Waves and Oscillations
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Lecture – 08
Damped Oscillator and Q-factor

Welcome back. We were looking at the Damped Oscillator. In this module also, we will continue to do so. In particular, we are going to look at how do we characterize the damped oscillator.

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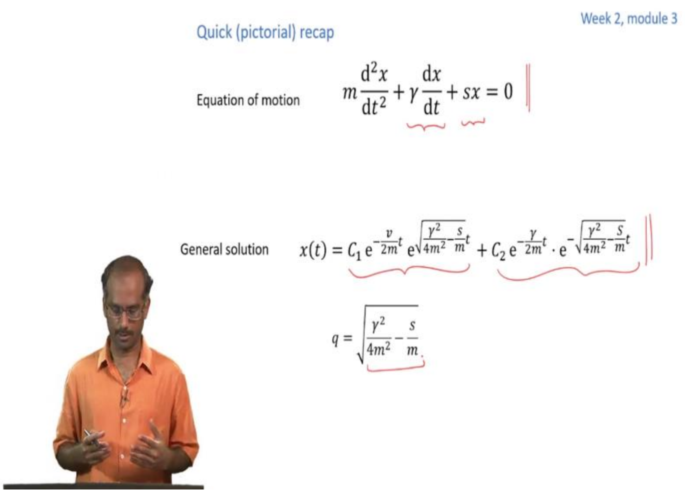

But before we do that let us quickly do a recap of what we did in the last module. If you remember we wrote down the equation of motion for a damped oscillator and I have this here, this is the equation of motion. And the important terms to notice are the term that involves the dissipation coefficient γ here and of course, this is the term that gives us oscillations in the first place.

So, we have a dissipative oscillator and we also solved this equation of motion and the general solution for this equation of motion is given here. So, it is displacement as a function of time. And once again I would like to draw your attention to the fact that you what we have written down here is a combination of two linearly independent solution. So, this is one linearly independent solution and the other one is this, C_1 and C_2 are arbitrary constants in general.

Now, this general solution will reduce to various possibilities and to what it will reduce to depends on this quantity here, inside the square root. So, depending on whether this quantity is either positive or negative or 0, we are going to get different kinds of dynamics for the damped pendulum.

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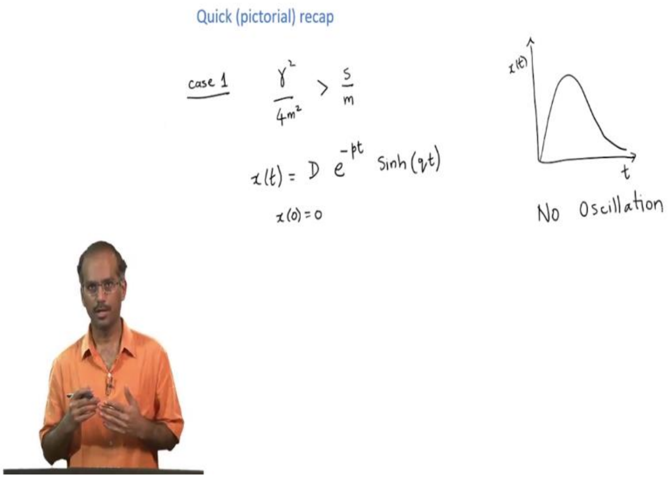

Let us quickly see each on them. So, we saw that in the last module when γ^2 is, $\gamma^2/4m^2 > s/m$ this is what we called as case 1. So, this is the case where dissipation is dominant compared to the restoring force or the stiffness coefficient.

So, in this case we saw that you could write the solution in the following manner. So, $x(t)$ would be let us say some constant, e^{-pt} and $\sinh q\,t$. To get this particular form of the solution we had put in this initial condition that displacement at 0 =0. I am plotting displacement as a function of time. So, we might get some solution that might look like this for instance. So, this is time here on the x axis. So, this is the case where there is no oscillation. You just give a push to the system, a system which is experiencing huge amount of dissipative force. It just comes back to the equilibrium position, no further movement is possible.

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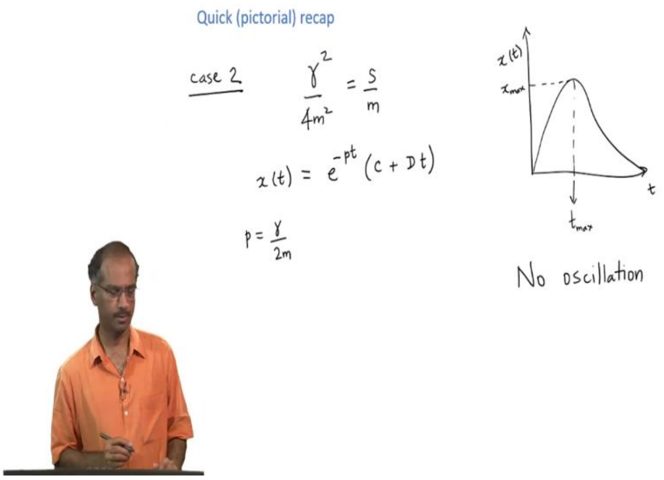

So, let us go to the parametric regime that we called case 2, just in the earlier module. So, in this case there is a precise balance between the dissipation represented by $\gamma^2/4m^2$ and the restoring force like coefficient or the stiffness coefficient that is s/m. In this case we were writing the solution as e^{-pt} and $C + Dt$. And just to remind you we use this p to mean $\gamma/2m$.

So, again this is a case where no oscillations are possible. We can quickly sketch the solution. Displacement as a function of time would again go something like this, ok. And this is the time at which t max, at which the maximum displacement happens. And I had left it as an exercise for you to find out the maximum displacement. So, here again the central result is that no oscillations are possible.

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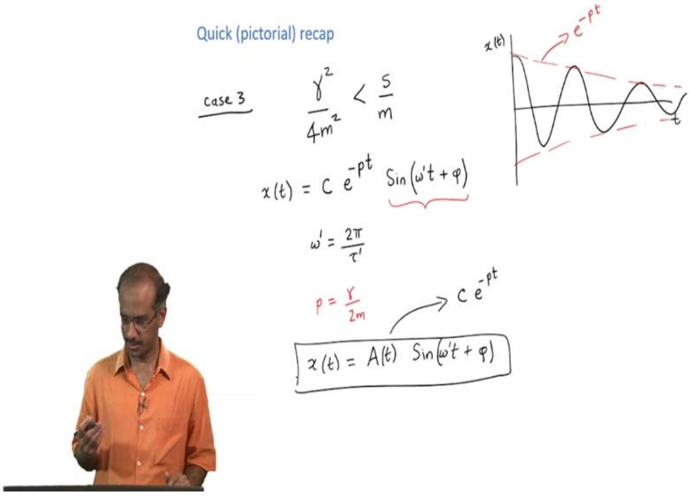

Then we looked at what we called case 3. In this case dissipation is lesser, so $\gamma^2/4m^2$ smaller than stiffness coefficient divided by the mass. In this case, if you notice the term inside this would be negative, in which case q would be complex number. So, again you can write down the solution in this case. One possible solution that we wrote down was to say that $x(t)$ would be some C times $e^{-pt}\sin(\omega' t + \phi)$ in general. So, this is a possible solution for this choice of parametric regime, and ω' is the angular frequency of the damped oscillator, so that would be simply equal to $2\pi/\tau'$.

So, you can see that this is an oscillatory part; there is an oscillatory part to the solution which is modulated by the exponentially decaying part. So, I can sketch the solution here. So, I am going to have the oscillatory part doing this, whereas, the exponentially decaying part would go like this. So, this would correspond to e^{-pt}. So, just to remind ourselves $p = \gamma/ 2\,m$.

And it is also important to realize that, you can think of this solution as $x(t)$ being equal to some amplitude which is time dependent times $\sin(\omega' t + \phi)$. So, this $A(t) = Ce^{-pt}$. So, when you look at the solution in this form it looks very similar to the solution that we had obtained for the standard undamped oscillator, except that in this case the amplitude is time dependent and as we can see the amplitude keeps decreasing as a function of time.

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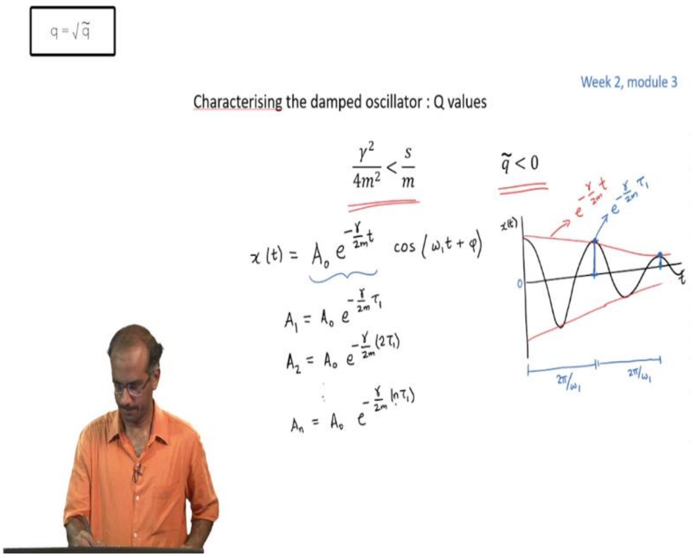

With this background having obtained the solutions for the damped oscillator let us see how we can characterize the damped oscillator. In particular we will be introducing the idea of q values.

In this part we will be primarily interested in the case when $q = 0$ which implies that $\gamma^2/4m^2 < s/m$ or more physically the dissipation is somewhat light, it is not the case of a heavy dissipation.

Now, in this case let us start off with one solution. So, let me say that this is a possible solution. Now, if I try to sketch this solution it might look like this. So, again this is a cos function. So, it should probably go like this and it is modulated by this exponentially decaying function which probably goes like this, something like this. Here we can identify a few things. For instance starting from here; so, this is time t equal to 0, and from here to here would correspond to one time period, so that can be called as $2\pi/\omega_1$ and from here to of course, here would correspond to two time periods

So, again this is also equal to $2\pi/\omega_1$, from 0 to this point would be twice the time period and so on. In fact, the value here for instance, the value of the displacement here would simply be equal to $e^{-\gamma/2m}$ the value of time period which we shall call τ_1. And here of course, it will be at this point it will be the same thing $e^{-\gamma/2m}$multiplied to τ_2 and so on.

What is important for us is how the amplitude is decreasing, otherwise the factor here is simply an oscillatory function, just like as it was in the case of a undamped oscillator. So, mostly we are going to be worried about how the amplitude is changing as a function of time. Let me give it a name A_0 here to be consistent with the notation that is going to come. So, I could say that $A_1 = A_0 e^{(-\gamma/2m)\tau_1}$and similarly I can write an expression for A_2. So, A_1would be this amplitude and A_2 would be of course, this amplitude A_2would be $A_0 e^{(-\gamma/2m)2\tau_1}$.

So, following this one can see the pattern you can directly write what happens at the nth step or the nth amplitude. So, that would simply be equal to $A_0 e^{(-\gamma/2m)n\tau_1}$.

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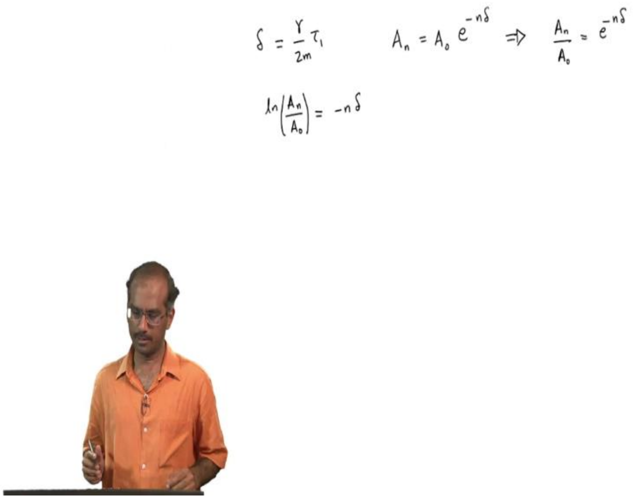

$(\gamma/2m)\tau_1$, I will call it δ. In which case the last equation that I wrote down would be $A_n = A_0 e^{-n\delta}$. And this would imply that $A_n/A_0 = e^{-n\delta}$. This quantity δ is something that characterizes the damped oscillator. So, I can extract this value by taking log on both sides. This would simply be equal to $- n\,\delta$.

And if you are doing and experiment where we track the displacement as a function of time, all one needs to do is to get the amplitude at every time period and take the ratio for different value of n. In which case, if we plot $\log{(A_n/A_0)}$ as a function of nwe should get a straight line with slope $-\delta$. So, that is one way of characterizing damping in an oscillator. In this larger the value of δ the stronger is the damping, in which case the decay of the oscillation is going to be much faster.

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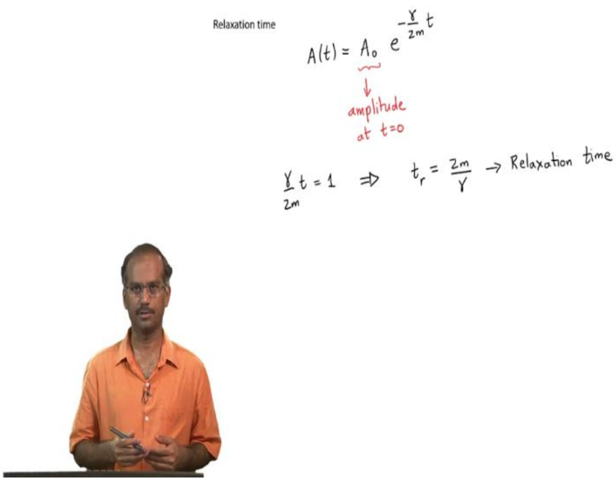

Another way to characterize a damping is through the notion of relaxation time. In other words relaxation is basically the approach towards the equilibrium position for a oscillating system which has been disturbed away from its equilibrium position. In fact, if you see the figure here where I have plotted the displacement as a function of time that is exactly what is happening. You displaced the oscillator a little bit at initial time and slowly the oscillator is damping and finally, would reach the equilibrium position corresponding to $x = 0$.

So, we would like to characterize it through relaxation time. Again it is going to involve only the amplitude part. So, I can write the amplitude as $A_0 e^{(-\gamma/2m)t}$. We could ask at what time the amplitude fall to $1/e$ of its value at initial time. The amplitude at initial time is this, at time $t = 0$ So, the question is when does it become $A_0 e^{-1}$ So, that is possible when $(\gamma/2m)t = 1$. This implies that this time scale t which I will call as t_r, to mean that its relaxation time would be equal to $2\,m/\gamma$. So, this time scale is called the relaxation time. So, this is another characteristic of a damped system.

And there is entirely another way of characterizing the damped oscillator in terms of energies. So, if you notice in the last two cases, we were looking at how the amplitude was decaying as a function of time and we essentially characterized the decay of the amplitude of the oscillation both when we defined a δ and when we defined this relaxation time. So, next we will go over to characterizing the damping of an oscillator in terms of energy decay.

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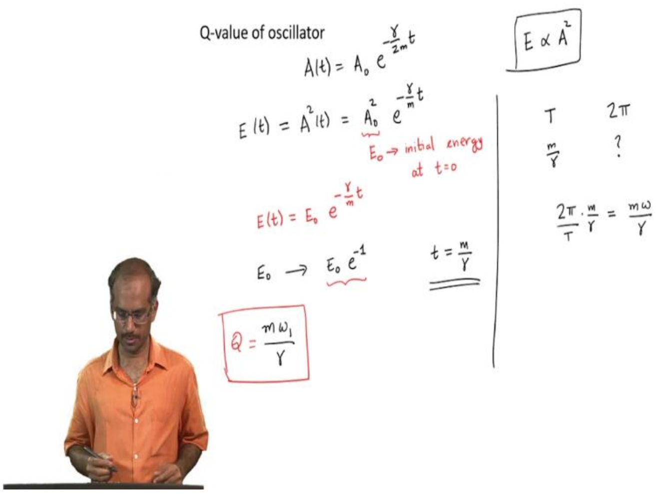

We will now see the notion of Q value for an oscillator. And again the starting point is the amplitude as a function of time. If you remember from our earlier treatment of undamped oscillator, at the end of the module I had said that energy is proportional to square of the amplitude and now we will have this occasion to use this idea. So, to define how energy is going to decay all we need to do is to say that energy as a function of time is equal to square of the amplitude which would be $A_0^2 e^{(-\gamma/m)t}$. And this quantity here which is A_0^2 I could re-designate it as E_0 to mean energy at time $t = 0$or the initial energy.

Therefore, energy as a function of time would look like $E(t) = E_0 e^{(-\gamma/m)t}$. We now ask the question, given that we are putting in energy equal to E_0 the oscillator at time $t = 0$ So, we are giving it some energy. When does that energy become $E_0 e^{-1}$? So, this will happen when $t = m/\gamma$, similar to what we worked out for the case of relaxation time. Now, in the time that the oscillator, oscillators energy reduced from E_0 to $E_0 e^{-1}$, the oscillator itself would have moved over certain radians, would have oscillated over certain radians. How much would it have oscillated? Let us compute that.

What we know is if T is the time period of the oscillator, in one full time period the oscillator would have oscillated through 2π radians. On the other hand if my time scale is T which is equal to m/γ, by what radians would it have oscillated? And that would simply be equal to $2\pi(m/\gamma)/T$ and this can be rewritten as T is $2\pi/\omega$ therefore, this will be $m\,\omega/\gamma$. In time T which is equal to m/γ, the oscillator would have; oscillated through $m\,\omega/\gamma$ radians, so my answer to this question is that it is equal to $m\omega_1/\gamma$ radians, since ω_1 is the angular frequency of the damped oscillator. Now, this is the quantity that we call the Q value.

And we will see the physical meaning of it shortly, but before that we will make it a little more robust. So, the Q value is dependent on ω_1 which is the angular frequency of the damped oscillator.

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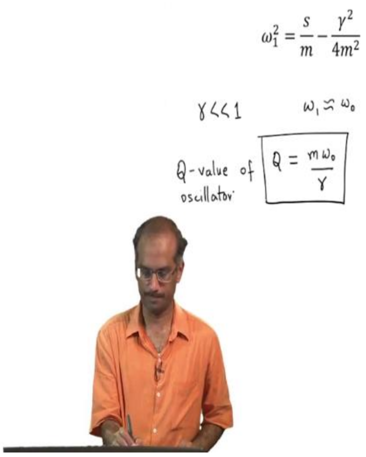

We remember from our earlier definition of ω_1 this is what it is which is given here. Now, when we consider the case of γ much less than 1 or when dissipation is really small then we could say that ω_1 is approximately equal to ω_0. In such a case, I could define my Q factor or Q value to be $m\omega_0/\gamma$. This will be our definition of Q value, Q value of oscillator.

To get some physical field for this Q value, let us go back to the equation that we had written down earlier.

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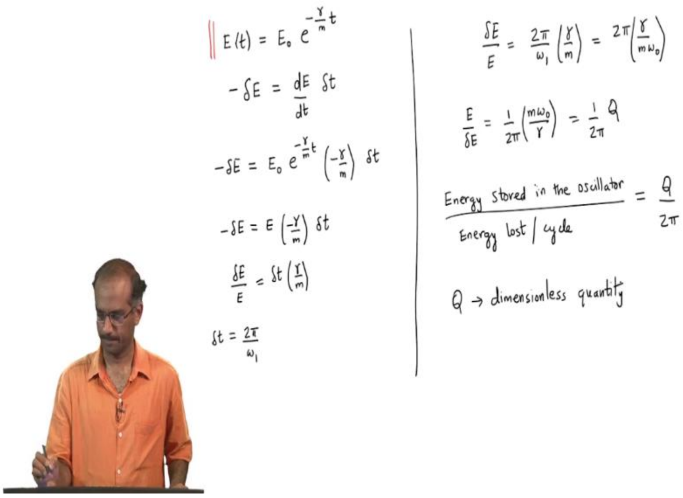

That $E(t) = E_0 e^{(-\gamma/m)t}$. So, now, let us ask the question how much of energy is lost in one cycle of oscillation. After all it is a damped oscillator, so energy is being dissipated out. Energy that is lost I will represent by δE and δE would be from basic calculus $(dE/dt)\delta t$. As time increases energy is lost or energy decreases, so I need to put a negative sign here. $-\delta E$ will then be equal to dE/dt, I can easily calculate starting from here. If I do that I will get $E_0 e^{(-\gamma/m)t}$multiplied by $(-\gamma/m)\delta t$.

And this $E_0 e^{(-\gamma/m)t}$is simply equal to E itself. So, what I have is $\delta E = E(-\gamma/m)\delta t$, and negative and negative will cancel and I will take $\delta E/E$ to be equal to δ t. Now, this δt could be the time period of your damped oscillator. In which case δt would simply be equal to $2\pi/\omega_1$, and if I plug in all these I will have $\delta E/E$ is equal to, I have missed out γ/m here, $(2\pi/\omega_1)\gamma/m$. And we will use the approximation that $\gamma$is really small in which case $\omega_1 = \omega$ approximately, so I will have $2\pi(\gamma/m)\omega_0$.

And let me write this equation other way around in terms of $E/\delta E$. If I do that, if I do that $E/\delta E$will be equal to $(1/2\pi)m\omega_0/\gamma$ and you will notice that this $m\omega_0/\gamma$ is precisely the definition of Q factor, and I can replace this into Q. In other words, to write it in words it is the energy stored in the oscillator divided by the energy lost per cycle of the oscillator and that is simply equal to $Q/2\pi$. This gives us a physical prospective of the Q factor for a oscillator. So, it is the ratio of the energy stored in the oscillator at some initial time divided by the energy lost per cycle. So, clearly larger the Q value, the oscillator is able to sustain the energy that is given to it for a very long time. Smaller the Q value it is going to dissipate energy very quickly, and any energy that you give it to it is going to be lost very soon.

So, in some sense it is a measure of how well the oscillator is able to efficiently spend the energy given to it. And at this point we should note that Q factor being the ratio of energy and another energy, Q is really a dimensionless quantity.

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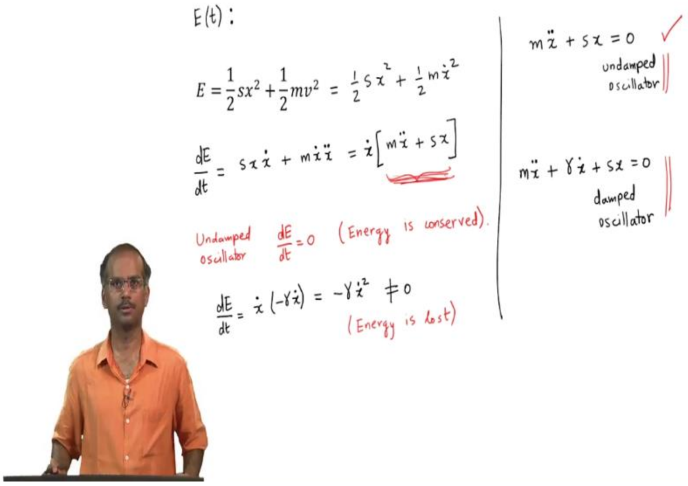

Finally, let us ask how is energy changing as a function of time. So, I will start from the expression for energy of the oscillator which is $(1/2)sx^2 + (1/2)mv^2$. Should remember that both x and v the displacement and velocity are both functions of time. Let me write it as $(1/2)sx^2 + (1/2)m\dot{x}^2$. Let us compute $dE/dt$for this case, remembering that both x and $\dot{x}$ are functions of time I will get $sx\dot{x} + m\dot{x}\ddot{x}$

If I take $\dot{x}$ common I will get $m\ddot{x} + sx$; now, at this point let us write down our standard equations of motion. This is for the undamped oscillator. So, this is for the damped oscillator. Now, let us substitute for this quantity here inside the square brackets. So, if I substitute the value from here, from the case of undamped oscillator this is identically equal to 0. So, all that it tells me is that for undamped oscillator $dE/dt = 0$ that comes from this equation. So, you put it in here in this equation you are going to get 0 identically all the time. So, that is just another way of saying that energy is a constant or energy is conserved.

And this is the idea that we already emphasized in the last week. But now if you come to the damped oscillator you substitute for this quantity $m\ddot{x} + sx$ from here. So, that would give me that $dE/dt = \dot{x}$ into, and if I substitute for this I will get - $\gamma\dot{x}$. So, that is $-\gamma\dot{x}^2$. So, in this case in general this is not equal to 0 So, the rate of change of energy with respect to time is actually negative and in general it is not equal to 0, so which means energy is not conserved and of course, energy is lost, this is something that we saw throughout this module.

With this we will conclude this module. And in the next module we will look at some problems related to the Damped Oscillator.