Physics through Computational Thinking
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Lecture 22
Solving Initial Value Problem with Mathematica

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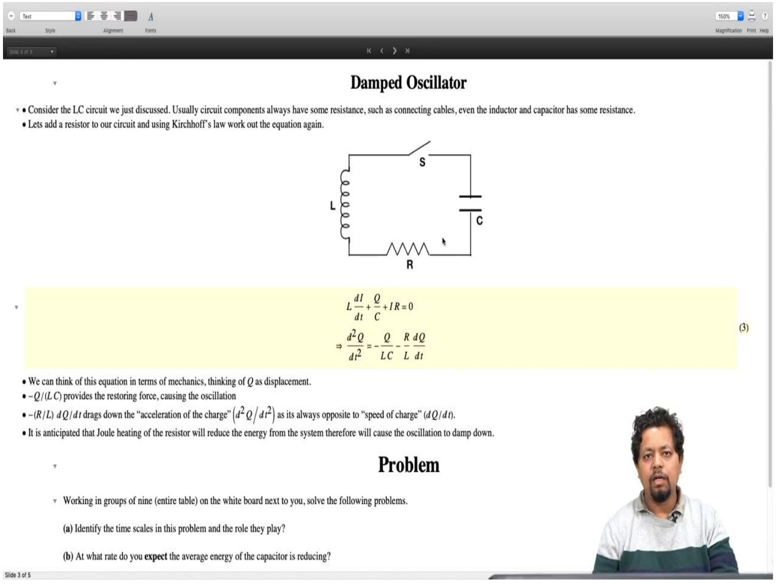

In the last video, we saw how to solve the damped oscillator analytically and how to verify that this is a solution of the damped oscillator analytically. What we will do is we will do this now with built-in Mathematica tools, using some tools of calculus also, to see how we can do this with Wolfram language. So, let us go ahead and get started with that. So, our job was to find out whether this satisfies a solution for equation number 3 over here and in that process we wanted to find out the constants A, α, β and ϕ.

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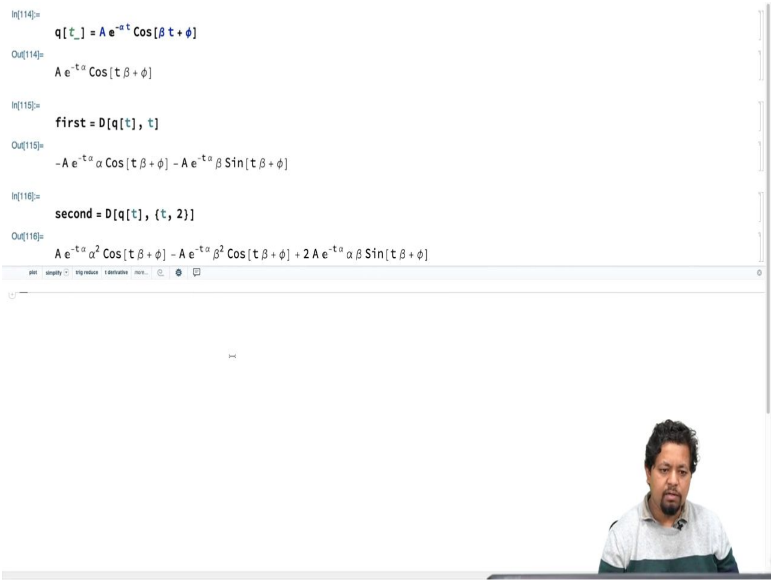

In order to do this what we will do is we will go ahead and define in the context of the mathematical definition of function $Q(t)$ and we will give it the form $A\,e^{-\alpha t}\cos(\beta t+\phi)$. That is $Q(t)$, we want to find $Q'(t)$, we will call that as first. That is the first derivative of $Q(t)$, we will say find D[$Q(t)$] with respect to t and I will just give the first derivative.

And you can, actually, there is a mistake $Q(t)$ is $A\,e^{-\alpha t}\cos(\beta t+\phi)$. So, first derivative is going to give me 2 terms, 1 with $-\alpha$ and the other is $-\beta$ and the *sin* over here. That is what we got when we did things analytically. And, for the second derivative we will again do D[$Q(t)$], with respect to t but this time the second derivative, so that gives me these 3 terms and Mathematica has automatically combined these 2 terms over here.

Now, the differential equation we have was over here, it was $Q''(t) = -Q/LC - R/L\, dQ/dt$ and we can go ahead and put that in here but it is useful to actually go ahead and non-dimensionalize this because wherever we do things in Mathematica, it is always better for non-dimensionalize this and work because that simplifies our system and can avoid a lot of confusion.

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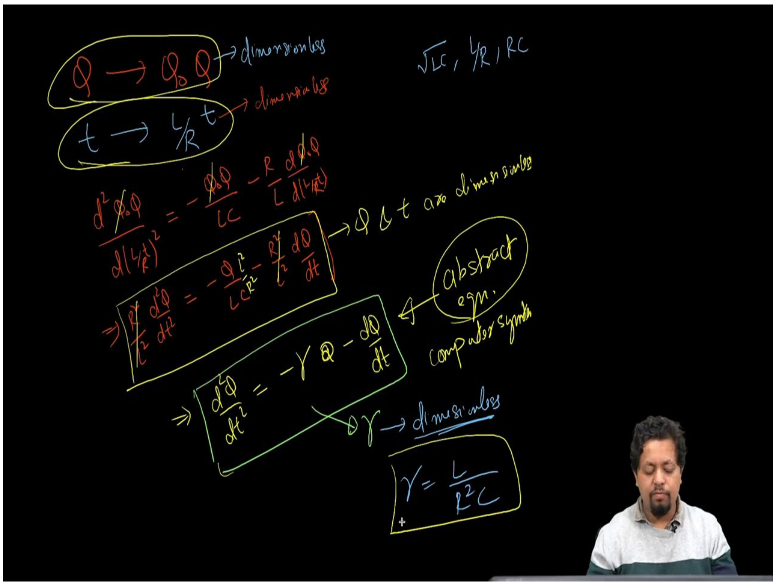

So let us go ahead and do that, we have got a charge scale Q which we are going to replace by $Q_0 Q$, when we do that, we are essentially drawing the scale Q_0 out of Q, so this Q is effectively dimensionless. And we also see that the time scales involved in the problem were $\sqrt{LC}$ and R/L, rather L/R. So, we can go ahead and make t go to $L/R*t$. So, that this t is dimensionless.

This makes that t dimensionless and with those 2 substitutions my equation becomes, d^2Q becomes $Q_0 Q$, dt^2 becomes $d(L/R*t)^2$ = -$Q_0 Q/LC$ and I get $-R/L$. dQ/dt is again d/dt of $L/R*t$ and $Q_0 Q$. Now, from here Q_0 is a constant, so Q_0 cancels out and I can pull out L/R from here and I get from here d^2Q/dt^2 is $R^2/L^2 = -Q/LC - R^2/L^2\ dQ/dt$.

In this equation, Q and t are dimensionless because we pulled out the dimensions of Q and t earlier. Now, we see that R^2/L^2 cancels between this term and that term, which means that I should multiply L^2/R^2 over here. And this simplifies to d^2Q/dt^2 equals minus, we get L/R^2C*Q - dQ/dt. Now, in this equation not only Q and t are dimensionless but we end up getting this constant which we are going to just call γ.

And this constant γ is also dimensionless because we know that L/R was a time scale and so was RC. So, this is the ratio of 2 time scales L/R and R and RC and therefore, this is dimensionless, so γ is also dimensionless. The advantage of doing things this way is that we get a simple equation, simply $d^2Q/dt^2 = -\gamma\, Q$. So, we simply get $-\gamma\, Q$, $-dQ/dt$ and that is a abstract equation and this is translated to be solved on a computer system.

This is good to be used on a computer system because it does not suffer from physical dimensions. So, this is the equation we are going to take for our system and after removing dimensions what happens is we are measuring Q in the units of Q_0, t in the units of L/R. Another consequence of that, I get my equation to be dimensionless, and it depends over a single parameter γ, which is also dimensionless. So, we are going to solve for everything in terms of γ, only a single parameter.

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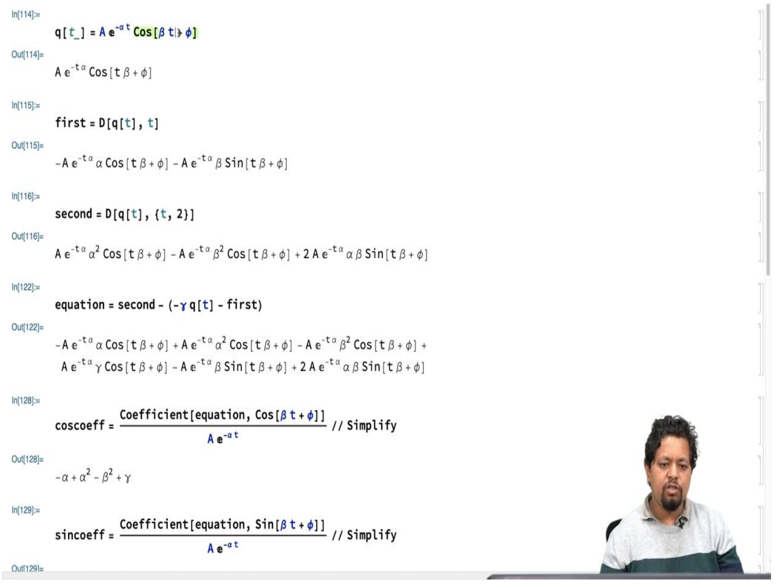

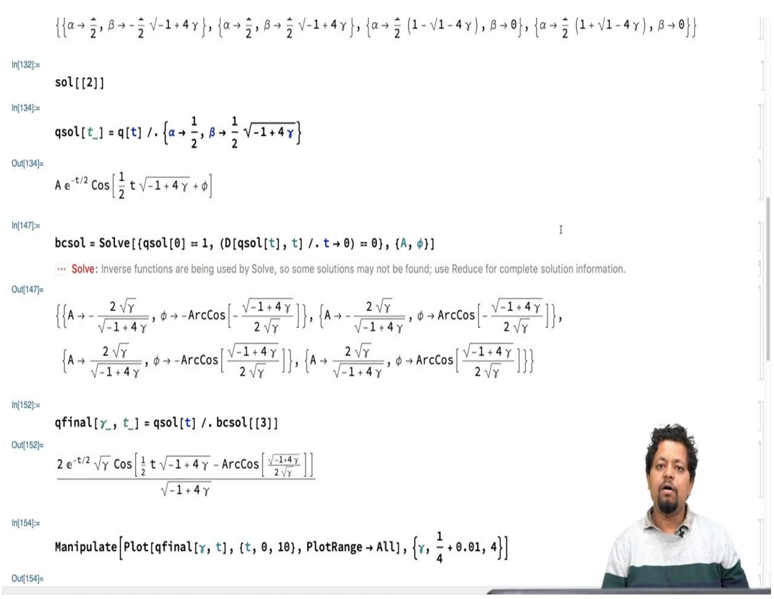

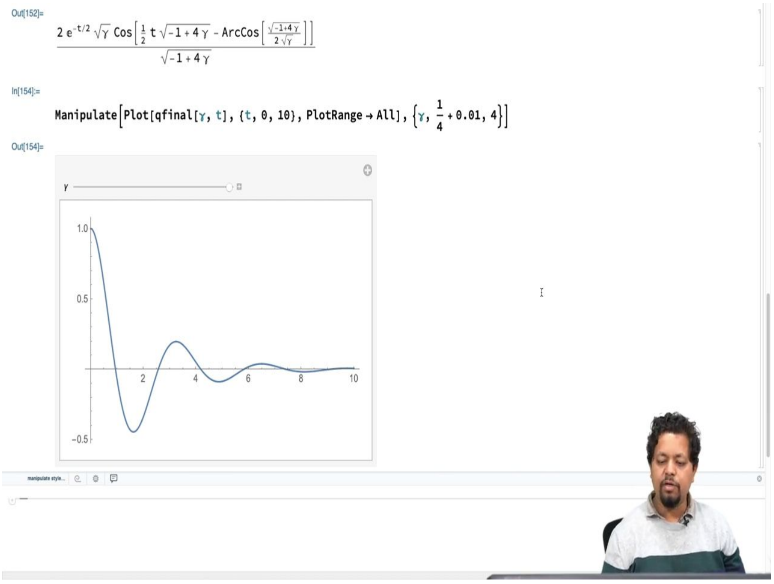

So, let us go back over here and say, put in the equation now. So, my equation was, the second derivative was equal to, so again I will use double equal to and that was equal to $-\gamma\ Q$ which is simply $Q(t)$ and I have got minus, look it up from here $-dQ/dt$. So, that is minus the first derivative. And I am going to go and execute that to see what I get and that is what I get and this is the equation I want to solve.

Now, this is simply a linear equation I want to solve and I can try solving it using the Solve function and I will just simply say solve this for α and β. And let us see what happens, that seems to be a lot of garbage, well that was not the most efficient way of doing it. Let us go ahead and look at this again, this is the equation we had, it is saying that this term is equal to that term and what you want to do is we want to read out the coefficients of cosine and sin.

But first, we will clearly see it from here $A\ e^{-\alpha t}$ is just going to factor out of this thing. So, what we should do is we should look at this equation; we should put a minus sign over here. Put this in the bracket, equal to 0 that is the equation we are looking at and I want to, I can go ahead and remove the right-hand side equal to 0. So, this is essentially the function we are looking at and we want to find the roots of this function.

Where does this function vanish? For that what you want to do is, this is, I mean to call as my equation and I am going to read out this sin coefficient or the cosine coefficients of this equation using the built-in function called coefficient. And I would say give me coefficients of the equation with respect to $cos(\beta t+\phi)$ and that is a much simpler result and I know that A and $e^{-\alpha t}$ are simply factoring out.

So let me divide by that. I expand that out or simplify, you can also use simplify and get the same result. So, this is the coefficient of cosine and similarly, you want the coefficient of sin term. So, let me go ahead and get the coefficient of sin term over here. I want to set this to 0, so let me go ahead and say this is 0 and I want to solve for this, well I should not do it like this, okay let me go ahead and give it a name.

So, let me call it cos-coefficient and let me go ahead and call this as sin-coefficient, now I will solve for the set of equations cos-coefficient equals 0 and sin-coefficient equal to 0. I will say that solve this set of equations for α and β and these are simply algebraic equations, so this is going to give me some solution, again I see that this time I get 4 solutions, 4 sets of solutions, every time we need 1 value of α and β.

And here I have got a solution corresponding to $\beta=0$, I get the branch of solution, the two values of α I get for the branch of solution $\beta=0$. For $\beta=0$, we have a overdamped oscillator, there is no oscillation. It just damps out, then the damping factor in the exponential is α and there are two values of α that are there, one is ½ $1+\sqrt{1-4\gamma}$, another is ½ $1-\sqrt{1-4\gamma}$.

So, this is the overdamped case, we were right now not interested in that, let us go ahead and look at the underdamped case, again for the underdamped case there is 2 solution, one is when we have got $\beta=-½$. This time it is the square root and plus ½ , the square root and you see that these 2 solutions are identical; in both the cases α is simply half. α is a constant which is half.

In both these cases β is dependent on γ and there is a difference between a sign of β. Now, that does not matter because remember what solution was $cos(\beta t+\phi)$, if I change sign of β, cosine is an even function, I can pull the sign out of, that just changes sign of ϕ. So, changing sign of β is equivalent to changing sign of ϕ and ϕ is just a integration constant and therefore, these two solutions are equivalent.

I am only interested in one of them, so let me just take this one because this gives me a positive β, of course, the condition for positive β is $4\gamma > 1$. $4\gamma = 1$, this vanishes and that is the case of critically damped solution in which case these 2 solutions become identical and when $4\gamma > 1$, sorry when $4\gamma > 1$, I get underdamped solution.

When $4\gamma < 1$, I get over-damped solution with 2 positive values of α and when I have critically damped solution $4\gamma = 1$. And in both these cases all these 4 solutions become identical to each other. That is about overdamped, underdamped and critically damped solutions, I want to extract this particular solution out, so let me go ahead and call this as my solution. And I am going to go ahead and extract the second item out of solution which is this.

And at this point, I can go ahead and say *q(t)/.* α is this and β is that and that is what I get for *q(t)*, this I am going to call as *qsol(t)* equal to, I never defines the function of t so that is my solution function now. I need to know determine what is A and ϕ, so for that, I am going to use the boundary conditions. The boundary conditions say that *qsol(0)* = Q_0, so that is simply $A\cos(\phi)$.

So, I am going to go ahead and set that to 0, that is my 1 equation and the second equation is $qsol'(t)$, so I am going to go ahead and say, let me do this separately first, so I want the derivative of *qsol(t)* with respect to t and I want to set this, I want to set $t=0$, so I can go ahead and simply say t goes to 0 and that gives me this and I want to set this whole thing must be 0. That is my boundary condition.

So I have got 2 boundary conditions, one is solution $q(0) = 0$, or rather sorry, $q(0)$ is some constant *q0* and derivative of $q(0) = 0$. So, that is the second equation which I will put that together over here. So, now that gives me this set of equations which I should simultaneously solve to figure out what is A and ϕ. Again, for that I will use solve and I will simply say, go ahead and give me, for these 2 set of equations give me A and ϕ and that should give me A and ϕ in terms of *q0*, here we go.

So we get a bunch of solutions again and if you closely examine them, most of these solutions are equivalent to each other and we will just take this solution in which A is positive. Which is these 2 cases because changing sign of ϕ, changing the sign of β, I can change the sign of A or adding a phase of the $\phi/2$ or 2ϕ, you can change the sign of A. So, let me just go for positive branch solution which is where A is positive, so these 2 cases and over here, the only difference in the 2 solutions is the sign of ϕ which is positive here and negative here.

And therefore these 2 solutions are equivalent because cos is a even function, so I can take any one of these solutions, so I am already interested in this particular branch of solution. I can go ahead and extract this out, again this is, I will go ahead and call this boundary condition solution and from *bcsol* I want to extract the third item, which in this particular case. I will go ahead and in my *qsol(t)*, I will substitute this condition.

And this gives me the final function which I will go ahead and call as *qfinal(t)* as a function equal to, and of course, at this point, you may want to substitute a value of gamma also. So, substitute some value of $\gamma > ¼$ because we are looking at the underdamped case. So, let me go ahead and put in $\gamma = 2$ and that gives me the solution. You can go ahead and make a plot of this?

So, plot *qfinal(t)* with respect to *t* from 0, 10, now there is a *q0* so let me substitute *q0* also to some value. Let me just say this is 1 because *q0* was actually 1 in our dimensionless case, in fact that is what we should have done here. Q_0 was 1, so then I do not know it, *q0* and now I can go and make a plot of *qfinal*, we get a very beautiful plot here for damped harmonic oscillators.

To add to the fun, we can actually go ahead and realize that this *qfinal* is a function of γ, γ is the ratio of time scales, that involve the problem ratio of damping time scale and the oscillation time scale. So, we will go ahead and call this as a function of α and *t*, that is that and we will go ahead and manipulate this and add manipulation parameter, sorry it was gamma and not α.

This was $\gamma(t)$ and manipulation, add γ here. So, plot is with respect to *t* but I am going to manipulate the parameter of γ here and I will choose some small value of γ, let us see what γ mean, γ was *L/R*C*, so a large value of γ means oscillation is dominant, a small value of γ means damping is dominant. So, we will take γ to point 1, 2 where γ has to be ¼ .

So, we will take it to be ¼ onwards to a large number like 4. Of course $\gamma = 0$ it does not like so we could have set here, slightly larger value than 4. So, 0.01 and that is just about critically damped and then as I increase γ, oscillation starts to dominate and we start to see the damping become smaller and smaller.

There we go, we have solved this problem completely using Mathematica's built-in functions, derivative function, solve function, coefficient function. And you see that it helps you avoid a lot of analytical mistakes that you can make when you are doing, solving problem by pen and paper. This will be all for today and in the next lecture, we will take another example of a damped oscillator.