Physics through Computational Thinking
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Lecture 21
Damped Harmonic Oscillator – LCR Circuit

Welcome back to Physics through Computational Thinking. Today we are going to talk about damped oscillators and in this video, we will work out through examples of damped oscillators, so let us go ahead and get started.

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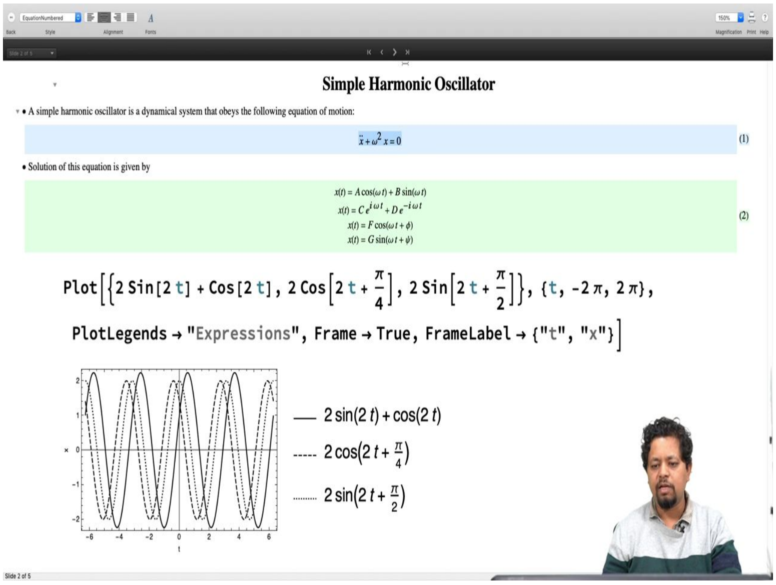

For a quick review we talked about a simple harmonic oscillator in the second week, this is the equation for simple harmonic oscillator and its solution can be written in these 4 different ways. And we plotted some of the styles of solutions and we saw that all of them represent oscillatory sinusoidal behaviour. Let us go and make this equation a little bit more complicated by adding a damping term to that and to do that we will take the example of LCR circuit.

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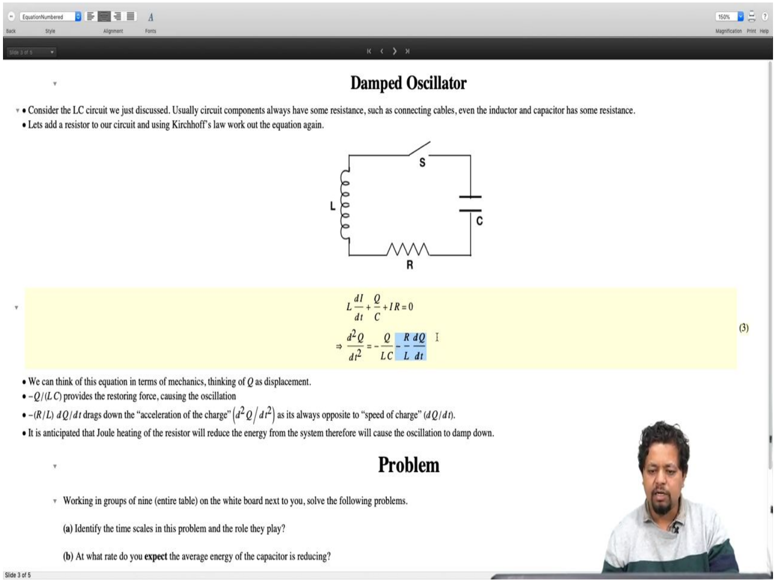

So, we will take a damped oscillator which is damped by resistance and this is a very natural example to consider because it is extremely difficult to make a pure LC circuit. All the components are made out of metals, inductors, and capacitors; they come with some amount of resistance. So, there is not really a perfect inductor or perfect capacitor. So, wherever you build a LC circuit, there is always a small resistance component to that and that is being represented by this resistance R over here.

So, we will take this circuit to be an LCR circuit, where inductance L, pure capacitor C and resistance R in the circuit. And for the LC circuit, we saw that the equation was $L\, dI/dt + Q/C = 0$. Using the Kirchhoff's law of loop, when we add a resistance, the potential drop across the resistor is $I*R$ and adding that terms give me the equation for the damped oscillator or this LCR circuit is $L\, dI/dt + Q/C + IR = 0$.

Now, the current I can write as dQ/dt, so when I write current as dQ/dt, I get this equation over here. And d^2Q/dt^2 is now $-Q/LC$, this is a restoring term and $-R/L\, dQ/dt$ which is the damping term because dQ/dt is like change of current and it is always in the negative direction, negative to change of current and therefore this is the term that causes damping. And actually, you know when you add a resistor to a circuit. There is Joules heating, that joules heating is responsible for losing the energy of the system.

And therefore the oscillation amplitude or the charge in the capacitor is going to die down and that is why we understand this term as a dissipative or a damping term. So, let us go ahead and look at this problem over here. There are 4 or 5 different parts of this problem and what we will do is, we will go ahead and solve out these parts and understand how to go about it.

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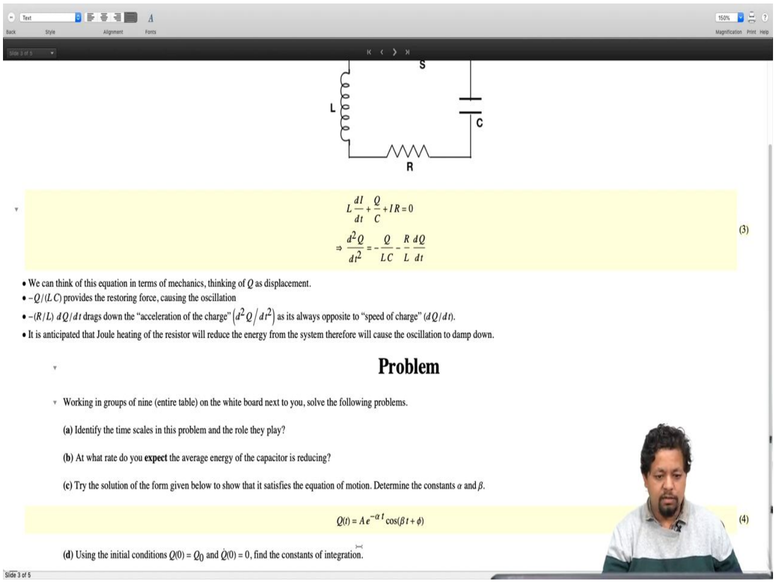

So, the first question here is to identify the time scales in this problem and the role they play. I have got an equation, I have got a circuit, this equation has got a bunch of time scales, you have to figure out what are the time scales in this problem and what is their importance? Then, we have to calculate at what rate do we expect the average energy in the capacitor to reduce. And then, the third part of the problem is try the solutions of the form given below. This is the solution that is given.

We will try out whether this is an acceptable solution for this differential equation and often enough a good way of solving differential is knowing a solution or guessing a solution. This is what we are trying to do here, we are guessing a solution, we are assuming that the solution is out of this form and see if this satisfies the differential equation, in that process, we will fix some of this unknown constants in the problem.

Later on, we will try to solve this as an initial value condition with the initial condition that $Q(0) = Q_0$ and $\dot{Q}(0) = 0$ and Q that is the charge in the capacitor at 0 is Q_0. And I think this last questions are repeat of b, at what rate average energy the capacitor is decreasing assuming a very small resistance in the circuit, so I think that is just the repeat of this, so we can remove that. Let us go ahead and solve this problem, so we will go back to the blackboard.

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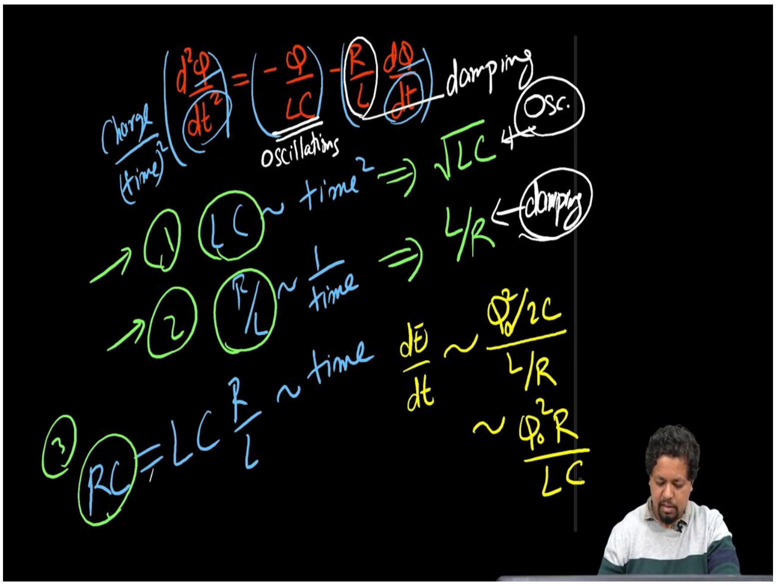

The equation that we have got here is $d^2Q/dt^2 = -Q/LC - R/L\, dQ/dt$. We have to find out what are the time scales in this problem. Now, we can read out the natural time scales that are present in the problem using the differential equation itself, each of the term here has dimensions of Q/t^2. You see that this term has dimensions of charge by time square.

And, if this equation is right then each of these terms has to have the same dimensions and that tells me that *LC* must have dimensions of time square and similarly here I see this is a 1 term of time present here that 2 of here is *Q n* the numerator, there is other *Q* in the numerator, so 1 *t* dimensional cancel out. So, I must have that *R/L* should have dimensions of *1/t*. That is the only way the time will make sense.

Because *LC* dimension of time square and *R/L* has dimensions of time. I see that if I multiply *LC* and *R/L*, so *LC* and *R/L*, if I multiply them together, I get dimensions of time. But, this is simply *RC*, and it is not surprising we know from *RC* circuits that *RC* is the time constant for the *RC* circuits. Therefore, we get 3 dimensions, 3 scales in the problem or 3 dimensionful quantities in the problem that are dimensions of time.

Made up with the constants of the problem, so this what we call as time scales in the problem, so there are 3 time scales in the problem. We have got first time scale LC, second time scale R/L, where LC is not the time scale, the time scale here is $\sqrt{LC}$ and the time scale here is L/R. And the third time scale, I have here is RC which is in some sense an emergent time scale given the time scale, given this is first 2 time scales.

So, we have got 3-time scales in the problem, let us see what this time scales are doing and for that, we have to go back to physics, we have to look at the equation and ask what is each term doing and we can note from here that, this term over here this is responsible for oscillations. So, this must be the primary time scales for oscillations and clearly, resistance is the damping factor, this is the damping term.

So, L/R must be responsible for damping. It is not evidently clear directly what is RC doing but it is an emergent time scale, it is a product of oscillation in damping time scale. So, these are the 2 important time scales in the problem. The oscillation time scale and the damping time scale and they are doing interesting dynamics that is what we are going to explore in this problem.

Let us go to the next question says that what rate do you expect for the average energy in the capacitor to be reducing and that we can do the estimation based on dimensional analysis that we have done over here. Let us go and think about that, so I want to find what is dE/dt. To estimate that, I can use dimensional analysis and dimensional analysis simply says that numerator is dimensions of energy.

So, we can simply go ahead and estimate the dimensions of energy. So, energy in the capacitor is given by $Q^2/2C$, so that is an estimation. I can write the energy scales 2 square by 2C. But, Q itself is given by Q_0 which is the initial charge in the capacitor. So, we can estimate approximately energies of the order $Q_0^2/2C$ and time, there are various different time scales present here question is which time scale to choose.

And we might be tempted to say that let us choose the damping time scale L/R because the time responsible for damping is L/R. So, let us go ahead and put that time scale over here and that gives an estimate for dE by dt or the rate at which energy is changing in the capacitor.

That is 1 estimate and this approximately values out to Q_0^2, the constants are not important. So we get Q_0^2 *R/LC*.

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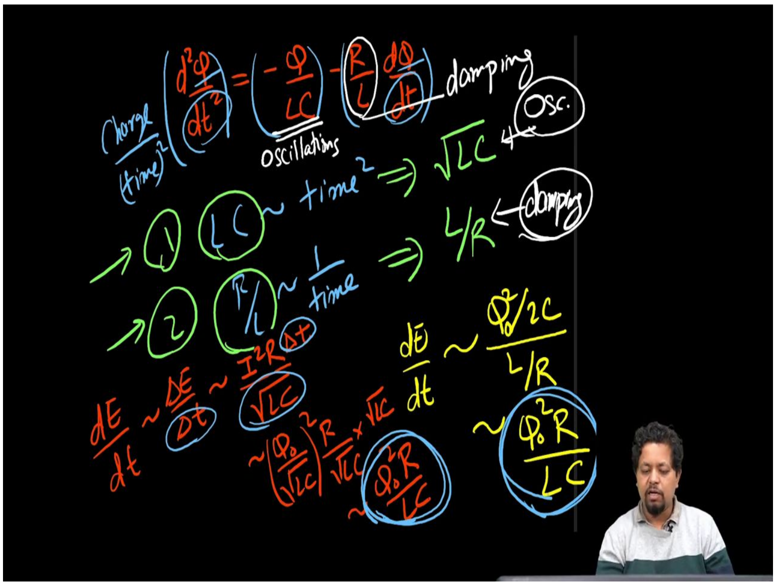

Alright, let us go ahead and look at another way of the same problem and for that, I will make some space here, I will erase this part. And let us go and think of another way of doing this problem and I can say dE/dt can be estimated by, especially when the capacitor, when the resistance is very small, joule heating is very small, damping is very slow. I can say that dE/dt is $\Delta E/\Delta t$, where ΔE is rate at which energy is changing.

So, energy is lost in one cycle and in one cycle because of joules we know the energy lost is IR^2. It is the order of IR^2 and the time taken for that is the oscillation time period which is of the order of $\sqrt{LC}$ and we can go ahead and put in units for I. I is current, so it is again current over time. So, current is Q_0, current is charge over time, charge is Q_0, time is $\sqrt{LC}$.

So, I get one another $\sqrt{LC}$ factor and that squares up because I already have 1 $\sqrt{LC}$ factor and resistance squared, so that gives me $Q_0^2 R/LC$. I think, sorry this is not IR^2, it is I^2R, so let me go and correct that. So, let me go ahead and correct that, joule heating is $I^2R/\Delta t$ which is the $\sqrt{LC}$ and I get I^2 is $(Q_0/\sqrt{LC})^2$ and I get.

Actually, the power is I^2R, so I have to multiply ΔE is joules heating which is I^2 times, the time, Δt which is again of the order of oscillation time period. So, I^2 is $(Q_0/\sqrt{LC})^2$, times $R/\sqrt{LC} * \Delta t$ which is of the order $\sqrt{LC}$.

So, as a consequence, I get here $Q_0^2\, R/LC$. Interesting thing is that by both logic we found the same result. So, the naive dimensional analysis over here gave us *d/dt* simply based on dimensional analysis gave us $Q_0^2\, R/LC$ and when we took into account physical input that damping is happening because of joule heating.

Joule heating is I^2R, it happens for a time period for 1 time period of oscillation that is Δt and that Δt is given by $\sqrt{LC}$. As a consequence, we got substituting all of that and estimating current is Q_0 over the oscillation time period because Q_0 of the charge is going out of the capacitor over the oscillation time period.

The estimated current is $Q_0/\sqrt{LC}$ and putting out altogether we get the same result, so physical input, and naive dimensional analysis using simply dimensions of energy on time give us the same result. So what we learn from here is dimensional analysis is a very powerful tool, sometimes we can get fantastically accurate results by simply doing dimensional analysis.

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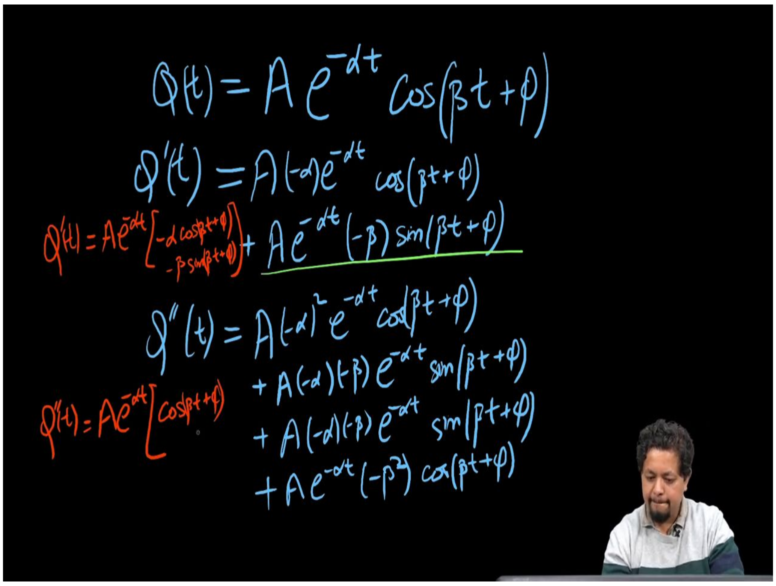

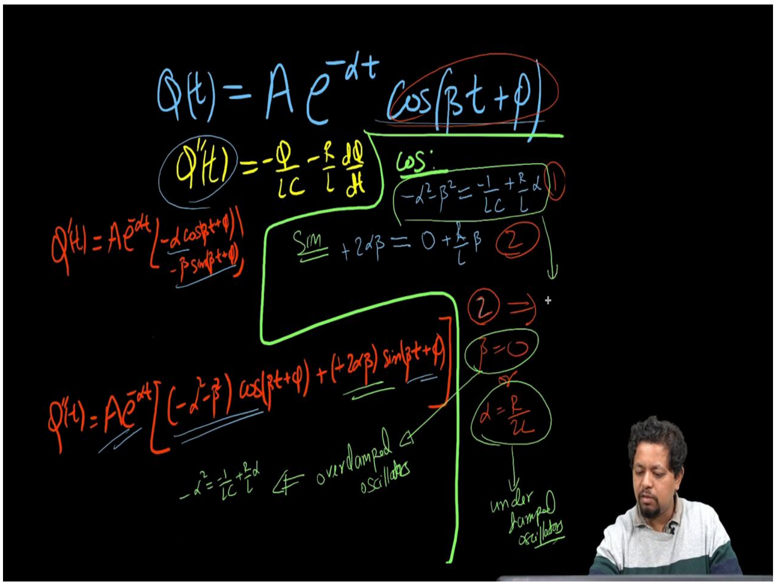

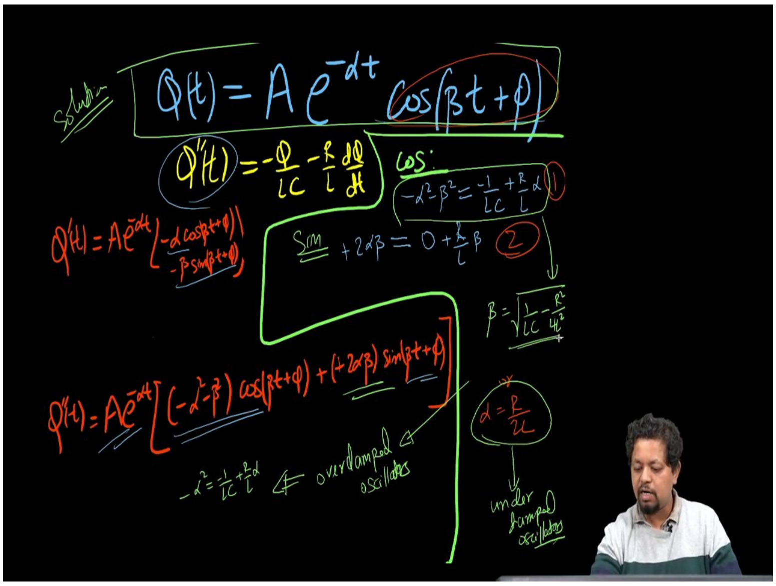

Let us go ahead and look at the next part of the problem which says that try this form of the solution and see if this works out as a solution for this differential equation. So, let us go ahead and try that out. And we found the solution that you want to try out is $Q(t) = A\ e^{-\alpha t} \cos(\beta t+\phi)$ and in order to do this, I will need to find out what is $Q'(t)$ or $\dot{Q}(t)$ which is A.

For the first term I get $-\alpha\ e^{-\alpha t} \cos(\beta t+\phi)$ and taking the derivative of the cosine term I get $A\ e^{-\alpha t}$, the derivative of cosine is a $-sin$. So, I get the $-\beta \sin(\beta t+\phi)$. So, that is your prime and I also need $Q''(t)$, so for that, I am going to take $Q'(t)$ and take one more differentiation and that is going to give me A, again when I differentiate it with respect to α. I get $-\alpha^2$ and $e^{-\alpha t} \cos(\beta t+\phi)$.

Taking the derivative of the cosine term gives me $-\beta$, so I get $A(-\alpha)(-\beta)\ e^{-\alpha t}\ sin(\beta t+\phi)$. Coming over to the second term over here, I get plus A, I am taking derivative with respect to α, I get $(-\alpha)(-\beta)\ e^{-\alpha t}\ sin(\beta t+\phi)$ and taking the derivative of the *sin*, I get $A\ e^{-\alpha t}$. I get derivative of cosine, and I get β out, so it is $-\beta^2$ and I get $cos(\beta t+\phi)$ back.

So, that is $Q'(t)$ and $Q''(t)$. Let me go ahead and clean up and to clean up, what I will do is I will go ahead and write $Q'(t)$ as $A\ e^{-\alpha t}$. And in the brackets, I can write $-\alpha\ cos(\beta t+\phi)$ and $-\beta\ sin(\beta t+\phi)$. And, again collecting terms in $Q''(t)$, $A\ e^{-\alpha t}$ comes out. I get 2 sets of term one is $cos(\beta t+\phi)$ and that is multiplying, $-\alpha^2$ and $-\beta^2$.

So, I can get rid of that and $sin(\beta t+\phi)$ term is multiplying $-\alpha\beta$ twice. So, I can get rid of that also. So, the cosine term is, comes with $(-\alpha^2)\ (-\beta^2)\ cos(\beta t+\phi)$ and sin term comes with $-2\ \alpha\beta\ sin(\beta t+\phi)$. Now, I can go ahead and get rid of this stuff and now what I want to do is, I want to look at the differential equation. So, that is my $Q(t)$.

Let me write down my differential equation, differential equation was $Q'(t) = -Q/LC - R/L\ dQ/dt$. So let us go ahead and substitute Q', dQ/dt and Q in this equation and compare on both sides the coefficients of sin and cosine terms, and, so if I look through all of that and compare coefficients of cosine term from $Q''(t)$, I get $(-\alpha^2)\ (-\beta^2)$ from $Q'(t)$ which is over here times $e^{-\alpha t}$ which is going to cancel out, so rather coefficient of $e^{-\alpha t}$ times cosine which was linear in the way to form αt times sin. So, comparing the coefficient, I get from the left-hand side $-\alpha^2+\beta^2$ and that must be equal to the right-hand side which is $-Q/LC$ from Q, I get the cosine coefficient as simply one that is multiplied by $1/LC$.

So, I get that and from dQ/dt, I have got a cosine term over here whose coefficient is $-\alpha$. So, I get minus minus plus, $(R/L)*\alpha$. That is my equation 1 and comparing coefficients of sin, I get from the left-hand side Q'' is the coefficient of sin, I get $-2\ \alpha\beta$ and that is multiplying $A\ e^{-\alpha t}$ times sin, coefficient of that is this and over here. On the right-hand side I have got $-Q/LC$ which does not have a sin term.

It only has a cosine term so we do not give any contribution from there. But, from dQ/dt, we have got a sin term, so I get $(-R/L)*(-\beta)$, so it becomes $+\beta$ and that is equal to let me put that back in. So, this is the left-hand side, the right-hand side the first term gives me no contribution, the second term gives me $(-R/L)*(-\beta)$ so that it becomes plus, gives $(R/L)*\beta$. So, this is equation number 2.

Now solving 1 and 2, we get, actually from 2 we get two solutions, equation 2 gives me two solutions, it gives me either $\beta=0$ or $\alpha=-R/2L$. I think there is a sign mistake, I should get $\alpha= R/2L$. So, there is a small mistake here, this should have been a plus sign, this should have been a plus sign, so now from equation 2 we get two solutions either $\beta=0$ or $\alpha= R/2L$.

Now, $\beta=0$ means that this term becomes simply a constant $cos(\phi)$ which can be absorbed inside A, therefore this term is meaningless and the solution is only $e^{-\alpha t}$, which simply means that it is exponentially decaying kind of solution. And that is a solution without any oscillation and this is called the overdamped case. So, $\beta=0$ gives me over-damped oscillators.

And that happens, when that happens the first equation gives me the value of α and so for overdamped oscillators I get $\beta=0$, so I get $-\alpha^2= -1/LC+(R/L)*\alpha$ which is a quadratic equation, we can solve and you get 2 values of α. So, the 2 exponential decay rates for 2 different branches of α and our general solution will be a linear combination of those 2. I will leave that for you to figure out.

Let us go ahead and talk about the other case where you get this branch of solution and this corresponds to underdamped oscillators and for this case I can go ahead and substitute $\alpha = R/2L$ inside the first equation and when I solve for the β, I am going to get, I need some space for that, so for underdamped, I am going to get $\beta = 1/\sqrt{}LC - R^2/4L^2$.

Therefore, what I see is that this equation provides me a solution for my differential equation urges me more than 1 branch of the solution, it gives me a branch of the solution when I have got $\beta=0$, that means no oscillation, just exponential decay, then I have got under damped oscillators where alpha is non-zero and beta is non-zero, I have also got a third mode which is called critically damped and that happens when I get exactly this term.