Waves and Oscillations
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LecturE_09
Damped Oscillator: Problems

Welcome to this 4th module. So, in this module, we will try and do some Problems originating from the Damped Oscillator; so, a very quick review before we plunge the problems.

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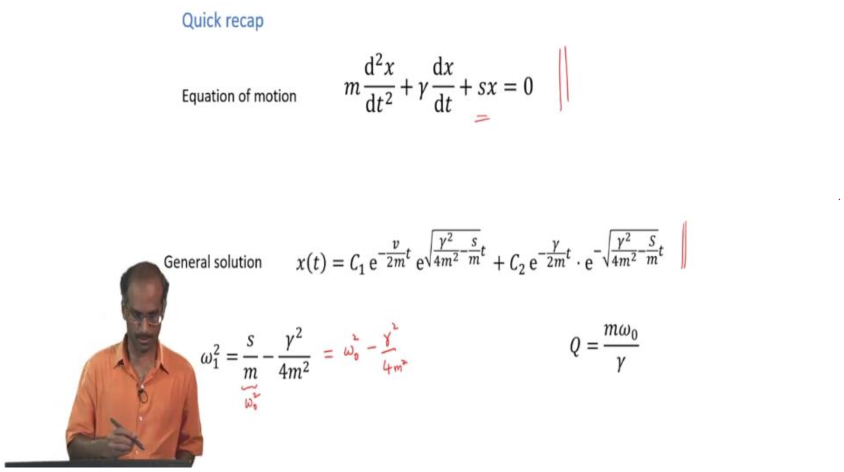

So, as usual we begin with the equations of motion, and they are here. And just to remind you γ is the dissipation coefficient and s is the stiffness constant. And in general you can obtain solutions for this we spent last three modules doing that, and here is the general solution. And two other things that we will be needing are this ω_1 which is the frequency of the damped oscillator, and in fact this quantity here is the is ω_0^2 which is the natural frequency of the undamped oscillator. So, I can write this as $\omega_0^2 - \gamma^2/(4m^2)$ And we also learnt about the Q-factor or the Q value which is $m\omega_0/\gamma$. So, this is the approximate value which is valid for valid when the dissipation is small. So, with this background let us quickly get the first problem.

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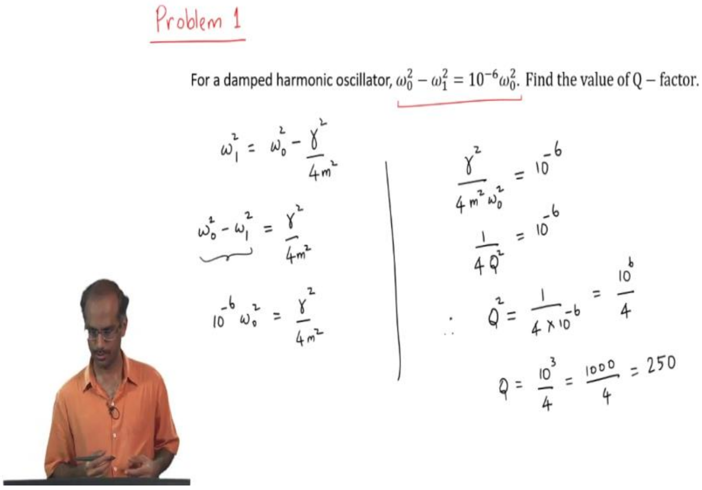

First problem is among the simplest. For a damped harmonic oscillator, we are given that $\omega_0^2 - \omega_1^2$ is equal to this value, 10^{-6} ω_0^2 . We need to find the Q-factor. So, all we need is the formula that is given here. And we just take rearrange this equation, $\omega_0^2 - \omega_1^2$will be equal to $\gamma^2/(4\mathrm{m}^2)$. And we are told that this quantity here is simply equal to $10^{-6}\omega_0^2$ that is $\gamma^2/(4\mathrm{m}^2)$.

So, if i bring ω_0^2 down here, I will have $\gamma^2/(4\mathrm{m}^2\omega_0^2)$ is equal to 10^{-6}. Therefore, the quantity I have here is 1 over 4, and this is simply q^2 in the denominator. Therefore, q^2 will be equal to $(1/4)10^{-6}$ which is equal to $10^6/4$. So, Q-factor is $10^3/4$ that is 1000 / 4 equal to 250. And remember that Q is dimensionless quantity. So, it is a ratio of two energies, hence it has no dimensions or no units.

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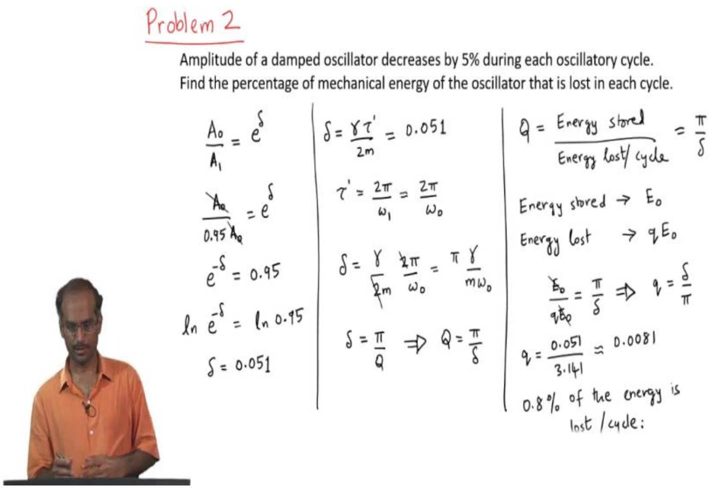

In the second problem the question is amplitude of a damped oscillator decreases by 5 percent during each oscillatory cycle find the percentage of mechanical energy of the oscillator that is lost in each cycle. So, remember that energy is simply square of the amplitude for our purposes. To do this problem what we need is the ratio of the amplitudes A_0/A_1 is e^{δ}. So, hope you remember the definition of δ that we had which was $\gamma\tau'/2m$, where τ' is the time period of the damped oscillator.

Now, A_0 is the initial amplitude of the damped oscillator. And the statement here says that after one period it decreases by 5 percent in which case A_1 will simply be equal to $0.95A_0$ and that is equal to e^{δ}. So, A_0 and A_0 will cancel out, and I will get $e^{-\delta} = 0.95$. So, if we take logarithm on both sides, we will get δ to be equal to 0.051. You can do this calculation yourself with the calculator and check that this is the value of δ.

Now, we know what is the value of δ which means that we know that this is equal to 0.051, you also know that $\tau' = 2\pi/\omega_1$. And as usual we are going to assume that the damping is sufficiently small so, that this is approximately equal to $2\pi/\omega_0$. So, now, if I substitute this in the value of δ, I will get $\delta = (\gamma/2m)2\pi/\omega_0$, 2 and 2 will cancel out, I will get $\pi\gamma/m\omega_0$. So, this tells me that δ here is equal to π/Q. So, if you remember $m\omega_0/\gamma$ is the Q-factor.

From this I can rewrite this equation as $q = \pi/\delta$. We know that Q is equal to energy stored by the energy lost in one cycle. So, what we are interested in is this energy lost per cycle and this is equal to π/δ. So, let me say that energy stored initially is E_0 and energy lost is some fraction of it which I will call qE_0. So, with this in place all I need to do is to plug in these two in this equation both E_0 and q times E_0.

If I do that, I am going to get the following. This will imply that again $E_0$and E_0will cancel out, and it tells me that q is equal to δ/π If I substitute the value for δ I can find out the value of q. So, δ is 0.051 $/\,\pi$ is approximately 3.141. So, this will give me a number which is 0.0081. So, this implies that about 0.8 percent of the energy is lost per cycle. So, that is the answer to this question.

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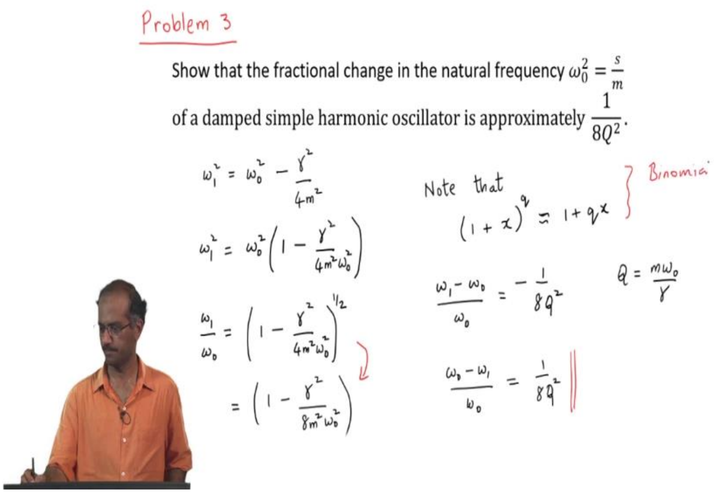

So, in the third problem the question is show that the fractional change in natural frequency which is stiffness constant divided by mass for a damped simple harmonic oscillator is approximately equal to $1/q^2$. So, the starting point to solve this problem is we should be able to relate the frequency of the damped oscillator to the frequency of the undamped oscillator. So, from here we just need to do some few simple manipulations. Starting point for this is the formula which we had seen before. So, we will just slightly rearrange this equation. And if I take square root, I will get ω/ω_0 is $\left((1-\gamma^2/(4m^2\,\omega^2)\right)^{1/2}$.

So, here is where we will do a binomial approximation. And if I do the binomial approximation, I will get $1-\gamma^2/(8m^2\omega_0^2)$. So, note that if x is small for a function like this, binomial approximation would mean that $1+\ q\,x$. So, this is the case when you terminate the binomial approximation at the end of the first term, which is exactly what I have done here. And from here we recognize that $m\omega_0/\gamma$ is q, and we can take one on the other side. If we do all that, we should be able to get $(\omega_1 - \omega_0)/\omega_0 = -(1/8)q^2$. Notice that Q is equal to $m\omega_0/\gamma$.

And now with the slight rearrangement $(\omega_0 - \omega_1)/\omega_0$ which is the fractional change in the natural frequency would be equal to$1/(8q^2)$. So, this is the answer that is required. And the approximation here is was done here applying binomial approximation.

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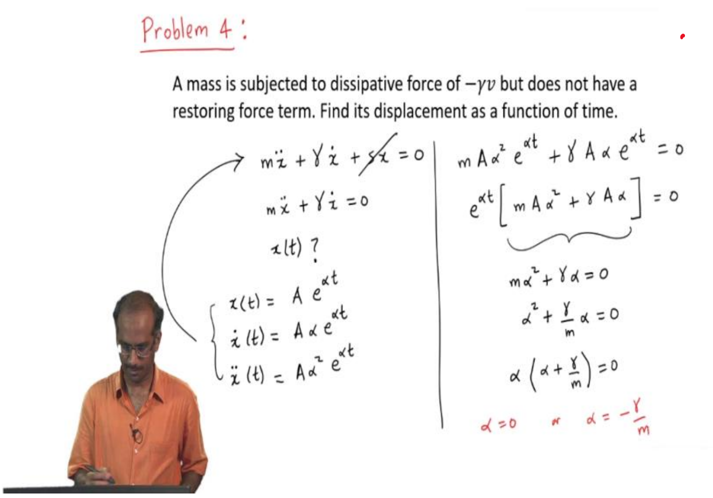

Let us start with the next problem in this case a mass is subjected to a dissipative force equal to $-\gamma v$, where v is the velocity, but it does not have a restoring force term in other words it does not have a sx term. So, we need to solve this equation, and find the displacement as a function of time. So, let us do this. So, normally this is your equation of motion for the damped oscillator. However, in this case, we are told that this term does not exist. So, the effectively the equation is $m\ddot{x} + \gamma\dot{x} = 0$. And we need to find solutions which means I need to find x (t) that is the question.

So, the method of solution is same technique as the one that we adopted for solving the dissipative harmonic oscillator. So, let us assume solution x (t) to be A which is a constant $e^{\alpha}t$, and we will determine this α in a while $\dot{x}(t)$ would be $A\alpha e^{\alpha t}$; $\ddot{x}(t)$would be $A\alpha^2 e^{\alpha t}$.

Now, we need to simply substitute these things back in this equation. If I do that, I will have $mA\alpha^2 e^{\alpha}t + \gamma A\alpha e^{\alpha}t = 0$, and take e^{α}toutside. For this equation to hold identically and $e^{\alpha t}$is not 0 for arbitrary values of α and t. Hence the condition is that this quantity is equal to zero. And you see that there is a in both the terms of this equation which can also be removed. So, I will have $m\alpha^2 + \gamma\alpha = 0$. And I can for example, divide throughout by m in which case it will be, so I take α out I will have $\alpha + \gamma/m = 0$. So, the solutions are easy either $\alpha = 0$ or $\alpha = -\gamma/m$. So, these are the two possible values of α.

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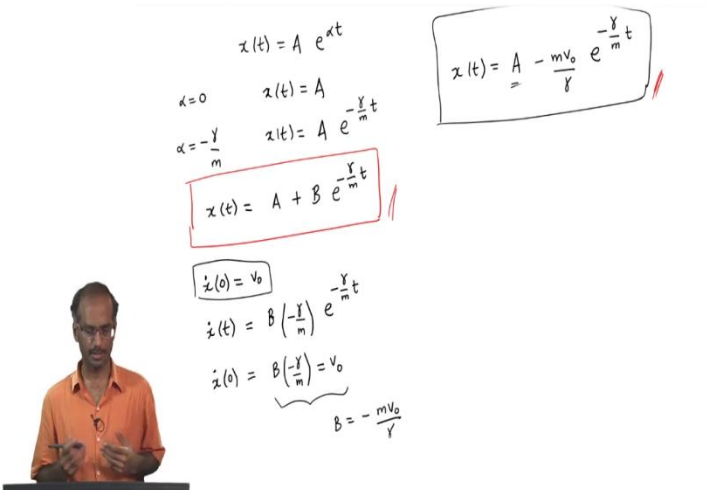

I can now plug this back the equation, the equation that I assume $Ae^{\alpha t}$. If I take α to be equal to 0, $x(t)$ would be equal to A, which is a constant. On the other hand, if $\alpha = -\gamma/m$, x (t)would be $Ae^{-(\gamma/m)t}$. So, these are two linearly independent equations. These are two linearly independent solutions. You cannot obtain one from the other just by multiplying a constant. So, there is one constant which is A, which is already there in this equation.

So, the general solution for displacement as a function of time would be e power this one, because we were solving a second order differential equation which is this. We should have two independent constants and that is satisfied by this form of the solution. If we had some initial conditions, we can plug in that initial condition and reduce one of the constants. For instance, let us say that I know that $\dot{x}(0)$ velocity at time 0 is equal to some v_0. So, in that case, one of the constants can be eliminated as follows. I need to find first $\dot{x}(t)$from here $\dot{x}(t)$ would be $B-(\gamma/m)e^{-(\gamma/m)t}$.

And if I put $\dot{x}(0)$, so if I put t equal to 0, I would get $B(-\gamma/m) = v_0$. And this implies that the constant $B = -mv_0/\gamma$. Hence my solution would be $A - (mv_0/\gamma)e^{-(\gamma/m)t}$. So, this is a solution which incorporates one of the initial conditions which is this. So, still we are left with one more constant which is A here, and that can be determined provided we have one more initial condition. So, this is a sort of general technique that you could follow for any problem. In general when you have two such constants, you need two independent conditions to determine them.

And if you think about this problem a little more physically then as a mathematical problem, it tells you that the restoring force term is not that which means that you are not going to have oscillations in the first place. And when you look at the solutions that we have obtained either this or this, they are not oscillatory solutions. So, this is an exponentially decaying solution so is this an exponentially decaying solution. So, in the absence of restoring force term, you are not going to get oscillations, and the solutions pretty much tally with that expectation.

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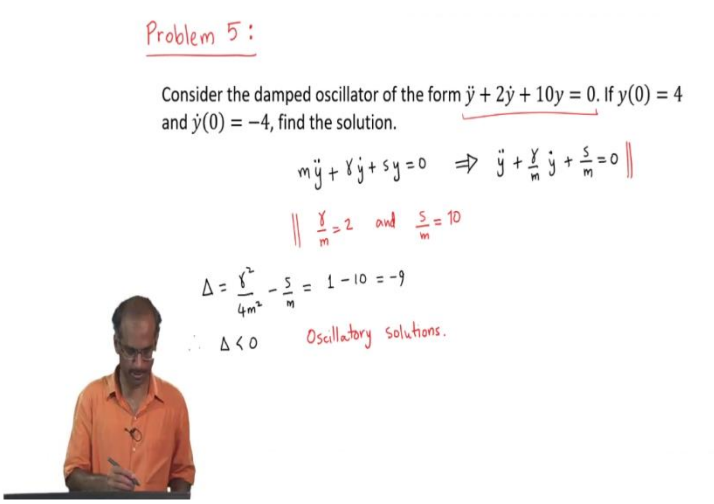

The last problem for this module is here. So, in this case, we have the equation of damped oscillator given to us which is $\ddot{y} + 2\dot{y} + 10y = 0$; and two initial conditions are given. We simply need to write down the explicit solution. Let us start from the equation of motion for the damped oscillator $m\ddot{y} + \gamma\dot{y} + sy = 0$, and I will divide throughout by m, so that I can make it similar to the equation that is given here. So, it will be $\ddot{y} + (\gamma/m)\dot{y} + s/m = 0$.

Now, when you look at this form of the equation and this equation given here, they are exactly identical provided we make the correspondence that $\gamma/m = 2$, and $s/m = 10$So, before we find the solution, we need to find out in which regime we are. Are we in the heavy damping regime or in the critical damping regime, or are we in the damped oscillating regime, and that will be determined by these set of parameters.

The quantity of interest there is let me call it δ, it will be $\gamma^2/(4m^2) - s/m$. So, if I substitute the numbers here, γ/m is 2. So, this would be 4 by 4, $1 - s/m$ is 10, so that is minus 9. Therefore, the quantity $\delta < 0$; hence we should expect to see oscillatory solutions. So, it would be an oscillating solution which will be damped. So, purely from the parameters of the problem, you could determine whether you are in the regime of heavy damping, critical damping or damped oscillations. So, here we are in the regime of damped oscillations. The next step is to write the solutions.

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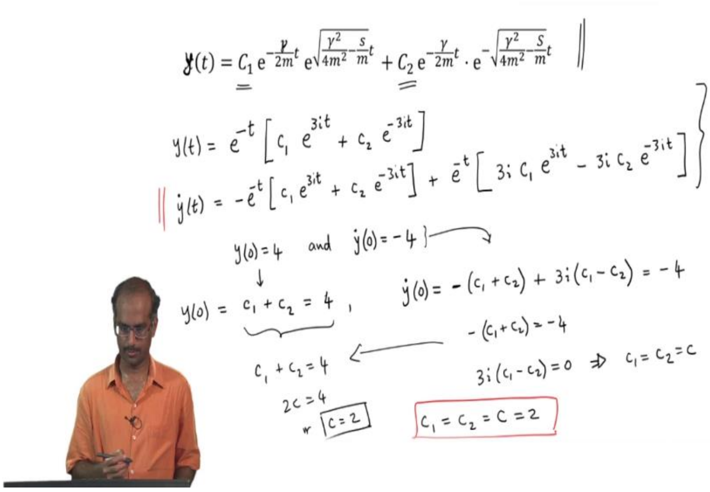

Since we are in the oscillatory regime or damped oscillatory regime, the solution that is going to be relevant for us is given here. And you will see that it has two paths to it with two constants C_1 and C_2. Now, we can plug in the numbers from the parameters of the problem. And if I do that I should be able to write my solution as follows. Once I have written down the solution for y, I can also write the solution for $\dot{y}$. I have an expression for

$y(t)$ and $\dot{y}(t)$ and we also have two initial conditions which is $y(0) = 4$, and $\dot{y}(0) = -4$. I just need to plug these things back in these two solutions; $y(0)$ will be equal to $C_1 + C_2$. And according to the first initial condition it is equal to 4 so that follows from here.

And $\dot{y}(0)$ if I put in t equal to 0 in this equation, I have this expression for $\dot{y}(0)$, and according to this initial condition this is equal to minus 4. So, this straightaway gives you that $C_1 + C_2 = 4$. Whereas, this second condition on $\dot{y}(0)$ gives you complex number, and whereas the initial condition says that it has to be a real number, it has to be equal to minus 4. So, this would then we can equate the real parts and imaginary parts, hence it would give me minus $C_1 + C_2 = -4$, which is same as the condition that we had gotten earlier.

And the second one which is $3i(C_1 - C_2) = 0$, because there is no imaginary part in the initial condition that is given. So, this would imply that C_1 is equal to C_2. And so we just need to put let us say that C_1 is equal to C_2, and it is simply equal to C in which case this would become 2, $C = 4$ or $C = 2$. Hence the constants are $C_1 = C_2 = C$, which is equal to 2. So, we have determined these constants. Now, all that remains is to write the solution explicitly.

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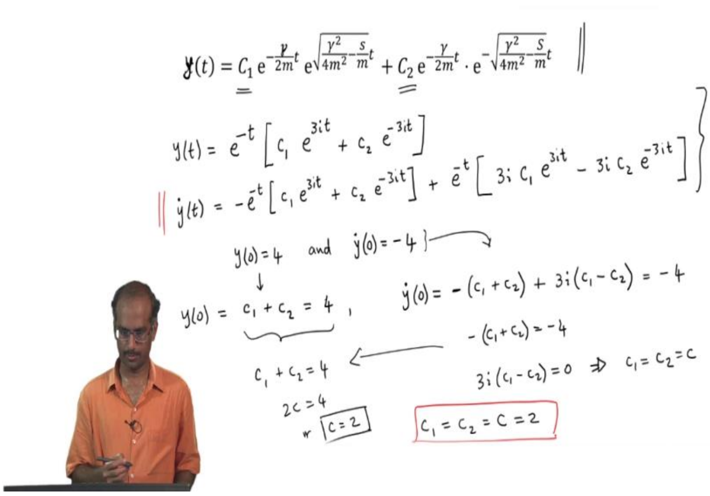

So, this is the explicit solution we wrote down earlier for displacement as a function of time. The undetermined constants are C_1 and C_2. And we now know that its equal to both are equal to 2, hence our explicit solution will be $y(t) = e^{-t}(2e^{3it} + 2e^{-3it})$. Now, you can simplify it a little more, you can take 2 outside and get $2e^{-t}$. And this suggests that multiply and divide by 2, then the quantity here is a cos function, hence we can write a very compact result like this $y(t)$ would be equal to $4e^{-t}\cos 3\,t$. This would be the final result in its compact form. So, you will notice that this has the dissipative part and the oscillatory part as we expected based on the parameters that was given for this problem.

So, with these five problems, I will close this session. But I encourage you to try out the problems that is given in the assignment.