Waves and Oscillations
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Lecture – 07
Damped Oscillator: Part 2

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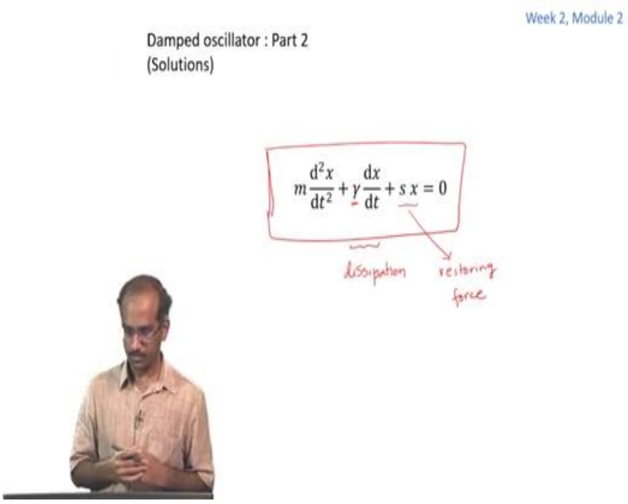

Welcome back, we are still looking at the Damped Oscillator the last class the last module we derived this equation corresponding to the damped oscillator. So, just to recall once again the terms here are the one corresponding to dissipation, here is the dissipation coefficient this here.

And this s multiplied by x that is the term corresponding to restoring force and s is the stiffness constant. And we need this restoring force because without that you are not going to get oscillation in the first place and we motivated ourselves by saying that to get more realistic we need to add dissipation to the system and that corresponds to the second term.

So, our model of dissipation here is that the dissipation coefficient multiplied to velocity would be the dissipative force. Another way of saying that is, is that the dissipative force is proportional to velocity. From all these considerations you obtain this equation which is right in front of you here.

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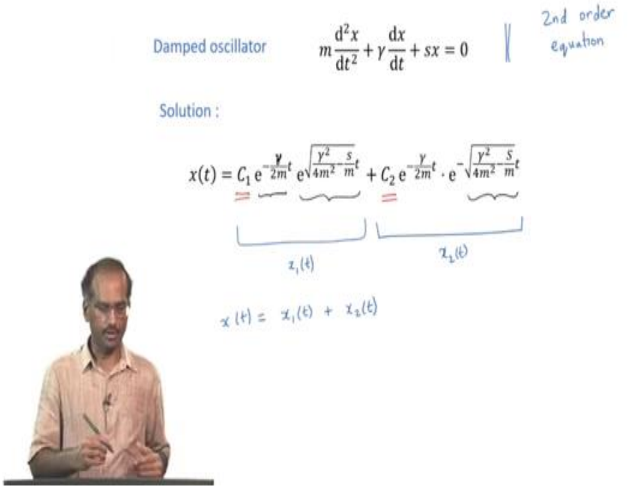

And now in this module we will try and obtain solutions for this equation. In fact, in the last module we had already obtained solution for this equation and I have written it down here. And you will notice that there are these two constants C_1 and C_2 just as it happened in the case of the undamped oscillator.

The standard oscillator both these constants would be determined from the initial conditions and we will worry about it when we do a particular problem. And then there is a first term which essentially corresponds to dissipation simply because the dissipation coefficient γ is right there in this term and this is the term which we are going to now analyze this and this. So, what happens to this term would depend on what happens to this quantity inside the square root?

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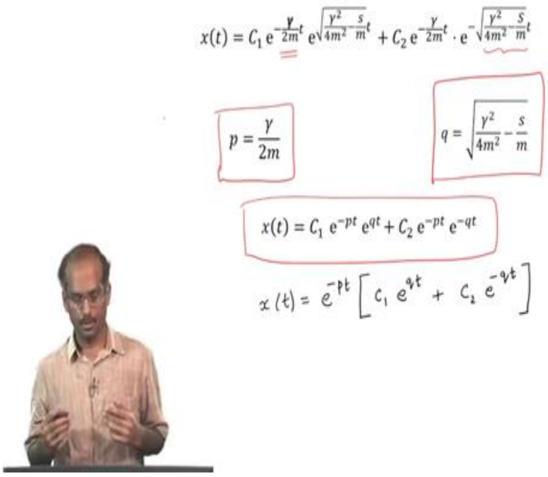

So, to do any further analysis let us simplify this equation a little bit simplify in the sense of notational simplification. So, I am going to call this quantity $\gamma/2m$ as p and this quantity here under the square root. In fact, the entire quantity along with the square root I would like to call it as q . So, it is just a change of notation and when I do this and substituted back in the equation this is the equation that I get.

So in fact, I can slightly simplify it little more by saying that $x(\mathrm{t}) = \mathrm{e}^{-\mathrm{pt}}[\mathrm{C}_1\mathrm{e}^{\mathrm{qt}} + \mathrm{C}_2\mathrm{e}^{-\mathrm{qt}}]$. So, we are going to work with this equation and in particular we are going to be worried about this quantity q. Since it is under the square root we would like to know what are the various possible dynamical behavior that the or damped oscillator can display for various possibilities of q.

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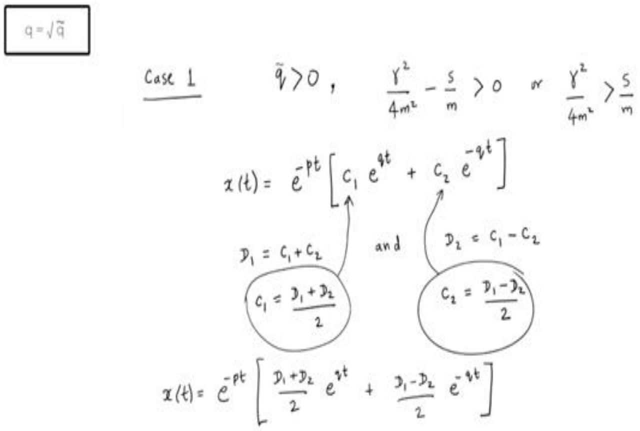

Now, let us begin with the case one which implies that it is $\gamma^2/4\mathrm{m}^2 - s/m$ is greater than 0 or you could say that it is dominated by dissipation simply because γ^2 term is greater than the stiffness coefficient. In this case I can write the solutions I can copy the solutions back again so $x(\mathrm{t})$ would correspond to e^{-pt} then I have $\mathrm{C_1 e^{qt} + C_2 e^{-qt}}$

So, in principal this should be the solution that we want, but we want to get it in a form that is that is a little more clearer for us to understand what is going on. Ideally I would like if my solution look something like this some $A\ sin\ \omega$t maybe $\tilde{\omega} t + \phi$ because this is very clear us to what is happening ok.

So, let us try and write this solution something that would look like this so that is the next few steps of our work. To do that we have these two constants $\mathrm{C_1}$and $\mathrm{C_2}$ to make it easier, I am going to replace those two constants $\mathrm{C_1}$ and $\mathrm{C_2}$ by two other constants $\mathrm{D_1}$and $\mathrm{D_2}$. So, I have that freedom to do that the way I would do that is to define new constants $\mathrm{D_1}$ as $\mathrm{C_1 + C_2}$ and $\mathrm{D_2}$ is $\mathrm{C_1 - C_2}$.

So, with these redefinitions so I am basically going to replace two constants $\mathrm{C_1}$ and $\mathrm{C_2}$ by two new constants $\mathrm{D_1}$ and $\mathrm{D_2}$ and now let me write $\mathrm{C_1}$ and $\mathrm{C_2}$ in terms of $\mathrm{D_1}$ and $\mathrm{D_2}$. So, it is a question of two constants basically two unknowns which I need to determine. So, I can write $\mathrm{C_1}$ as $(D_1 + \mathrm{D_2})/2$ and $\mathrm{C_2}$ would be $(D_1 - \mathrm{D_2})/2$.

So, the next step is substitute these two quantities here this and this here. So, substitute this in C_2 and substitute this in C_1 and if I do that, I am going to get my equation that might look like this. So, the next step is simply collect D_1 terms and D_2 terms separately and then we will see how now in the new form, it will get to become much simpler.

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$$x(t) = e^{-pt}\left[\frac{D_1+D_2}{2}e^{qt} + \frac{D_1-D_2}{2}e^{-qt}\right]$$

$$x(t) = e^{-pt}\left[D_1\underbrace{\left(\frac{e^{qt}+e^{-qt}}{2}\right)}_{\text{Cosh}(qt)} + D_2\underbrace{\left(\frac{e^{qt}-e^{-qt}}{2}\right)}_{\text{Sinh}(qt)}\right]$$

$$= e^{-pt}\left[D_1 \cosh(qt) + D_2 \text{Sinh}(qt)\right]$$

At $t=0$ $x=0$

$$\boxed{0 = D_1}$$

Let us now separate the D_1 and D_2 terms in which case $x(t)$ would become like this. Now the next step is to recognize that this function that I have written down here is nothing, but a sin hyperbolic function or explain this case it is the cos hyperbolic function. Similarly this function that I have written down here is sin hyperbolic function. So, it is $\sinh q\,t$.So, you plug in these things back in the equation I get. So, this might look like it is a sort of good enough for us, but let us go one step further and make it a little more simpler but to be able to do that I will have to assume some initial condition. So, let me assume that at $t = 0$, $x = 0$. So, if I plug in this initial condition in this equation it will give me the following $\cosh q\,t$ will give me 1; whereas, $\sinh q\,t$ would be 0 at $t = 0$.

So, in which case this simply tells me that $D_1 = 0$, so by assuming one initial condition what we have achieved is to set one of the constants to 0, so that is the outcome. So, we will still be left with another constant to be able to determine that we still need one more initial condition, but for what we are doing we can still go ahead and write the solution without having to determine the second constant. With D_1 being equal to 0 let me write the solution.

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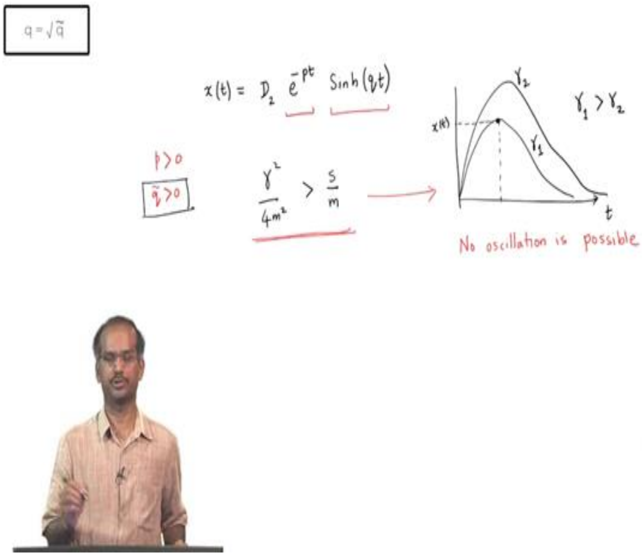

So, in this form our solution looks sufficiently simple enough that we can actually sketch the solution and see the behavior. So, it is a combination or actually the product of two possible functions; one is this exponential function as a function of time this quantity is going to decay exponentially. Let us remind ourselves that p is dependent on γ and m, but both are positive.

So, p is a quantity that is greater than 0. So, this guarantees us that this function is exponentially decaying function and similarly this function sin hyperbolic function is an monotonically increasing function. And we also know that in fact, one of our conditions was that q is greater than 0.

Now, what we have is a product of these two functions one which is exponentially decaying other one which is increasing quite fast. And when I try to sketch the product of these two functions, I will get the displacement for a damped system and that could look something like this.

Let us say that this is a curve that corresponds to value γ_1; where γ_1 is one of the possible values for the dissipation coefficient. And let me also for clarity plot one more curve something like this and let us say that this corresponds to γ_2 and this γ_1 and γ_2

both of them are two different possible values of dissipation coefficients. They are such that $\gamma_1 > \gamma_2$ ok. And physically what does it say? It tells me that if I keep increasing the dissipation coefficient or if I put an oscillator in a in a viscous medium and the viscosity is actually larger and getting larger and larger in which case the maximum displacement that you will get.

So, the maximum displacement corresponds to this value here, the maximum displacement would keep decreasing ok, which make sense simply because if the medium is more and more viscous you would not expect the oscillator to be displaced by a huge amount.

So, in that sense physically it does tally with our intuition of what is expected to happen when an oscillator is in a dissipative medium. Finally, we should also worry about what the solution itself is telling us that is $\gamma^2/4\text{m}^2$ was greater than s/m again to remind ourselves. So, the dissipation dominates over the stiffness coefficient.

So, in such a case the solution that you have written down and the one that you have sketched basically tells us that they are cannot be oscillations. So, if you remember the basic equation of motion for a standard oscillator gives you either a sin or a cosine function or a combination of the sin and cosine functions.

In either of these cases what you physically see is an oscillating solution. Here in the presence of dissipation at least in one possible case where this condition is satisfied you can have a situation where no oscillation is possible. So, for instance physically if you try and let us a have an oscillator set up in a highly viscous medium and you give it a push the only thing that would happen is that the system would simply come back to the equilibrium position which is what this graph is telling us.

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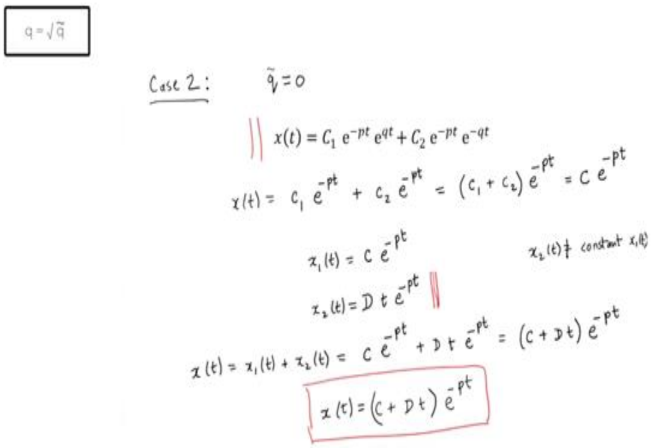

With this now let us go to the second case let us start again from the original solution that we wrote down here. So, I have it in front of me here so, I am going to put $q = 0$. So, if I do that this equation simplifies quite a bit. So, x(t)would simply become $C_1 e^{-pt} + C_2 e^{-pt}$ which would simply correspond to saying that it is $(C_1 + C_2)e^{-pt}$ and C_1 is a constant and C_2 is also a constant.

So, sum of two constants is another constant, so I do not need to maintain two different names for a single constant. So, I will just call it Ce^{-pt}. The earlier cases that we worked out we always had two solutions and $C_1$and C_2 were constants corresponding to each one of these solutions.

So, for instance the general solution for the damped oscillator had this two parts. So, there is this first part which if you would like you could have called it as $x_1(t)$ and there is the second part which you could have called it as $x_2(t)$. So, these are two linearly independent solutions.

So, if you remember from the first module on oscillator we kept saying that since the equation of motion is a second order ordinary differential equation you should have two linearly independent solutions and for the damped oscillator it is still a second order ordinary differential equation. So, this is a second order differential equation so this has two linearly independent terms such that you could say that the general solution is simply sum of these two linearly independent solutions.

And we found that when you analyzed the case of let say the case of $q > 0$ corresponding to this first case. So, here again we had two different linearly independent combinations as solutions and it is our choice of putting a particular initial condition here which reduced it to one particular solution which is this.

Now, here if you look at what we have done we seem to have only one solution whereas, again we are still looking at solutions of second order differential equation. So, ideally there should have been second solution as well. So, let me first start by calling this as x_1. So, there are from the theory of differential equations there are ways of getting the second solution given one solution here I already know one solution which is right in front of you here $C\mathrm{e}^{-\mathrm{pt}}$ and using this I can get the second solution.

So, that is a mathematical way of doing it and if you go back to any basic books on differential equations you can in fact, the book will tell you how to do that. So, I am not going to do that here on the other hand we can once again guess a second solution.

So, there has to be some basis for guessing a second solution the basis is that we already know that the two solutions that we get should be linearly independent. So, which means that I have one solution here and now my aim is to write the second solution and the condition on the second solution is that it should be linearly independent from the first. In other words $x_2(t)$should not be some constant times $x_1(t)$.

So, I cannot just multiply it by x multiply x_1 by another constant. So, what is the simplest thing I can do such that $x_2$will be linearly independent of x_1? The simplest thing I can do is to simply multiply by t. So, simply multiply $x_1(t)$ by t.

So, if I do that this is what I get, but in general I can put in a different constant here and let me call it D. Now I can write down the general solutions $x(t) = x_1(t) + x_2(t)$ which will be $Ce^{-pt} + Dte^{-pt}$which will be $(C + Dt)e^{-pt}$.

So, I have my solution and in fact, you could go back and verify that the second solution which we basically simply guessed by saying that we will make the simplest of change, or multiply a simplest simple function to $\mathrm{x}_1(\mathrm{t})$ and manufacture a second solution which is what I have done. But you could always substitute this back in the equation and verify that this indeed is possible solution, I urge you to do that yourself I will not spend time on that here.

So, my complete solution is and now I have two unknowns which is C and D both of which are constants which have to be determined from initial conditions. And again it tallies with what we have been saying right from first that if you are dealing with solutions of second order ordinary differential equations there is bound to be two constants which need to be determined from initial conditions.

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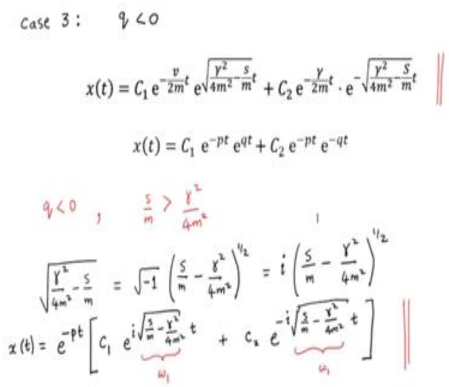

Now, to understand what is happening let us again sketch this solution and this is so much simple as a solution to sketch. Because the t dependence comes from these two terms this and this one of them is exponentially decaying very fast and this other one which is Dt is only linearly increasing. So, clearly it is somewhat easier to handle.

So, the displacement as a function of time could look something like this and you can also determine the time at which the displacement is maximum which is this. And let us call it t_{max} and I urge you to verify that $t_{max} = 2m/\gamma$. And using the value t_{max}you can also find out what is the value of largest displacement which would correspond to this value here. And if you call it x_{max}you can find that value as well and again I leave it as an exercise for you to determine the value of x_{max}.

So, all this you can do from the condition that at this point $dx/dt = 0$. So, again this is the second case which is often called the case of $q = 0$, but it goes by the name of what is called critical damping. So, in this case again there is no oscillation, so we already saw one case of the first case which was $q > 0$, there was no oscillations in that case and again in this particular case of $q = 0$ the system is completely damped.

So, you give it a push to a system or an oscillator that is inside a viscous medium the oscillator is simply going to come back to the equilibrium position without showing any oscillations. Now let us look at the case of $q < 0$. So, I we have the general solution in front of us this one and now when $q < 0$, the quantity in inside the square root becomes negative. So, which means that this is the case since $q < 0$. So, this is the case of s/m being greater than $\gamma^2/4m^2$. So, in this case restoring force or the stiffness coefficient dominates over the dissipation coefficient. So, you can imagine this to be a case of weaker a dissipation. So, the I can rewrite this quantity in the under the square root differently.

So, let me start by doing that; I have this $\sqrt{\gamma^2/4m^2 - s/m}$. Since s/m is greater than $\gamma^2/4m^2$ this would correspond to $\sqrt{(-1)}$ multiplied by $\sqrt{s/m - \gamma^2/4m^2}$ and root of -1 is of course, our $i\sqrt{s/m - \gamma^2/4m^2}$.

Now, I will plug this in back in our general equation, in which case I would get something like this $x(t) = e^{-pt}[C_1 e^{i\sqrt{s/m-\gamma^2/4m^2}\ t} + C_2 e^{-i\sqrt{s/m-\gamma^2/4m^2}\ t}]$ Now when I assembled the solution here this last equation, this pretty much looks like sum of two exponential solution and of course, there is an i.

So, this does and is going to provide us oscillatory solutions in which case I can do the following. I can call this as I can call this by a different name let me call this as ω_1 both these quantities this will of course, allow us to simply it a bit. So, if I do that I will be able to rewrite the equation in a slightly simpler form from which we can easily understand what is going on.

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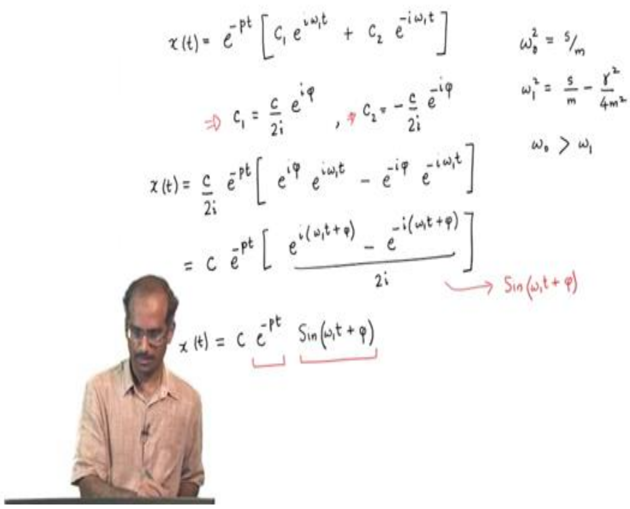

Once I plug in this change of notation basically ω_1 this is what I am going to get and to be sort of consistent let me also say that ω_0 would be s/m or ω_0^2 would be s/m and ω_1^2 would be $s/m - \gamma^2/4\mathrm{m}^2$.

So, this clearly tells us that ω_0 which is the angular frequency of the un damped oscillator is greater than ω_1 which is the angular frequency of the damped oscillator. Now to simplify it further I have these two constants C_1 and C_2 and this is the technique that we used earlier as well because we have freedom in choosing these two constants. I can take these two constant C_1 and C_2 and introduce two other constants.

So, let me call C_1 to be $(C/2i)\mathrm{e}^{\mathrm{i}\phi}$and let me call C_2 to be $-(C/2i)\mathrm{e}^{-\mathrm{i}\phi}$. Now if I plug in these two choices that I have made for $C_1$and C_2back in the equation I am going to get the following with the little rearrangement it would look like this. This can be further simplified $C\mathrm{e}^{-\mathrm{pt}}\mathrm{e}^{\mathrm{i}}$.

Now, when you look at this equation it is very clear that, the term inside thesquare brackets is simply the sin function. So, this can be written as $sin(\omega_1 t + \phi)$. So, just come from the basic trigonometry so now, I can easily write the solution for displacement as $C\mathrm{e}^{-\mathrm{pt}}sin(\omega_1 t + \phi)$.So, that is remarkable the sense that for the case when q is less than 0; where the stiffness coefficient dominates over the dissipation coefficient. What we get is one term which is the sin term for instance is the oscillatory function whereas; this $\mathrm{e}^{-\mathrm{pt}}$ is an exponentially decaying function.

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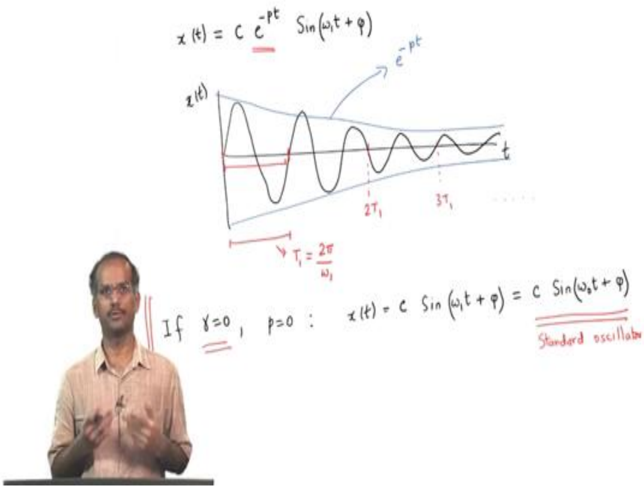

So, now if you look at the solution that I have it has two constants one is ϕ and other is of course, the constant C. So, initially we had two constants C_1 and C_2 here this and this and they were now replaced by these two other constants which is C and ϕ. An oscillatory function like $\sin(\omega_1 t + \phi)$ is being modulated by the exponentially decaying function which is e^{-pt}.

So, you should expect to get something like this sin function whose peaks something like this. So, the decaying profile here is e^{-pt} this distance in time from here to here we will defined for us one full time period until this point and that would correspond to let us call this time period T_1. So, this would correspond to $2\pi/\omega_1$ and of course, this would then be 2 times that time period $2T_1$ and this would correspond to $3T_1$ and so on.

So, when you look at the final solution you see something that seems to appeal to intuition you have an oscillatory solution which is modulated by and exponentially decaying profile. So, clearly the role of dissipation which is actually embedded here inside this exponentially decaying function because p is $\gamma/2m$; γ is your dissipation coefficient leads to displacements which are decreasing as a function of time. In fact, when you do any experiment for instance you take a simple pendulum and let it oscillate. This is exactly what you see in the presence of air was a dissipative medium, successively the maximum displacement of the pendulum essentially keeps decreasing and finally, it stops.

And this also tells us another point here that if $\gamma = 0$ which means that I do not have dissipation in the medium at all. So, any energy that I put in the system will remind there forever if that is the case then $p = 0$. And then the solution would simply be equal to and if you remember that $\gamma = 0$ would imply that $\omega_1 = \omega_0$; this would simply reduce to $C \sin(\omega_0 t + \phi)$ and this you would recognize is simply the solution for the standard harmonic oscillator without any dissipation.

So, what we see is that in the presence of dissipation you do get oscillation at least in one of the possible cases whose amplitudes are successfully decreasing. And if you take the limit of no dissipation corresponding to $\gamma = 0$ you recover the solution corresponding to standard oscillator. So, I will stop this module with this and in the subsequent modules we will look at also including effects such as forcing and oscillator in other words you keep supplying energy to the oscillator so that will come later.

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Damped Simple Pendulum Experiment

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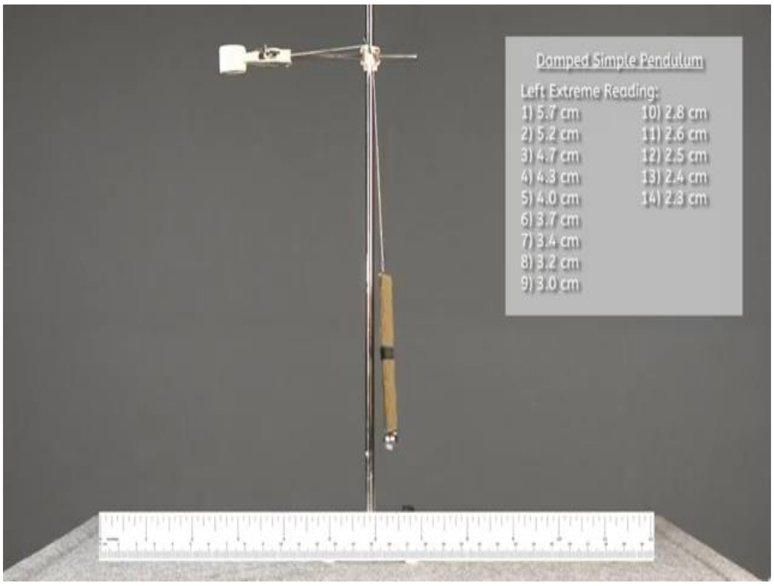

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- In this experiment, the length of the successive half amplitudes (left extremes) of the damped simple pendulum is measured.
- The pendulum is damped by air resistance which is amplified by attaching a flat piece of cardboard to the pendulum perpendicular to its plane of motion.
- The successive left extremes are time-separated by the time period of the pendulum.
- The length of the successive half amplitudes are measured using an on-screen length scale.
- Though the readings have been presented with the units of this scale (i.e. centimeters), the measurements are relative.

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- The following plot contains the graph of the length of the successive half amplitudes vs. their measurement number.
- It can be noted that the length of the successive half amplitudes decreases exponentially with time as expected from the theory.

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