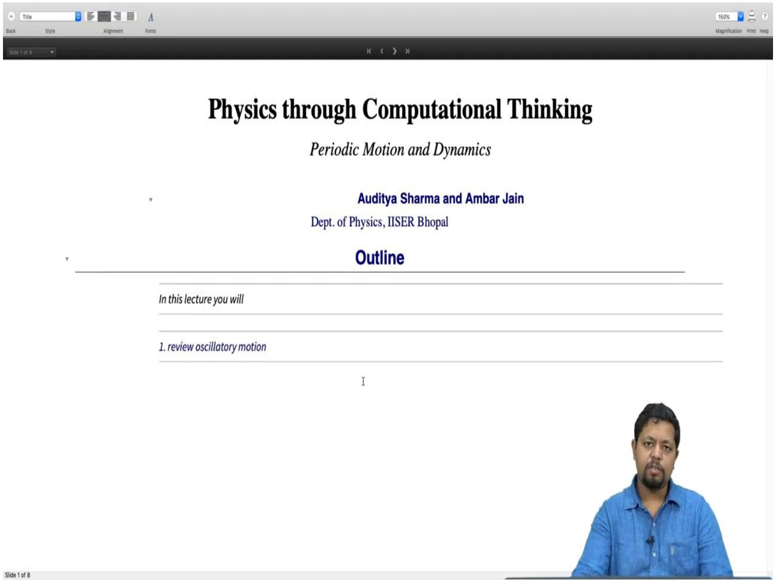
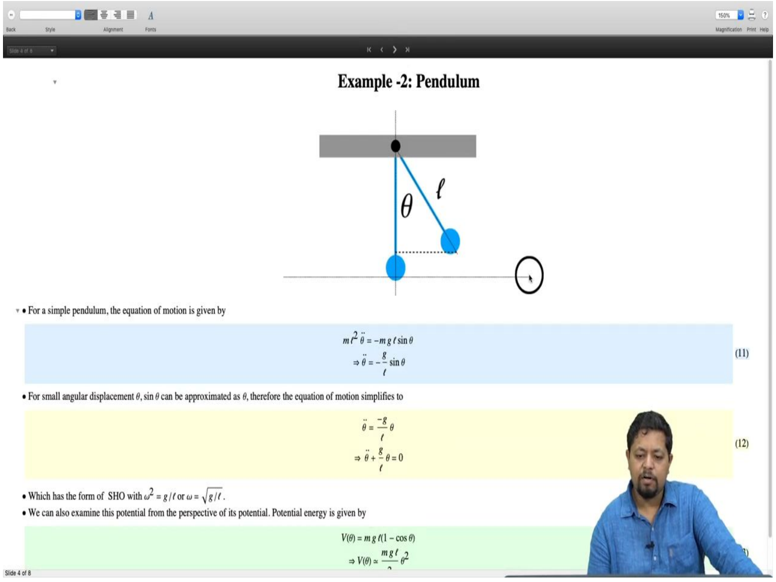
Physics through Computational Thinking
Professor Dr. Auditya Sharma & Dr. Ambar Jain
Department of Physics
Indian Institute of Science Education and Research, Bhopal
More Examples of Simple Harmonic Oscillator

(Refer Slide Time: 0:28)



Welcome back to Physics through Computational thinking. Today we will talk about periodic motion and dynamics. We will review oscillatory motion and periodic motion. Some of which you are already familiar with so this is going to be a quick review about the same, and we will slowly build upon to Anharmonic Oscillators. Let us get started. Let us go ahead and look at another example.

(Refer Slide Time: 0:48)

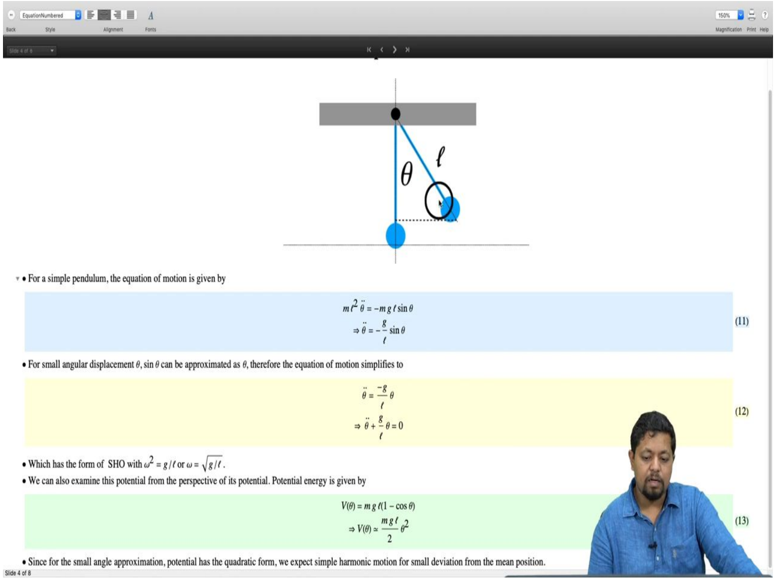


This time example of a Pendulum, a Simple Pendulum. As you all familiar for the simple pendulum, the equation of motion is given by ML^2 $\dot{\theta}$ = -MGL Sin(θ) where -MGL Sin(θ) is the torque that is the gravity is applying on this bob and ML^2 is the moment of inertia and $\ddot{\theta}$ is angular acceleration. So, I* $\ddot{\theta}$ is the torque and and from that we get $\ddot{\theta}$ = -g/L Sin(θ).

Now, that is not an equation of a Simple Harmonic Oscillator. So, Simple Pendulum is not actually a Simple Harmonic Oscillator. However, if you were to take the small angle limit that is small θ, we immediately say that Sin(θ) will reduce down to θ, and I will get a equation that is given by this line and I immediately see that this is the equation of Simple Harmonic Oscillator with g/L = ω^2.

So, from here I see ω^2 = g/L and therefore, ω = $\sqrt{g/L}$ the famous result that gives you frequency or time period for the Simple Pendulum.

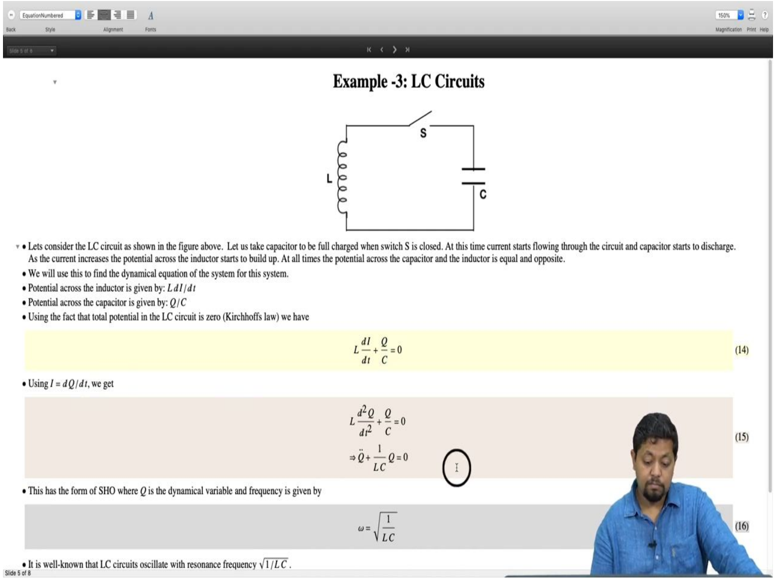
(Refer Slide Time: 2:08)



Now let us examine this problem from the perspective of a potential, for potential V(θ) = MgL (1 - Cos(θ)) because the potential here we can consider as 0. As a Pendulum goes up by angle θ, the height of the Pendulum gains is L (1 - Cos(θ)) and therefore, the potential becomes MGL (1 - Cos(θ)). And if you remember, we plotted this in one of the previous videos, when we plotted this we found that for small θ it behaves like a quadratic potential, and for small θ, you can write Cos(θ) as $1 - \theta^2/2$. And you see that $V(\theta) = MgL/2\ \theta^2$.

From that also you can obtain the same equation confirming that $\omega^2 = g/L$ or $\omega = \sqrt{g/L}$. So, therefore, Simple Pendulum also becomes a Simple Harmonic Oscillator in a small angle limit.

(Refer Slide Time: 3:02)



Let us take the third example of Simple Harmonic oscillation, this time from electromagnetism. And the example under consideration is LC circuits. So, let us take the picture shown over here. We have got a Circuit. On the left hand side we have got an Inductor. On the right hand side we have got a Capacitor and they are connected by a wire and there is a Switch S in between.

Let us assume that the Capacitor is fully charged in the beginning and there is no current flowing in this Circuit. At that moment, I close the Switch S. What happens? As I close the Switch S we have a complete circuit, where current starts to flow through this circuit. And as a current flows, there is a potential drop across the Inductor given by minus L dI/dt.

And after some time as the Capacitor discharges, dI/dt maximizes, the Inductor becomes fully charged and the Capacitor becomes completely discharged. And after some time as dI by dt changes sign Capacitors also charge again Capacitors becomes fully charged. And there is no current to the circuit.

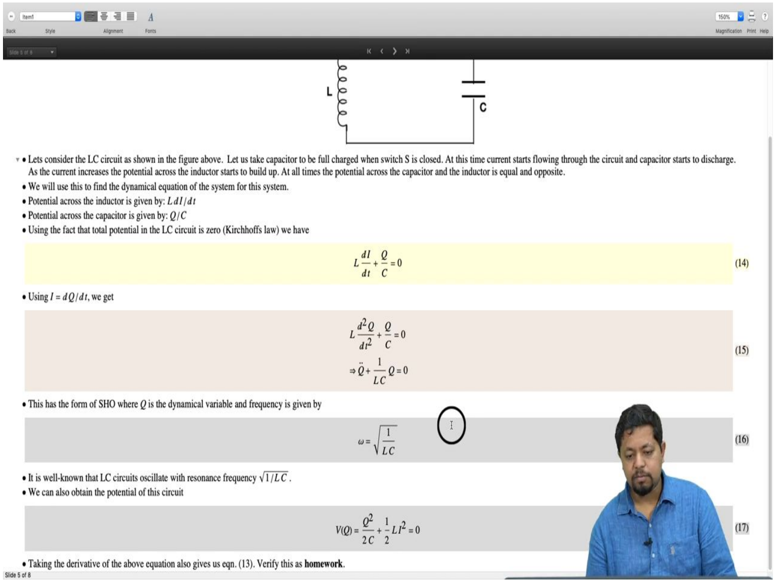
And eventually this flips back so there is an Oscillation going on over here, the charge on the Capacitor oscillate, there is a +Q over here and -Q over here after some time the charge on the Capacitor plates will flip and the current in the Inductor will also, direction of the current in the Inductor will also change. There is Oscillation going on between the potential is transferring from Capacitor onto the Inductor and from Inductor back to the Capacitor.

So, in this problem we will explore exactly that. In terms of equations, we can use the Kirchhoff's law and write down the equation for this Circuit. Using the fact that the total potential in the LC Circuit is 0. We have the potential drop across the Inductor that is L dI/dt + Q/C that is the potential drop across the Capacitor must add up to 0. Therefore, L dI/dt + Q/C = 0.

Now, using the fact that current I is dQ/dt, I can replace this I with dQ/dt, I get this equation $Ld^2Q/dt^2 + Q/C = 0$. I immediately see that this becomes an equation of Simple Harmonic Oscillator where Q as a dynamical parameter, if I divide this entire equation by L, dividing this entire equation by L, I bring the equation to this form where my dynamical parameter $\ddot{Q} + 1/(LC)\ Q = 0$.

So, charge is what becomes a dynamical quantity and it is the charge that is Oscillating. Remember this charge Q was a charge we took on the Capacitor. The charge was the charge that we took on the capacitor, so we find that the capacitor charge is Oscillating. But because Capacitor charge is Oscillating and current is dQ/dt, we will see that the current will also Oscillate. So, let us go ahead and work this out.

(Refer Slide Time: 6:15)

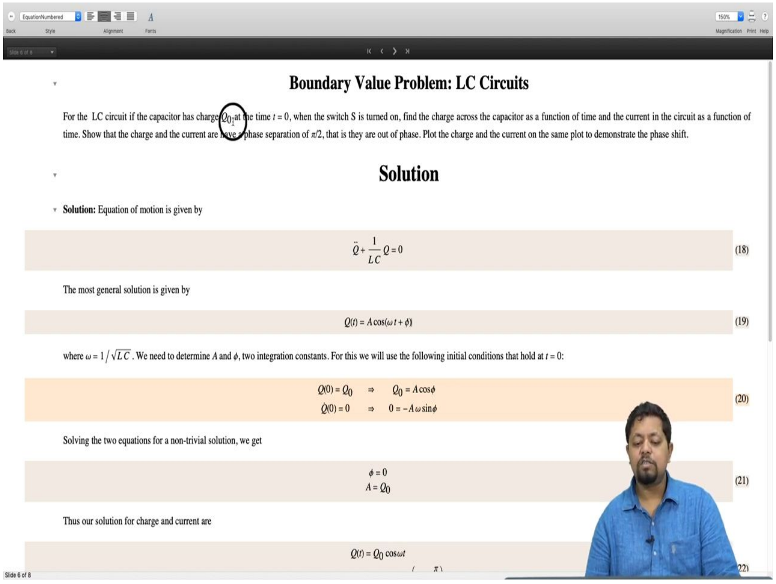


So, since $1/(LC) = \omega^2$, we get the result that $\omega = \sqrt{1/LC}$. Let us go ahead and analyse the system from the view of the potential, for the system, you can work with a potential by simply adding the potential for the Capacitor and potential for the Inductor, the potential across the Capacitor is ½ Q^2/C for the Capacitor and for the Inductor is a ½ LI^2.

So, this is my total potential $Q^2/2C$ + ½ LI^2 and if I do $-dV/dQ$, you can verify that you will reproduce this equation back. Therefore, potential formulation also helps in this case. So these three were examples of Simple Harmonic Oscillations. We looked at a spring mass system, we looked at a simple pendulum in the small angle limit. And we looked at the LC circuit.

So, we see that the Simple Harmonic Oscillation appears in many diverse examples from various areas of physics, even in quantum mechanics, classical mechanics, electrodynamics, in almost every field of physics that you see, you will come back again and again at Simple Harmonic Oscillation.

(Refer Slide Time: 7:28)



So, let us go ahead and look at now a boundary value problem for the LC circuits. So far, we were just discussing examples, in general. But let us now go ahead and ask. Let us go ahead and solve a specific boundary value problem so that we can work out the constants. So, let us go ahead and think of our LC circuit that the Capacitor having a charge Q_0 at time t = 0, as shown over here. So, let us assume that our Capacitor was fully charged at time t = 0, which is when the Switch is turned on.

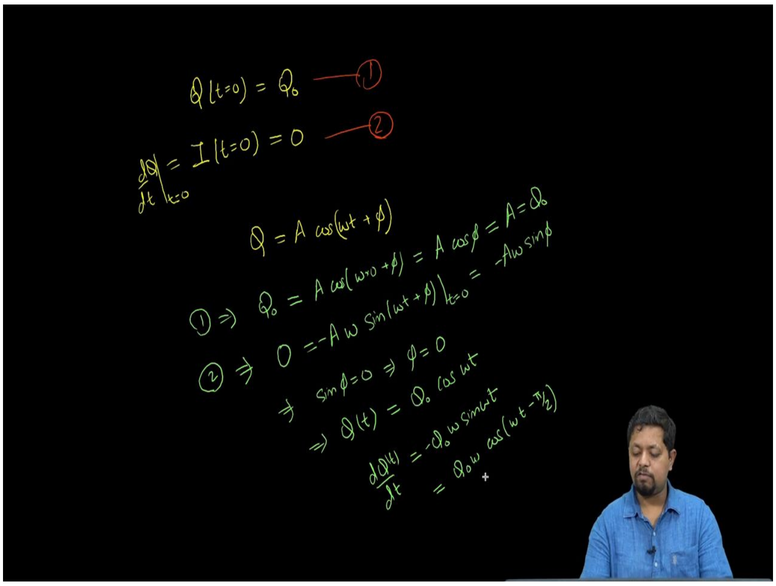
Now, let us find the charge across the capacitor as a function of time. And current in the circuit as a function of time, then show that the charge and the current have a phase separation of п/2. That is, they are out of phase.

Two Oscillations are in phase when they are Oscillating together, but here, what is going to happen is that the charge in the current are not going to oscillate together, charge will peak at a different time and current will peak at a different time. As a consequence, there is going to be a phase shift between charging current and we have to work out that that that, we have to show that that phase shift or phase separation is п/2. Plot the charge and current on the same plot to demonstrate the phase shift.

So, let us go ahead and execute this problem again taking our Computational thinking approach. So, let us first look at the equation of motion is $\ddot{Q}$ + 1/(LC) * Q = 0, solution of that is straight forward is Qt = A Cos(ωt + φ). Now we have to find out what is A and φ, this is the solution of this equation. But that does not tell me what is A and φ.

In order to find out A and φ, I need the boundary conditions. And in this case boundary conditions are given to me. Looking at the boundary conditions at t = 0, my charge is Q_0 and Switch is just turned on. So, there is no current flowing through the circuit.

(Refer Slide Time: 9:28)



Therefore, I can say that charge in the capacitor at t = 0 is Q_0 and current in the circuit at t = 0 is 0. So, these are my initial conditions, current is dQ/dt = 0 at t = 0.

So, these are my initial conditions given to me, using and the solution is Q(t) equal to Acos(ωt+φ) to find out what is A and φ, so, let me use the conditions here, using the first condition over here, so from 1, I get, Q_0 that is charge at t = 0 is Acos(ω*0+φ) I have substituted t = 0, therefore I get this equal to Acos(φ) . So Acos(φ) = Q_0 as the first equation for me.

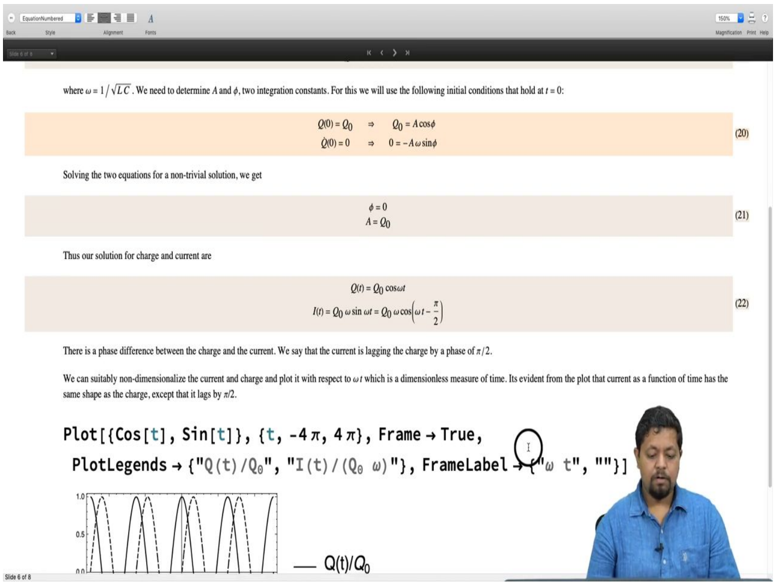
Second equation gives me dQ/dt that is current at t = 0 so dQ/dt, current is 0, but dQ/dt from here I get is Aω and derivative of Cos is -Sin so -Sin(ωt+φ) and this is 0 therefore from this at t = 0. So, this I have to evaluate at t = 0. So, I get this equal to minus -Aω Sin(φ) .

Now, I obviously cannot have A = 0 because thing A = 0 I get the trivial solution Q = 0. So, I must have from the second equation Sin(φ) = 0, Sin(φ) = 0 means φ = 0 and therefore, my solution is simply $Q(t) = Cos(\varphi)$ becomes $\pi/2$ as, I substitute $\varphi = 0$ it becomes $Cos(\varphi) = 1$ so this becomes A.

So, $AQ_0 = A$ this is Q_0 and I get my solution as $At = Q_0\ Cos(\omega t)$ now that is my charge and I can work out dQ/dt also now that I know charge is a function of time. I can work out current as a function of time and taking that derivative I get Q_0 - $Q_0\omega Sin(\omega t)$.

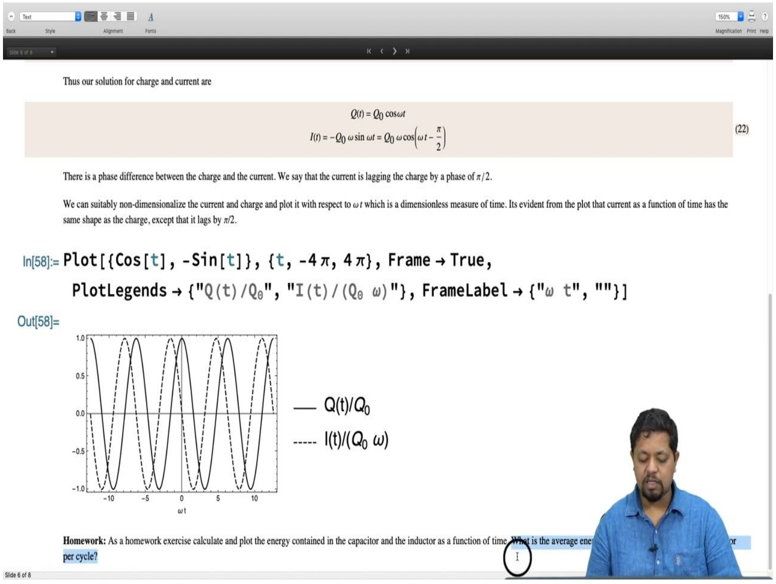
At this point, I can go ahead and write down $Sin(\omega t)$ also I can write down in terms of $Cos(\omega t)$ and I can write this down as $Q_0\ \omega Cos(\omega t - \pi/2)$. You can work out that $Cos(\omega t - \pi/2)$ is nothing but $-Cos(\omega t)$. Let us go ahead and verify this.

(Refer Slide Time: 12:42)



So, I have worked out these conditions for you on the blackboard. I got $\varphi = 0$ and $A = Q_0$ and that gave me $Q(t) = Q_0\ Cos(\omega t)$ and $I(t) = Q_0\ \omega Cos(\omega t - \pi/2)$. So, we see that from these 2 solutions $Q(t) = Q_0\ Cos(\omega t)$ and $I(t) = Q_0\ \omega Cos(\omega t - \pi/2)$ that these 2 solutions have a phase shift and current is lagging the charge by a phase of $\pi/2$. Let us go ahead and verify this by plotting over here.

(Refer Slide Time: 13:20)



So, here is my plot of $Q(t)/Q_0$ and $I(t)/(Q_0\omega)$. In order to do the plotting I need to non-dimensionalize and the way I do non-dimensionalization is I divide $Q(t)/Q_0$. So, Cos(ωt) is a dimensionless function $Q(t)/Q_0$, Q has dimension of charge, Q_0 is of dimensions of charge. So, $Qt(t)/Q_0$ is dimensionless. So, what I am plotting on the Y axis over here is $Q(t)/Q_0$.

Similarly, for the charge, for the current I am plotting $I/(Q_0\omega)$ because $Q_0\omega$ has dimensions of current. Q_0 is the natural scale for the charge present in the problem. ω is the natural scale or $\omega = 1/\sqrt{LC}$ is the natural scale for inverse of time present in the problem. So, natural scale for current in the problem becomes $Q_0\omega$. So, I divide the current by $Q_0\omega$, and that leaves me with Cos($\omega t - \pi$).

And that is what I am plotting over here, I am plotting over here, Cos(t) and Sin(t). And this plot gives me I am using a minus sign over here. So, let me put that minus sign. Now, this is correct, and I should put current is -Sin(t) plot that and there we go. So, what we see is that the charge is given by the solid curve or the dimensionless. The charge on the capacitor and dimensions unit $Q(t)/Q_0$ is given by the solid curve and current is given by this dashed curve.

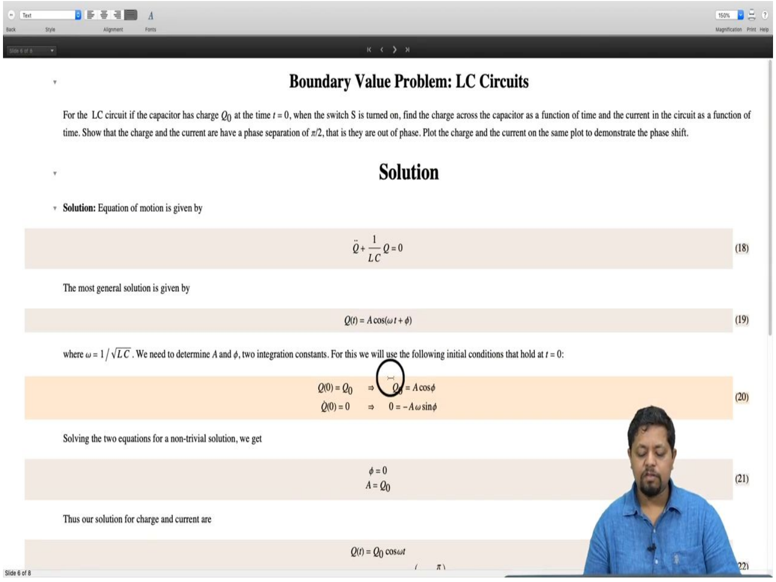
We see that the charge is always leading the current that is charge peaks first and current peaks later, current reaches this maximum after the charge. This is because first there is no current, the charge is maximum as the charge depletes to 0 that is when the current reaches this maximum and so on.

So, this is how by plotting the dimensionless quantities dimensionless current and dimensionless charge in the same plot we can find out that whether the current is leading or the charge is leading and we can see the shape between the charge and the current. As a homework exercise I want you to calculate and plot the energy in the Capacitor and energy in the Inductor as a function of time. Remember, energy in the Capacitor is $Q^2/(2C)$ and energy in the Inductor is $\frac{1}{2} LI^2$.

Now, you know the charge in the current so you can work out what is the energy in the Capacitor and energy in the Inductor and you can plot both of them on the same plot. What is the average energy contained in the capacitor and the inductor per cycle or the oscillations are very very fast.

So, you really cannot measure energy as a function of time but you will really see if you try to make energy measurement you will see average energy per cycle, when the frequencies are very high, that is, when the frequencies are few Kilohertz, the oscillations are so fast that you will not be able to measure the energy as a function of time but all you will see is average energy. So, think about how to calculate the average energy in the capacitor and the inductor per cycle.

(Refer Slide Time: 16:40)



So, this was an example of a boundary value problem where we solved a particular, given particular boundary conditions on initial value conditions we found the solution for the charge and the current in LC Circuit.