Waves and Oscillations
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Lecture – 06
Damped Oscillator: Part 1

So, welcome to the 2nd week, this is the very first module of this 2nd week. Entire last week we were looking at the standard oscillator which is probably one of the most important and simplest models in all of physics. So, in this week we are going to devote time to looking at Damped Oscillator. So, you should keep in mind that the standard oscillator that we met in the last week the total energy of the system was a constant. But when you look at a realistic oscillator, now you could for instance go to the garden and let the swing give an oscillation to the swing and you will notice that after some time it comes to a halt.

Or, if you have a block and a spring just give the block a push and it will start oscillating, but again after sometime it is going to come to a halt. So, all this is because you are not supplying energy to it continuously, the initial energy that you give basically gets dissipated as its working against some frictional forces. In many of these cases the resistance comes from the air surrounding the body which is oscillating. So, we need to add in those realistic effects and see what changes are effected in the oscillator.

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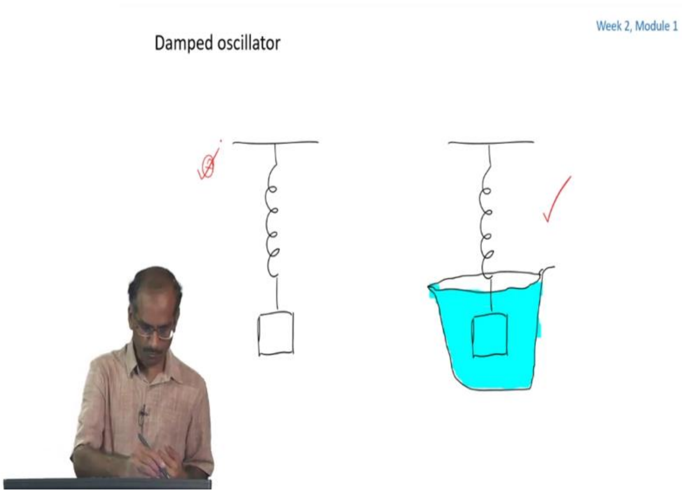

For instance one of the things that we might be considering is the following. So, here is on the left hand side the standard oscillator a block with the spring, but on right hand side I have the same oscillator, but put inside a some liquid maybe with sufficiently large viscosity. Now, if I said the second oscillator to work it is going to face a huge amount of friction from the molecules that surrounding the block and would come to a halt much faster than the oscillator that is on this right hand side. So, we are going to see the effects of these in our mathematical equations that we write down for the oscillator.

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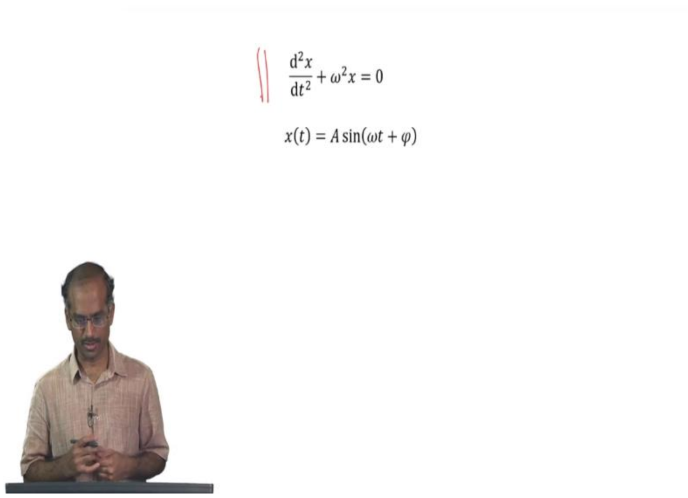

Now, before we do that we need to add some more tools to our mathematical equations more precisely the solutions. So, let us go back to the standard equation that we had written down. For the standard harmonic oscillator the equation was $\mathrm{d}^2x/dt^2 + \omega^2 x = 0$ Again, to remind ourselves this comes from the basic physics which is that restoring forces proportional to displacement. And, we also saw that you can write solutions for this; one possible solution is this $A\sin(\omega t + \phi)$, ϕ is the phase factor.

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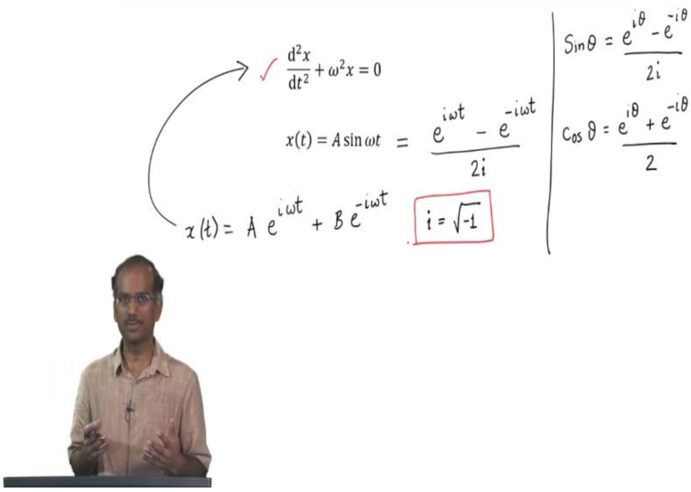

Now, let me get rid of this phase factor let us say that ϕ is 0 and now my solution is simply $A\, sin\, \omega\, t$. Now, let us rewrite this solution a little bit differently, I can rewrite this in the following manner. So, I have rewritten this in terms of exponential functions $e^{i\omega t}$ but note that there is that crucial i. So, what I have used is the fact that in general a function like $\sin\theta$ can be expressed in terms of functions like this. In case you are not familiar with this way of rewriting sin and cosine functions, I urge you to go back and get some background on how to write sin functions and cosine functions and indeed all trigonometric functions in terms of $e^{i\theta}$ and $e^{-i\theta}$.

Now, coming back to our main story if $\sin\omega t$ is a proper solution to our equation of motion, then all I have done is to rewrite $\sin\omega t$ using a different function. So, which means that each of these functions should also be a solution of my differential equation. In fact, that is correct it is true to say that not only is $\sin\omega t$ a solution, but in principle I could have taken my solutions to be $x(t)$ as $Ae^{i\omega t}$. And, I know that since my equation of motion is a second order ordinary differential equation there are two possible solutions. We in fact, discussed why there should be two possible solutions earlier on in the first module.

So, the second possible solution could be $e^{-i\omega t}$. So, I can write a more general solution as $Ae^{i\omega t} + Be^{-i\omega t}$. So, this is indeed a possible solution of this equation of motion. So, earlier we had sin's and cosines as possible solutions, now we see that not just sin's and cosines, but also $e^{i\omega t}$ and $e^{-i\omega t}$ a combination of this would also be a possible solution. And, again I urge you to go and substitute x back in this equation of motion and check indeed that it is satisfied.

For completeness let me also write the expansion for $\cos\theta$ here, it will be $\{e^{i\theta} + e^{-i\theta}]/2$. There is one question that you might want to think about $x(t)$ or x itself is displacement a very physical quantity. So, we are asking for what is the displacement as a function of time. Now, we see that we are able to write displacement in terms of these complex functions and this i which is $\sqrt{-1}$ makes this function a complex one.

So now, the question would be if my displacement is a real physical quantity, it has dimensions of length how can that be equal to some complex function with some imaginary part to it? So, clearly the answer is that you can use $e^{i\omega t}$ or $e^{-i\omega t}$ as possible solutions. But, when you want to finally, put in the initial conditions and match it with some realistic oscillator, it will always turn out that, it comes in such pairs. And, would cancel out in such a way that the final displacements, velocities and accelerations that you are going to get would always be real numbers.

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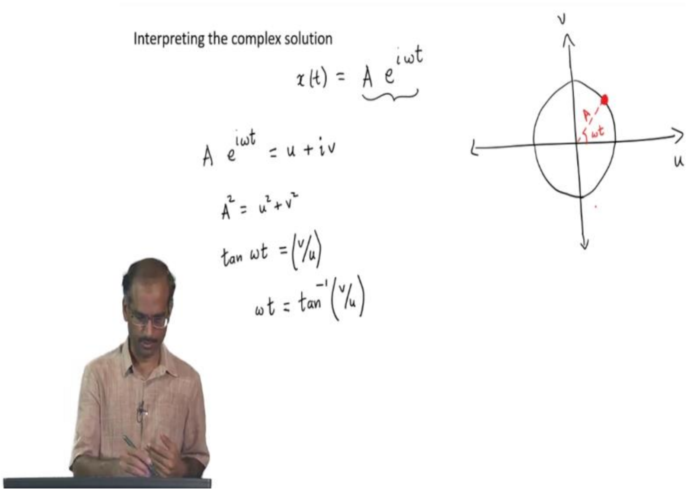

Let us try and understand a little more about this complex solution; what I had done was to state that maybe $x(t)$ displacement can be written as $e^{i\omega t}$. So, as I said this is one possible solution which you can verify that it is indeed a solution. How do we understand it from our oscillator perspective? So, the way to do that is to look at it in the complex plane itself; so, something like this it is a complex number. So, let me separate it out $Ae^{i\omega t}$, another way of expressing complex number is to say that it has an real part and an imaginary part $u + i\, v$.

And, from your earlier studies on complex analysis you must have already seen that $A^2 = u^2 + v^2$ and $\tan \omega\, t = v/u$. So, all this can be nicely visualized in the argand plane. So, this is the real part of our complex number, let us call it u and the imaginary part of complex number is v. So, you should not confuse this v with velocity, is just a complex number whose real part is u and imaginary part is v. And, $\omega t = \tan^{-1}(v/u)$

Now, I can picturize this $e^{i\omega t}$ as a complex number $u + i\, v$ might represent a point here in this argand plane. So, it could be a vector like this. You would see that the radius of this is a constant which is A which is $\sqrt{u^2 + v^2}$ and the angle that it makes with the x axis is ωt. Now, if you remember that t is something that is changing so, it is increasing in time. What we are looking at is essentially that this point, the red point here is moving along this circle; all the time keeping the value of A constant.

So, it is moving on this circle as time progresses and ω is the angular frequency. So, that tells you how many times the point moves around this circle every second or at least it is related to that quantity. So, you can keep this in mind as a sort of physical representation of the solution that we have obtained. With all these tools at hand let us now go to the next part of this module.

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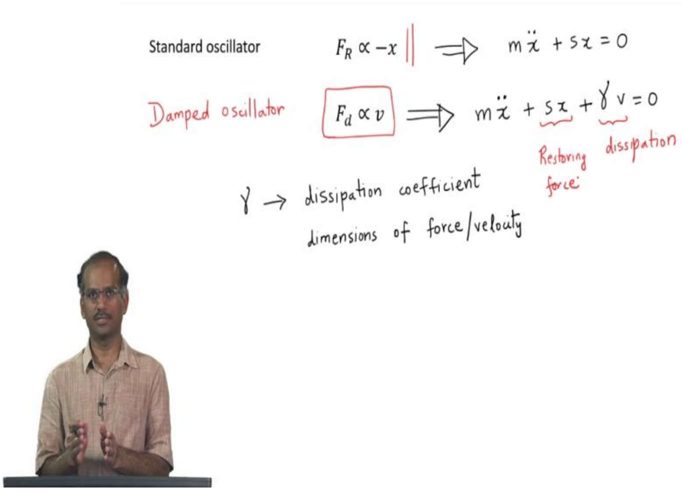

So, recalling again that for the standard oscillator the restoring force is proportional to displacement with the negative sign. This was the central physics and the main ingredient that is going to give us oscillations in the first place. Now, what we are after is how do we incorporate the idea that there is dissipation in a reality. The key physics there is what is written here ok. So, this is going to be our key physics for the damped oscillator. So, the damping force we will assume it to be proportional to velocity. So, this is the important ingredient which I am going to incorporate in the equations of motion.

So, if my standard oscillator gave me this equation of motion, here m is mass and s is the stiffness constant. As far as the damped oscillator is concerned I am going to extend this equation of motion. So, let me rewrite it in the following way. So, I will have $m\ddot{x} + sx$ and now I will introduce the force due to dissipation as being proportional to velocity. So, I will introduce a constant times velocity. So, I have introduced this constant γ which will be dissipation coefficient.

This quantity γ will have dimensions of force per unit velocity. So, now here is our additional term which accounts for dissipation in the system. And, this one was the original restoring force which we had all along which is anyway required for oscillatory motion in the first place. So, you can see how nicely the equation fits in and we can separate out various components.

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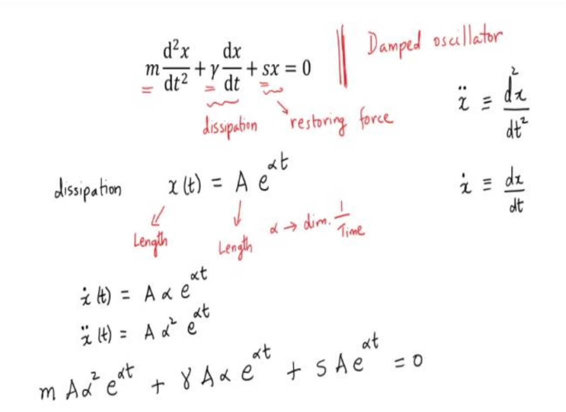

Now, I have my equation for the damped oscillator and just to remind ourselves this is the term that accounts for dissipation. And, our model of dissipation is that the dissipative force is proportional to velocity and this term is the usual restoring force. So, given this equation now the next step is to obtain solutions for this equation which is what we shall do now. So, this is a kind of second order differential equation for which a solution of this form is guaranteed to exist.

So, I have assumed my solution to be capital $A\mathrm{e}^{\alpha t}$, where t is my time and this is my anzats or my guessed solution for the displacement. Again like we did earlier on in the first module, in the first week we are going to obtain a solution for displacement as a function of time. So, once again let me state clearly that if I have differential equations of the type that is written here. So, these are a class of differential equations where the coefficients mean quantities like this m $\gamma$and s these are constants, they do not depend on x itself ok.

So, if you have a class of differential equations where the coefficients are constants, in such a case you can always postulate solutions of this type. Now, if you look at the solution on left hand side I have x which is displacement, it has dimensions of length and α is sitting on top of exponential. So, since we want this quantity to be dimensionless this implies that alpha has dimensions of 1 over time. So, only then it will cancel t which has dimensions of time here and of course, A should have dimensions of length. So, under these conditions the solution is consistent. I have introduced two constants here if you see A and α and I need to determine both of them.

And, the way to do that would be to simply plug in my solution back into the equation of motion and find out the value of α. So, which is what we are going to do right now and x(t) is $Ae^{\alpha t}$ and I need dx/dt and from now onwards I am going to use the following short forms. So, I am going to use $\ddot{x}$ to indicate d^2x/dt^2 and I am going to use $\dot{x}$ indicate dx/dt; so, we should keep that in mind. So, with this notation $\dot{x}(t)$ which is dx/dt would be $A\alpha e^{\alpha t}$ and $\ddot{x}$(t)would be $A\alpha^2 e^{\alpha t}$.

Now, I need to substitute all this back into my equations of motion. So, if I do that I will get the following $mA\alpha^2 e^{\alpha t} + \gamma A\alpha e^{\alpha t} + sAe^{\alpha t} = 0$

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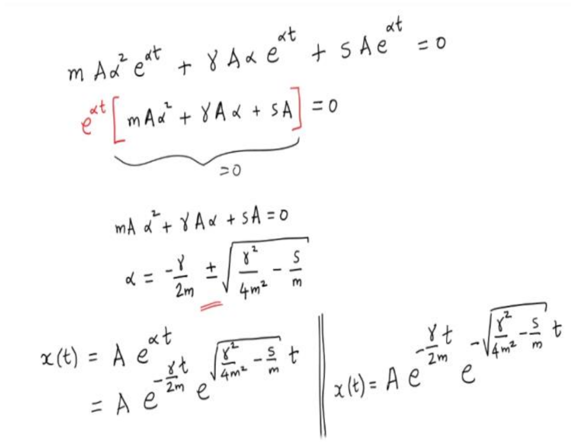

So, now I can rewrite this equation as $e^{\alpha t}\,[mA\alpha^2 + \gamma A\alpha + sA] = 0$. So, under these conditions we should take this quantity to be equal to 0 because, the other one which is $e^{\alpha t}$ is an exponential function and need not be 0 for arbitrary values of t. This equation as you see defines quadratic equation in α. So, if you remember I want to find out what is the value of α under the condition, that my assumed solution is true and my assumed solution is $Ae^{\alpha t}$.

This is a simple equation to solve and if I solve it, I will get the following result. So, I have got the possible values of α, with this now I can write the complete solution. And in fact, my equation has predicted two possible values of α; one with the positive and one with the negative sign here; obviously, it is a quadratic equation so, we have two possible routes. Now, let me finally, write down the solution. So, remember our original solution was our assumed solution was $Ae^{\alpha t}$ and this would mean that this can be written as $x(t) = Ae^{-(\gamma/2m)t}e^{\sqrt{(r^2/4m^2 - s/m)}\,t}$. When I use the positive sign I have one possible solution and when I use the negative sign I can in principle have the second solution which is $x(t) = Ae^{-(\gamma/2m)t}e^{-\sqrt{(r^2/4m^2 - s/m)}\,t}$. So, I have two solutions and you can verify that these two are independent solutions. In the sense that you cannot get one solution from the other just by multiplying a constant to it. Hence, my general solution would be of the following form.

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$$x(t) = c_1 e^{-\frac{\gamma}{2m}t} e^{\sqrt{\frac{\gamma^2}{4m^2} - \frac{s}{m}}\,t} + c_2 e^{-\frac{\gamma}{2m}t} e^{-\sqrt{\frac{\gamma^2}{4m^2} - \frac{s}{m}}\,t}$$

So, if you should look at the equation that we have written down. So, it has two parts to it, corresponding to our expectation that there should be two independent solutions; simply because we are solving a second order differential equation of this type. And, we are going to use this solution in the next class and analyze what are the various possible physical manifestations of this solution. And, clearly it is going to depend crucially on this quantity which is under the root here.

So, this quantity could be either positive or 0 or negative and corresponding to the sign of this quantity, we are going to see different kinds of dynamics. Whatever be the dynamics there is this first term here in both these solutions which is $e^{-(\gamma/2m)t}$ and γ if you remember is your dissipation coefficient. So, it tells you that the displacement is going to decay as a function of time, irrespective of what happens to the quantity that multiplies it ok. So, we are seeing first signs that dissipation is appearing in the solution. So, we will see the details of the solution in the next module.