time scales associated with oscillation and there is a time scale associated with damping. And then, finally, we are going to assume the friction is small and calculate what is the rate at which energy in the oscillator is changing and the rate at which amplitude is changing. So, let us go ahead and get started with that, so let us first analyse what is the equation of motion for the system.

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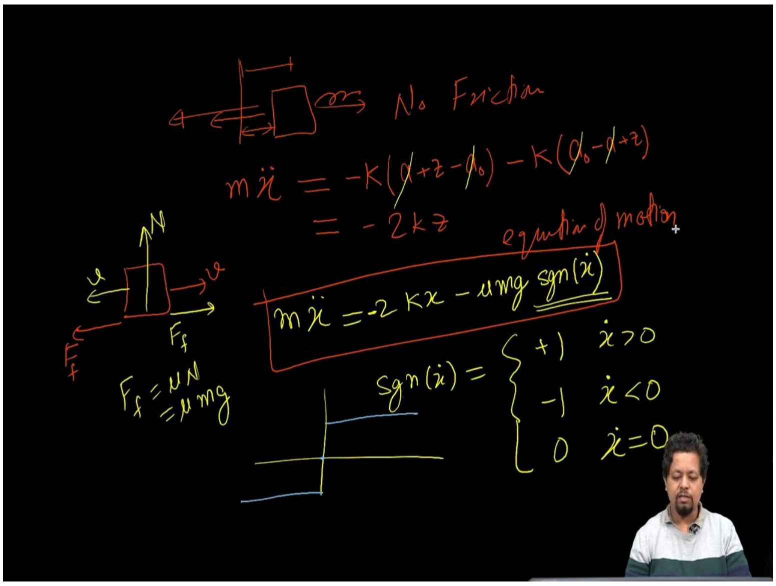

We have got a mass which is being displaced and we worked out last time that $m\ddot{x}$. Let us assume this, there is no friction and let us assume that from its mean position the mass has been displaced to the right-hand side as a consequence there is a stretch in the spring and the force because of the spring is to the left, so we have got $m\ddot{x}=K(a+z-a_0)$ where, stretching of the spring that is a $+z-a_0$ plus the elongation of the spring, the force and the negative direction, so that gives me that.

And because of the spring on this side spring on this side, again the force is going to be in the left direction because the spring will be contracted we are assuming a extreme case, so again I have got a minus sign for the force, then I have got K and I got the contraction is a_0-a+z. The contraction is $a-z$, the length of the spring is a -z and a_0 is the natural length.

So, $a_0 - a - z$ which is $a_0 - a + z$ is the elongation in the spring, with K using the force and if I add these two terms, I see that a cancels out, I see that a_0 cancels out and I get $-Kz$ from here. So my net force is $-2Kz$ that is what we did last time. Now, when we add friction, when the friction force always acts opposite to the velocity of the mass is moving this way.

And the friction force is acting in that direction and if the mass is moving this way, then friction force is acting in that direction, so the friction force is always opposite to the mass and that only requires adding a correction, so my equation becomes $m\ddot{x} = -2Kx$ which is a spring force and I have to add the friction force which is always opposite to the direction of velocity.

To write that, I would say and the friction force magnitude is μN , so $F_f = \mu N$ which is μmg , so it is $-\mu mg$ and it is always opposite to the direction of a force so is a signum function of $\dot{x}$. What is the signum function of $\dot{x}$? It is simply +1 for $\dot{x} > 0$ and -1 for $\dot{x} < 0$. 0 for $\dot{x} = 0$.

So, that is the signum function if the block is moving the friction force will always, will be acting opposite to that. And that is taken care of by the signum function. And as you notice that the signum function isis not a continuous function, if you make a plot of this, it is going to be +1 on this side and -1 on this side and 0 over here.

So, this is not a smooth force, this is not a smooth function. Therefore, we will find that it is hard to solve this equation analytically and therefore, we may have to solve this equation numerically. And that is why we talked about some of the numerical methods which are easy to solve which we will use in this example. But for now, that is our equation of motion, this is our equation of motion that we are interested in.

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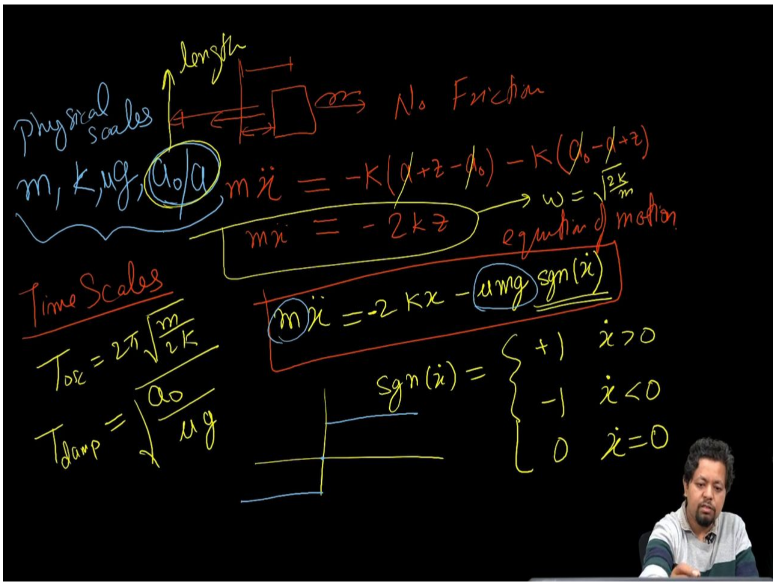

Now, let us talk about what are the time scales involved in this problem? The time scales involved in this problem are again of two varieties, let us first ask about the constants involved in the problem and what are the scales involved in the problem. The skills involved in this problem are mass, the spring constant K, of course, the gravitational constant g and rather since only mu g that comes together.

We can call it μg which is the deceleration due to friction. So, the so this scale is μg and those are the natural skills that are visible in the equation but there is also the scale associated with the amount of displacement that we make to the spring. So, that we can consider as a_0 or a, either of the two will work, because we are just doing a dimensional analysis, we are talking about dimensional analysis, we are only interested in the scales of the problem.

The actual displacement that we start for the system, in the beginning is going to be of the order of a_0 or of the order of a. So, these are the physical scales present in the problem. Now, coming to time scales in this problem, one thing that is obvious is if we look at this equation the oscillation frequency $\omega=\sqrt{2K/M}$ which gives me the oscillation time scale.

So oscillation time scale is $2\pi/\omega$, so it is $2\pi\sqrt{M/2K}$. That is the oscillation time scale in this problem and there is damping time scale, T_{damp}, for damping timescale we need to look at the damping term. The damping terms have got μmg which is the force divided by m, the mass gives me the expression which is simply μg, so I have got acceleration μg and if I want to extract a time out of this, I should divide the length scale and invert it.

So, the only length scale that is present in this problem is a_0/a. This is the only length scale in the problem. Apart from that, I do not have any other length scale. So, I take $\sqrt{a_0/\mu g}$, so length divided by acceleration is going to be t^2 and the square root of that will be t. So, this is my damping time scale and that is my oscillation time scale. So, these are two-time scales that are present in this problem. Given these two-time scales, now I can go ahead and estimate what is d by dt.

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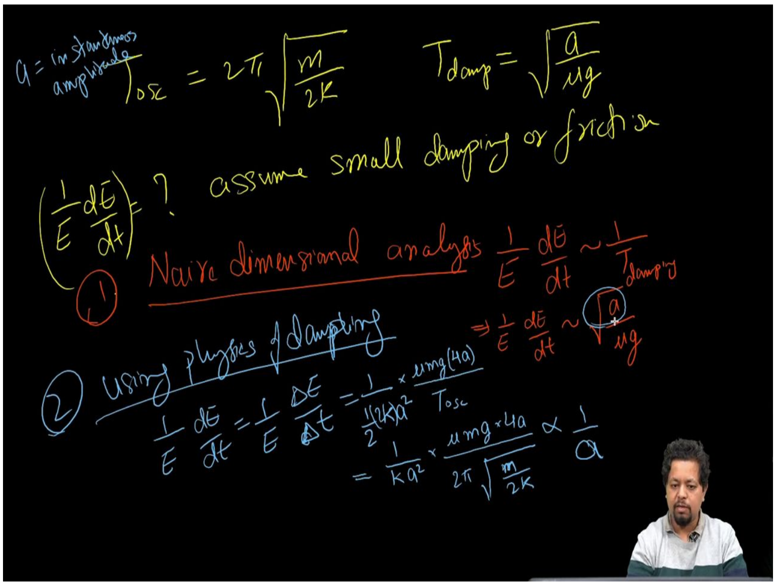

So, using this you want to estimate what is $1/E\, dE/dt$, under the assumption that assume small damping friction that is the problem statement. We are over here assuming small friction, so that damping is happening at a very slow rate, estimate the rate at which energy in the oscillator is changing at the rate at which amplitude is changing. Alright, so we let us start with the expression $1/E\, dE/dt$ because it is simple in the sense that it contains the only dimension of time.

So by using naive dimensional analysis, if there was no damping energy will be constant, but because of damping energy is changing. Naive dimensional analysis tells us as we did in the last example of the LCR circuit. We used naive dimensional analysis; it says $1/E\, dE/dt$ is simply at the order $1/T_{damp}$, which gives me the result that $1/E\, dE/dt$ is of the order of $\sqrt{a_0/\mu g}$.

Well, let us take a to be instantaneous amplitude and if I take a to be instantaneous amplitude, I can drop this a naught from here, drop this a_0 from here. As I just simply take, a as instantaneous amplitude at that point. So, as instantaneous amplitude changes, damping time also changes and we will make this a to be that instantaneous amplitude. So, we will assume over here that is a is instantaneous amplitude.

So, this gives me that $1/E\, dE/dt \propto \sqrt{a}$. Let us take another approach, so this was approach number one, just take another approach where we will use some physical input using physics of damping, physics of damping. We can be write $1/E\, dE/dt = 1/E\, \Delta E/\Delta t$. And this I can write as $1/E$ I can write as, $2/Ka^2$ because a is the instantaneous amplitude and spring constant is K, actually spring constant is effectively 2K.

So $2/2Ka^2 \Delta E$, ΔE is the loss of energy which is equal to the friction and the friction force is μmg . Work is done by it is 4 times, because the friction force drags, over one cycle of oscillation drags the mass by 4 times, a is the instantaneous amplitude. Therefore, that is the work done by the friction force, thereby time period of oscillation because we are talking about 1 full oscillation, the energy, amount of energy lost in 1 oscillation divided by 1 oscillation time period.

That is the $\Delta E/\Delta t$, there is a good approximation when damping is very slow. So, when we simply this we get $1/Ka^2 \mu mg 4a/T_{osc}$ and $T_{osc} = 2\pi\sqrt{M/2K}$. The important result here is that everything here is constant apart from the fact that a which is the instantaneous amplitude, we find that this result gives me something that is proportional to 1/a. In contrast to the result from previous where I got proportional to $1/a^2\, da/dt.$

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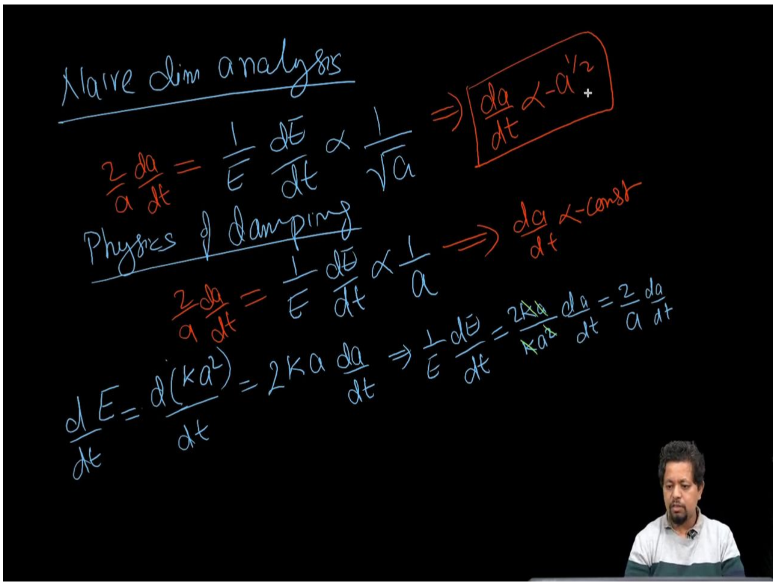

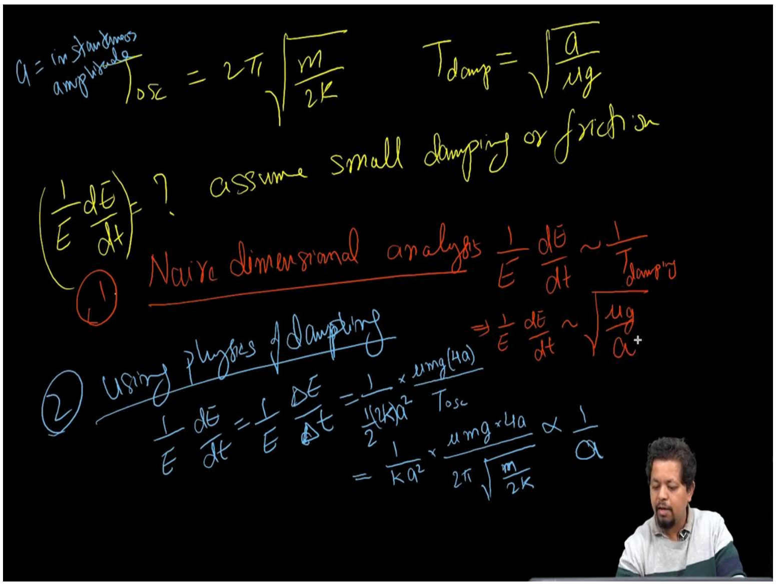

So the summary is that naive dimensional analysis gives me $1/E\, dE/dt \propto \sqrt{a}$, while physics of damping gives me $1/E\, dE/dt \propto 1/a$. Now, only one of these things could be correct, both of them cannot be correct. How do we find about which one of them is correct? Remember in the case of the LCR circuit, we got both the results to be identical, in both the cases we got the same expression, there the naive dimensional analysis and physics of damping gave me the equivalent result.

But, in this case, we do not find that to be true, so what is going on here, whether the naive dimensional is correct or it is the physics of damping which is correct. How do we find that out? For that we can only know, if we solve this equation and find out the energy and solve the equation of motion which is over here, solve this equation of motion, find out the solution of this and find out at what rate the energy is changing and compare whether it is changing with square root of instantaneous amplitude or $1/\sqrt{a}$.

Here, is one easy way of doing this, rather than talking about the energy we can also take, we can can start out, we can we can calculate $1/E\, dE/dt$ in terms of instantaneous energy. Instantaneous energy in the oscillator is $1/2Ka^2$ assuming the damping is small, so over the 1-time cycle, there is not a lot of energy that is lost. So, I can assume that energy is roughly constant over 1 oscillation and $1/2Ka^2$.

Then, dE/dt is d/dt of this, which is $Ka\, da/dt$ and if on both the sides we divide by $1/E$, we get Ka/Ka^2 which is the actually it was, there was no half here, the energy is . Spring constant is effectively 2K. dE/dt is $d/dt(ka^2)$ and there should be a 2 here. And therefore, although 2 is not important but let us keep track of that. So, we get $2Ka$ over here.

Energy is we divide that times da/dt and this becomes, I get a bunch of cancellations, I get K and a to cancel and I am left with at the end of the day $2/a\, da/dt$. So, what do I get from here? I get from here that $1/E\, dE/dt = 2/a\, da/dt$. From here I conclude that $da/dt \propto a^3$. This should have been…

There is a mistake here, this should have been inverse so to correct that, it should have been $\mu g/a$ and therefore over here, I get $1/\sqrt{a}$ and therefore I get here $a^{1/2}$ and coming to this one, this gives me, a cancels out and I get $da/dt \propto a\, const.$ It does not depend on a, da/dt does not depend on a. So, in one case naive dimensional says that da/dt is changing as the $\sqrt{a}$.

Well, actually it should be minus because we know that amplitude has to decrease as with damping and this is a negative constant, $da/dt \propto -const$. So, this says that the the amplitude is decreasing at a constant rate and this says that the amplitude is decreasing as a $\sqrt{a}$. So, let us find out which is the case and for that what we will do is, we will solve this system numerically using NDSolve.

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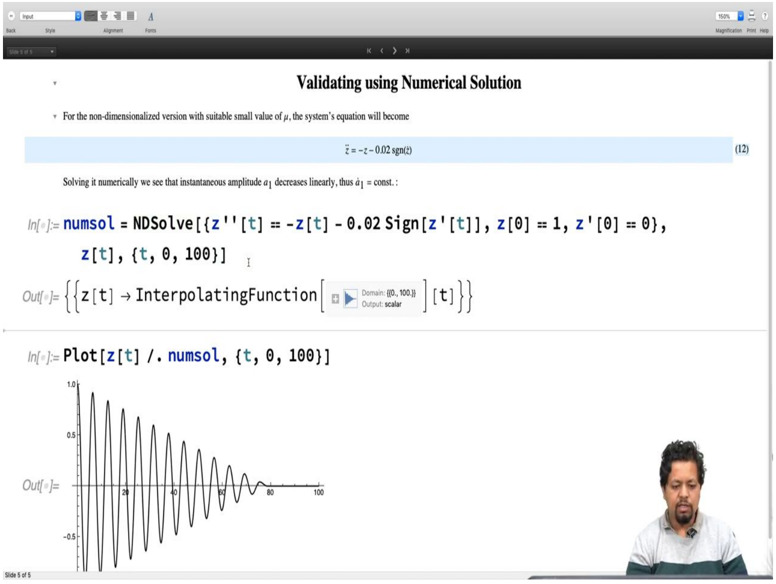

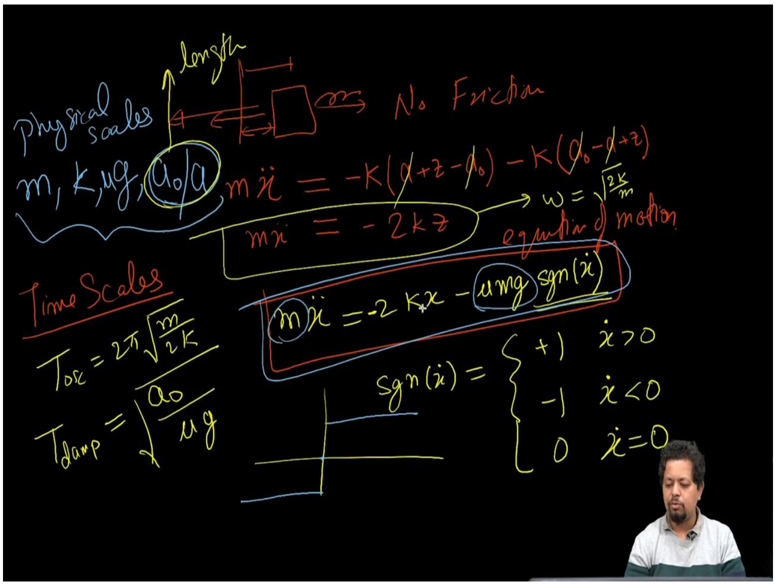

So, take out an equation to be, let us get rid of all the dimensions mg etc. So, this is our equation of motion was $m\ddot{x}=-2Kx-\mu mg\sin(x)$. If I remove, if substitute $m=1$, $2K=1$, $\mu mg=1$, et cetera., or $\mu mg=1$ with small coefficient as -0.2 or something like over here. This is identical to that equation except that we have removed the dimensions from the problem.

This is the equation that we wanted to solve, this is something we cannot solve analytically, what we have assumed here is that the damping term is small, so that the coefficient we have taken here is 0.02, you can go ahead and play around with this coefficient but this is how we do. We use NDSolve to solve this equation. We set up the equation; the sort of equation we are using is $z''(t)-z(t)-0.02\sin(z'(t))$.

So this is my equation, we are simply implementing NDSolve, this is my differential equation, the boundary condition $z_0=1$ and $z'(0)=0$. With these boundary conditions, I want to solve for z(t) from t, from 0 to 100, it is large value of time. When I execute this, again I get a solution in the form of replacement rule, from this I have to extract this interpolating function, that is what I have done here, by saying z t slash dot numsol.

Substitute z(t) for numsol and plot it from t from 0 to 100 and we get this beautiful plot over here which shows that the amplitude of oscillation is actually going down linearly. It is a very clear case of showing that when oscillation when damping is small the oscillation, the amplitude is simply decreasing linearly for the constant friction force case.

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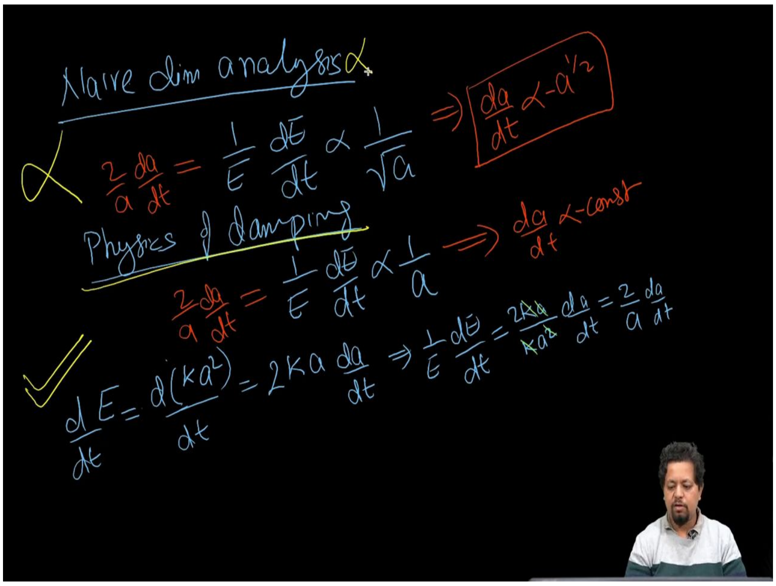

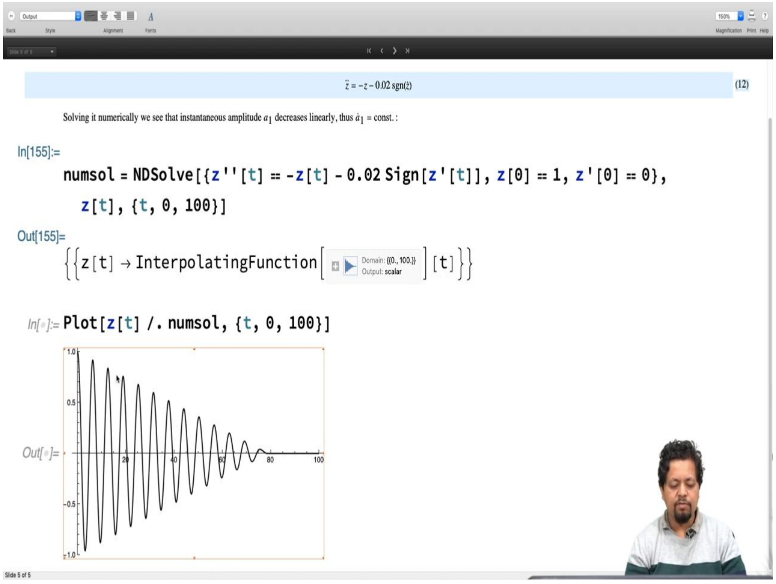

Therefore, what we learn from here is that the physics of damping gives me more accurate result. While naive dimensional analysis gets me an incorrect result. So, the physics of damping is right and Naive dimensional analysis is wrong, so what we learn from here is naive dimensional is not going to always give us the correct result. Most of the time, naive dimensional analysis works very well.

It gives me a good idea about what is happening in the problem. But, it is not always and this is an example where we find what happens to be true. We have to cross-check the naive dimensional analysis with physical input and we should trust naive dimensional analysis only when we can correlate or cross-check with some physical input. So, therefore when you use dimensional analysis you have to be a little bit careful.

And often, it is always useful to cross-check it with some known result or something you can validate across. In this case, when we validate we find that the naive dimensional analysis gives us the wrong result and it is the physics input of damping that we use, gives us the correct prediction for the rate at which amplitude is going to change. So, this is all for today and will see you in the next video.