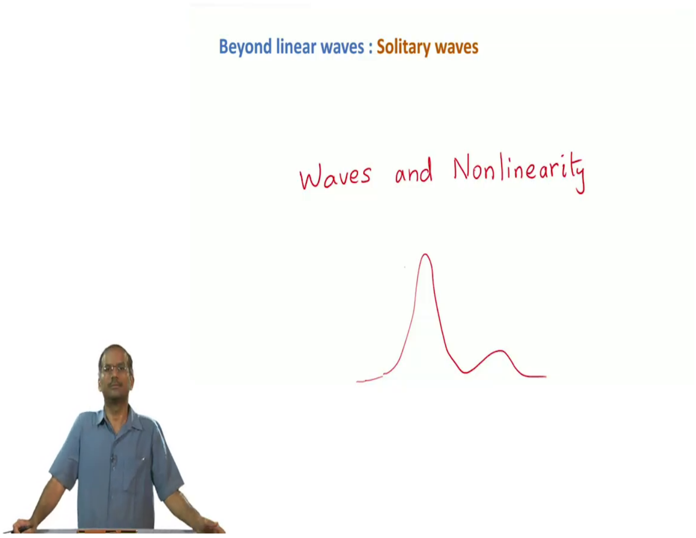
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

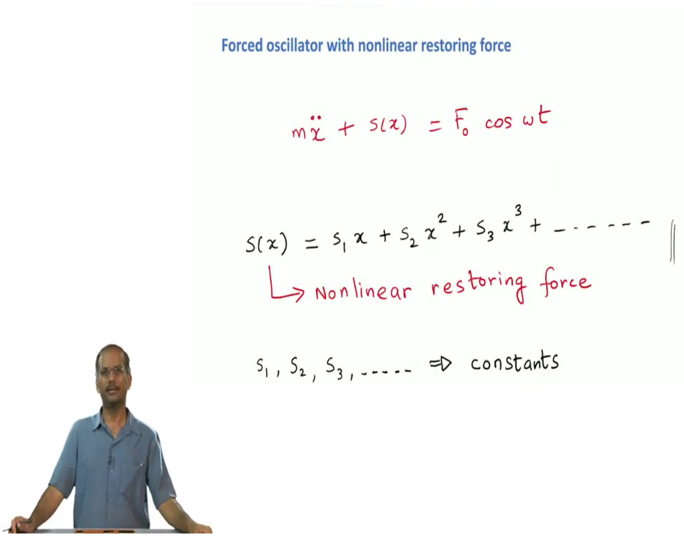
Lecture - 57
Beyond Linear Waves: Solitary Waves

(Refer Slide Time: 00:15)



Welcome and the theme of this week has been Beyond Linearity. So, we are looking at in some sense what happens if you go beyond linear assumptions as far as waves and oscillations are concerned. In the first two to three lectures of this week we saw what happens in the case of oscillations, when you have non-linear restoring force. So, we saw various effects of that, in today's lecture we look at what happens if you go beyond linearity as far as the waves are concerned.

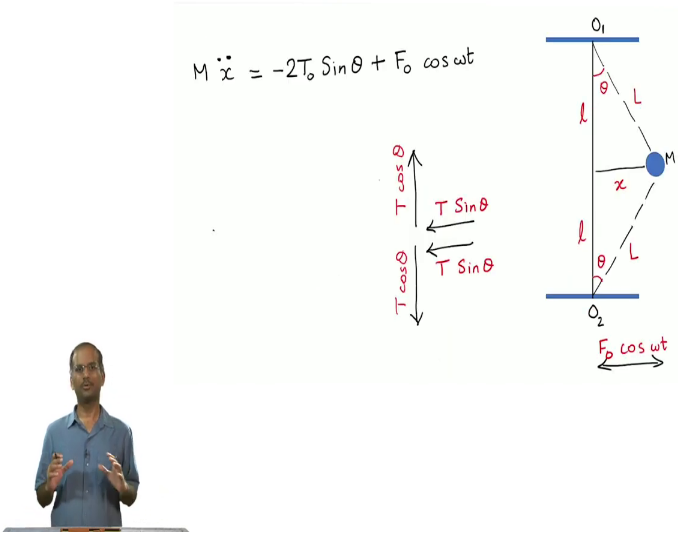
(Refer Slide Time: 00:57)



So, let us start by doing a quick recap of what we saw in the last one or two classes So, we were looking at non-linear restoring force models which would give rise to oscillations. So, one of the simplest things that we did was to look at a model of this type where you had a non-linear restoring force which is $S(x)$ and it was also driven.

And, in principle this non-linear restoring force has you can imagine infinite terms $S_1x + S_2x^2$ and so on. But, in practice we will restrict it to 2 or 3 and for the case that we analyze we just restricted to the first three in fact, only the first and third one.

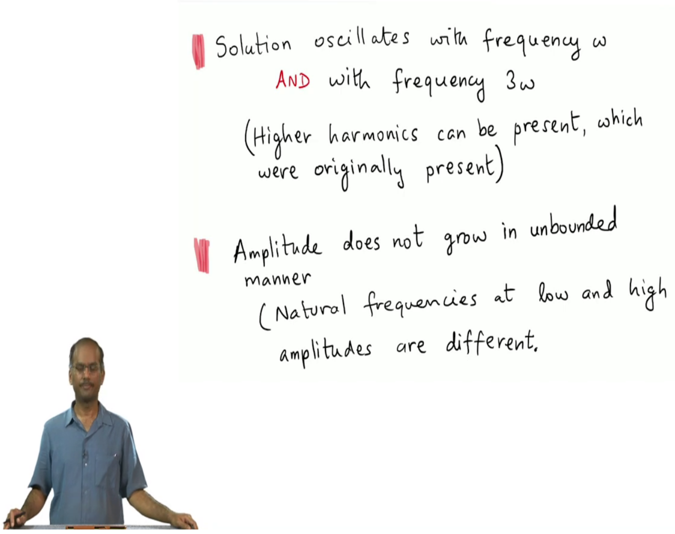
(Refer Slide Time: 01:45)



And as an example we looked at this problem of beaded string. So, here you have one bead which is held in place by 2 strings which are tied to walls the whole set up is mounted in vertical direction. And, what we do is simply pull the string apart and assume that the oscillations are in one plane and only in this direction. And in which case you can analyze this problem assuming that there is uniform tension in the string and so on; but there are two important deviation from what we did with similar problems in the past.

One is you should not assume that wherever $\sin\theta$ comes you can approximate it by θ and that even though the tension is uniform in the string you cannot assume that the tension in the string, when it is in its equilibrium position and when it is let us say moved apart a little bit are equal to one another. So, all this adjustments have to be made and when you will do that you will notice that the equation of motion that you get is actually non-linear.

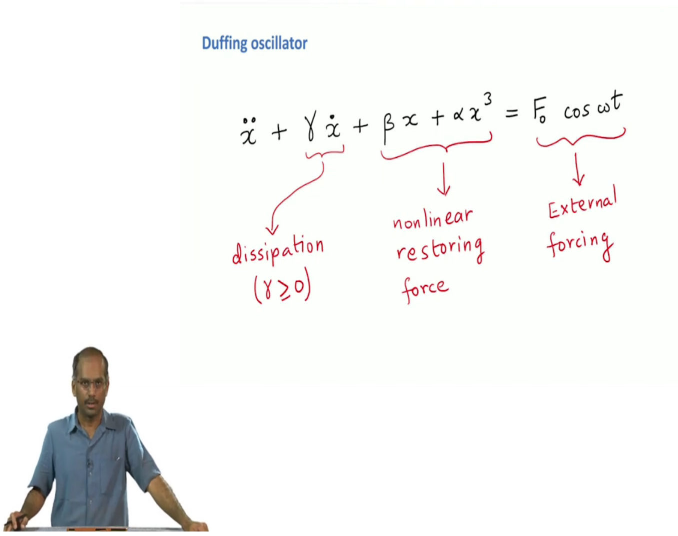
(Refer Slide Time: 02:58)



And, what we saw in that case was that if you are forcing your system with frequency omega the system oscillates as in the linear case with frequency ω, but also has additional frequency. For instant in this case it also showed oscillation with frequency 3ω.

So, in general higher harmonics can be present in oscillations and also that the Amplitude does not grow in an unlimited manner in such cases because the frequencies at low and high amplitudes are really different.

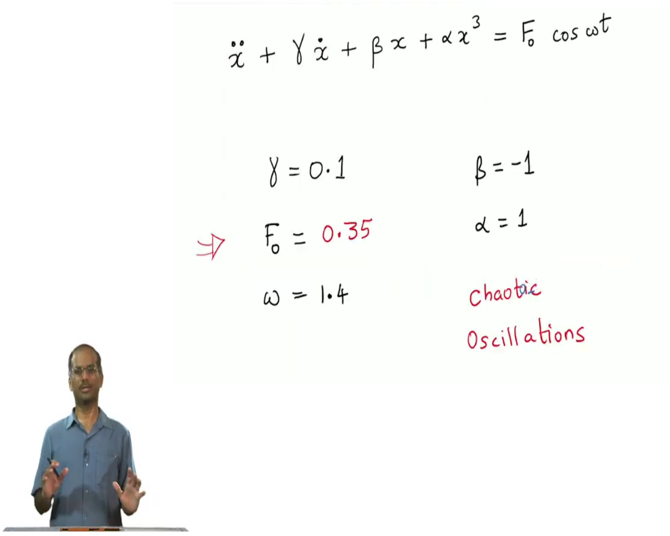
(Refer Slide Time: 03:38)



We worked out another model only in only partially the model was called the Duffing oscillator. So, here again you see this 3 important ingredients; you had dissipation, you have non-linear restoring force and an external forcing; external forcing is a sinusoidal forcing with frequency ω.

So, in this case while you can see many things, it is a very interesting system in itself, but the biggest highlight is that it can lead to chaotic solutions for certain choices of parameters.

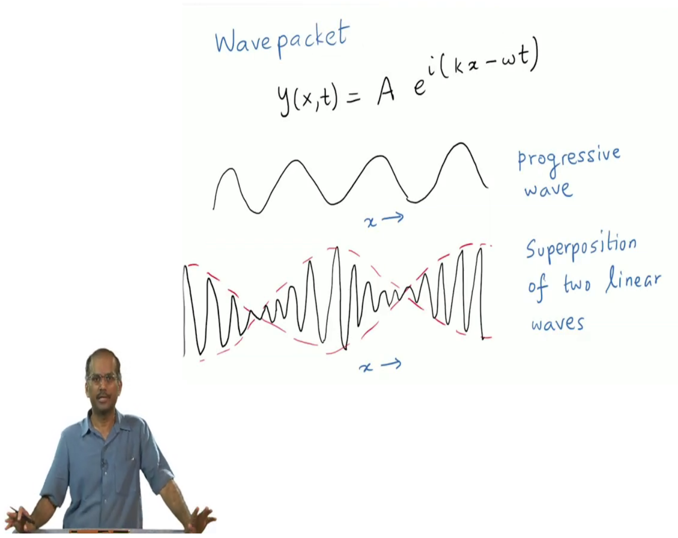
(Refer Slide Time: 04:09)



So, by chaos we mean that when you usually look at for instance the displacement as a function of time. It will look like a random graph are for example, more critically if you if you look at take two initial conditions and integrate the equations of motion the two initial conditions being very close to one another as close as you can make. But, you will see that initially they these two initial conditions would evolve close to one another, but after sometime they would diverge and they would diverge exponentially that is one signature of a chaotic system.

And, this you would not have seen in any another linear system. So, that is an important sort of new phenomena that emerges only in non-linear systems. So, with this background today we will look at waves, all these that we saw Duffing oscillator and beaded string and all that where actually oscillations, oscillations of single particle ok. Now, how does non-linearity affect the wave phenomena, that is the question that we like to explore today.

(Refer Slide Time: 05:34)

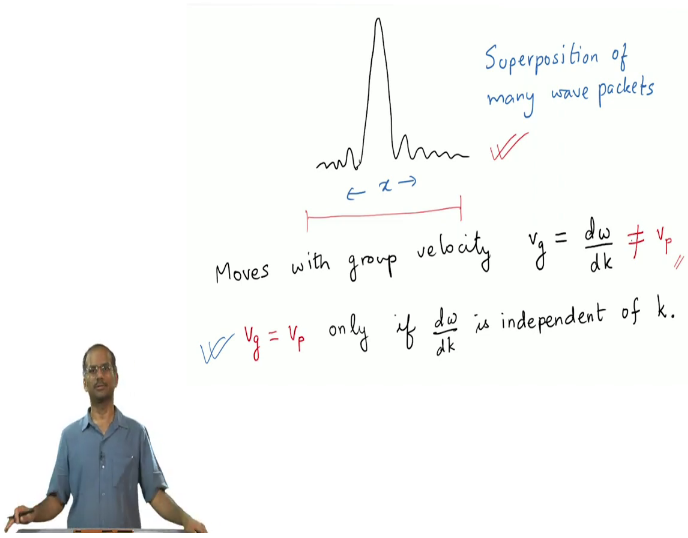


Let us go back to what we had studied several weeks back about waves. So, at some point we started with beaded string, went to the continuum limit, wrote down an equation of motion and we called it the wave equation. And, the solutions of the wave equation are one possible way of representing the solutions was this; y which is a function of position x along the string and t is of course the time. So, y is the amplitude of the wave at position x at time t is given by some amplitude A times this $e^{i\omega}$, $e^{kx-\omega t}$.

Now, if you try to sketch this function it is a waveform that extends all the waveform minus infinity to plus infinity sinusoidal wave. So, it will look something like this think of it as a progressive wave ok. So, I have just plotted it as a function of x, you could also see it progressing in time. So, this is as far as a single wave is concerned and remember that we got it from a linear partial differential equation. So, this is the solution of a linear partial differential equation.

Now, what if I superpose two of such waves with slightly different frequencies again its the problem that we had seen in many different ways before and here if I try to superpose two of them, do some manipulation using trigonometry identities what you will get is something like what is shown here. So, you will have a fast oscillation and the profile of that the overall profile which is shown in this red dashed curve here would be the profile that you will actually see.

(Refer Slide Time: 07:37)



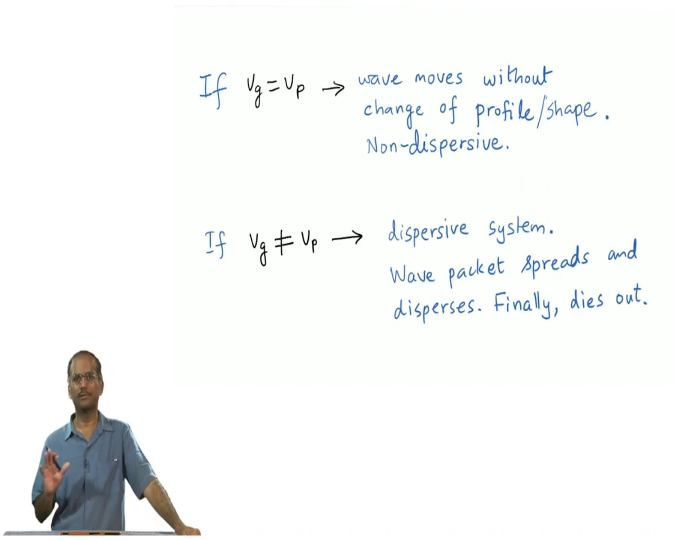
Now, what if I Superpose a large number of waves, when I say large number of waves I mean I am superposing large number of progressive waves. The difference is that each wave has a slightly different frequency from the other. If that case I am going to get a overall waveform which is like this. Now you can see what is happening. So, when I had one the physical extent of the waveform, where the amplitude is non zero is pretty much the entire x axis goes from minus infinity to plus infinity.

By the time I superpose a large number of these waves. The region over, which the amplitude of the total waveform is non-zero is very small it is only roughly about say this. So, beyond this it is for all practical purposes gone to 0 and you cannot even possibly detect it. So, now this looks like a localized wave packet. So, it is quite nice that something on one hand you had something that extends all the waveform minus infinity to plus infinity, but you could using these solutions of the linear partial differential equation construct wave packets of this type.

So, the nice thing about this is that this wave packet, that we constructed putting together many waves of slightly different frequencies. It progresses or moves with group velocity. And again this is a concept we have seen, and group velocity is given by $\frac{d\omega}{dt}$. Where ω is the angle of frequency, now in general this group velocity which is the velocity of the entire group is typically different from the velocity of the individual components of that group. So, the velocity of the individual components of that group is what is called the phase velocity and that is represented by this quantity V_p.

So, typically in a situation like this the group velocity is not equal to phase velocity. And the only time the group velocity will be equal to phase velocity is if this quantity $\frac{d\omega}{dk}$ is independent of k. So, again this is the idea that we had seen when we came across this idea of phase velocity and group velocity. So, keep in mind that pretty much in most of the realistic cases group velocity and phase velocity are not equal to one another.

(Refer Slide Time: 10:24)



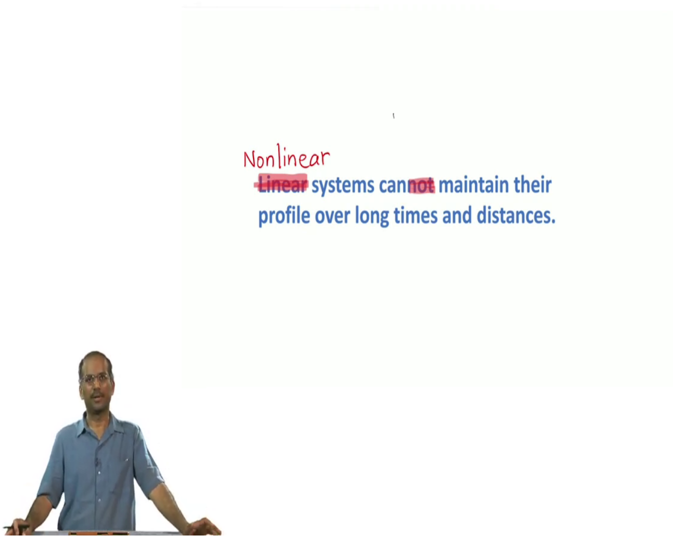
So, we can write down some conclusions based on this, that if the group velocity is equal to the phase velocity it is a very special case. The wave in that case can move without change of profile or shape. In other words if the shape of my waveform is like this fairly localized, it will move without any distortion to the shape. So, it will move as it is. And that is possible if $V_g = V_p$ and physically you can imagine why it should be so, because different components are moving at the same speed. So, the wave is not being torn apart. So, it maintains it is whatever shape that you gave at initial time.

So, such waves are called non dispersive so; that means, that you give it a shape at let us say time $t = 0$. It continues to move with the same shape and profile. So, that is why it is not dispersive it does not break apart and disperse. On the other hand if $V_g \neq V_p$ this would not happen. That is the case where the system is dispersive, again physically if you imagine you have many components you put together many waveforms to construct your localized wave packet like this.

And each of that component has a different phase velocity which means that each one is trying to go a different speed. So, clearly you can imagine that a waveform, which is constructed out of that is not going to stay together for a very long time, is going to disperse and after sometime you will not see any trace of the wave itself.

So, that is a case when a wave packet that you constructed initial time having such a sharp non zero value at some point. Begins to spread and disperse at later times, and finally after some amount of time it actually dies out. So, that is what happens when $V_g \neq V_p$. So, we can write this is out as a general conclusion.

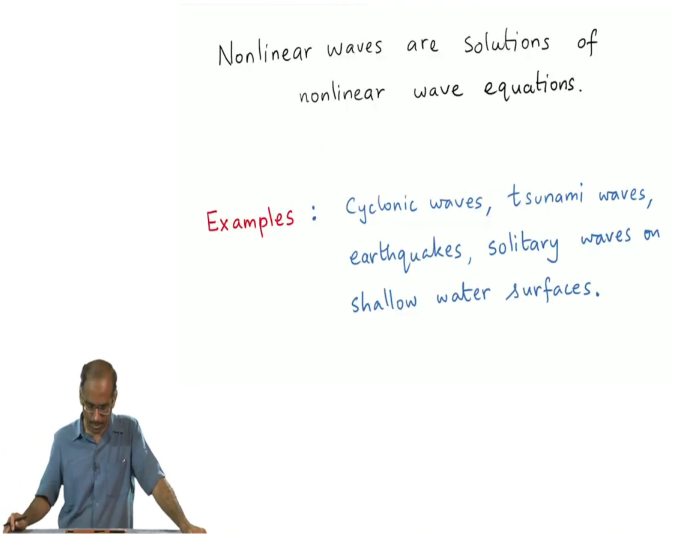
(Refer Slide Time: 12:52)



So, if your solutions of waveforms come from a linear partial differential equation. In other words if they are linear dispersive systems, they cannot the solutions cannot maintain that profile over long times and long distances. So, you should understand this to mean that a profile that you create initially, as it travels over time and over distance will disintegrate generally except in the special case where $V_g = V_p$, but that is a very rare and special situation, but otherwise in most realistic cases it is going to disintegrate.

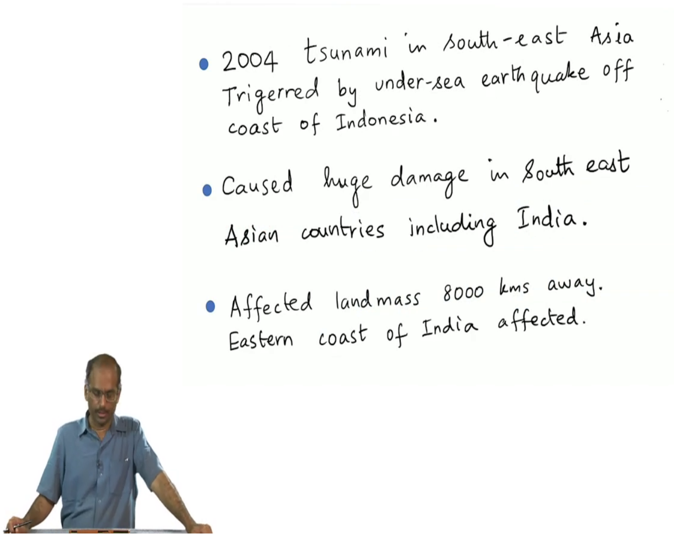
But here is the surprise. Let me first tell you a more general conclusion; Non-linear system by which I mean non-linear partial differential equation and their solutions. They can actually maintain their profile over long distance and long time. So, these are called the non-linear waves or in other words these are solutions of non-linear wave equations and again the solutions are going to be functions of position and time in one dimension.

(Refer Slide Time: 14:06)



And such solutions are called the non-linear waves and they come out as solutions of non-linear wave equations.

(Refer Slide Time: 14:14)

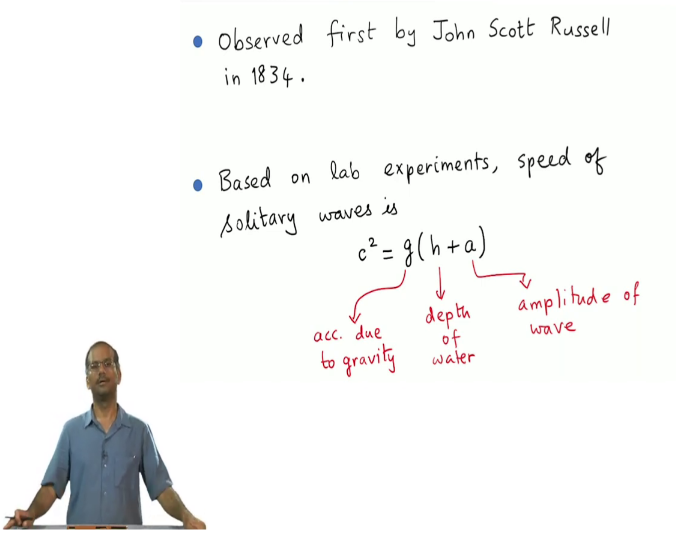


To give you an example of a non-linear wave some of you must have heard or even experienced the tsunami that came in the year 2004. It hit large number of countries in the South East Asia. So, it was actually triggered by an earthquake, an undersea earthquake near Sumatra are actually off the cost of Indonesia. So, there was sudden change in the level of water, which led to a wave being launched on the seas.

So, these waves are special or what would be call solitary waves, they can maintain their shape over long times and long distances. These can be described by solutions of non-linear wave equations. So, what happened somewhere in Indonesia came and affected the Eastern coast of India. So, really distances are more than 1500-2000 kilometers.

So, the wave actually travelled and for almost 45 minutes to 1 hour before creating havoc in east coast of India. So, it caused huge damage in South Asian countries in many parts of South East Asia almost it affected land mass, which is as much as 8000 kilometers away from the point of origin. So, really that is the staying power of these waves. They really can sustain themselves for that long and for that kind of distances they travel huge distances without being affected.

(Refer Slide Time: 15:58)



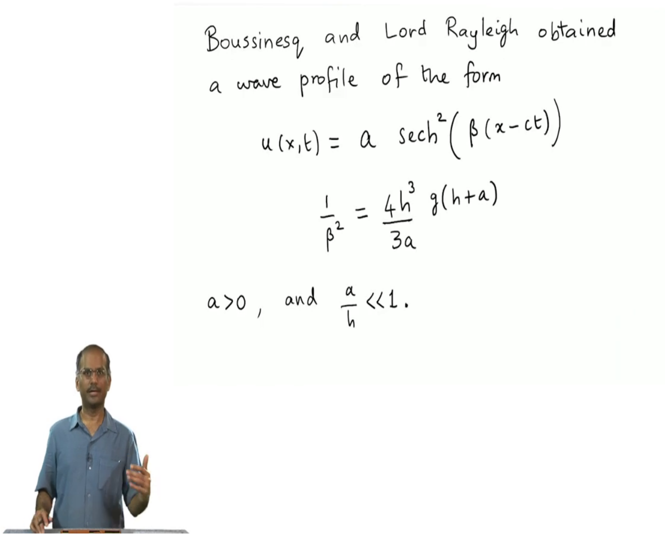
These kinds of waves are not really new; in the sense that they were first observed in 1834 by John Scott Russell in England. So, he was engaged by a shipping company and he was looking at one day he was going in his horse by the side of what is called the union canal. So, what he observed was it they were two boats which were also flying in that route and they suddenly stopped and when they stopped a small wave originated from there and what he noticed was that the wave that originated from that did not die out. So, normally your experience is the following that you throw a stone say at a quiet lake.

It starts waves all around the place and after less than a minute you will see that again it the waves have died out or in general for instance you let us say you play on a musical instrument like a guitar for example, the sound that you generate it last only for a short time and soon it dies out. Any wave that you generate like this generally dies out very fast, but what he observed was the wave that was triggered in that canal by the boat actually started moving in one direction and did not die out for very long time.

So, he was in his horse and followed the wave for almost close to 2 miles. And the wave did not die out either in shape or in time. So, over time of course, he could not follow the wave and lost it in the tributaries of the canal. But nevertheless it is a surprise that it a wave of that kind could sustain itself over such long distances.

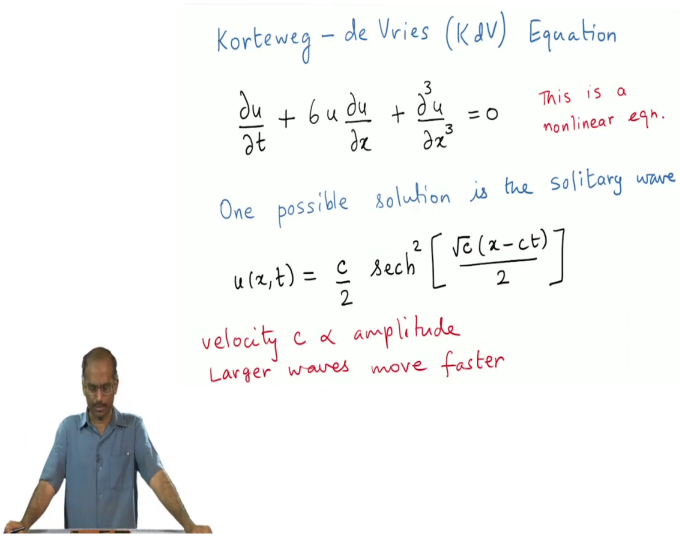
And then Scott Russell went back and did some laboratory experiment and based on all the experiments, he put together and wrote down this result that c^2 the velocity of the wave is equal to $g(h+a)$. So, here g is the acceleration due to gravity, h is the depth of the water and a is the amplitude of the wave. So, larger the amplitude faster it is going to be. So, that is something unusual about this there was no theoretical basis at that time for this formula.

(Refer Slide Time: 18:45)



So, the first time that any kind of theoretical basis was proposed was much later by Boussinesq and Lord Rayleigh. They obtained a wave profile of this form. So, the lesson at this point is that you can have this non-linear wave equation; whose solutions like this sec hyperbolic square would be wave forms that are actually localized in space and could travel without much distortion.

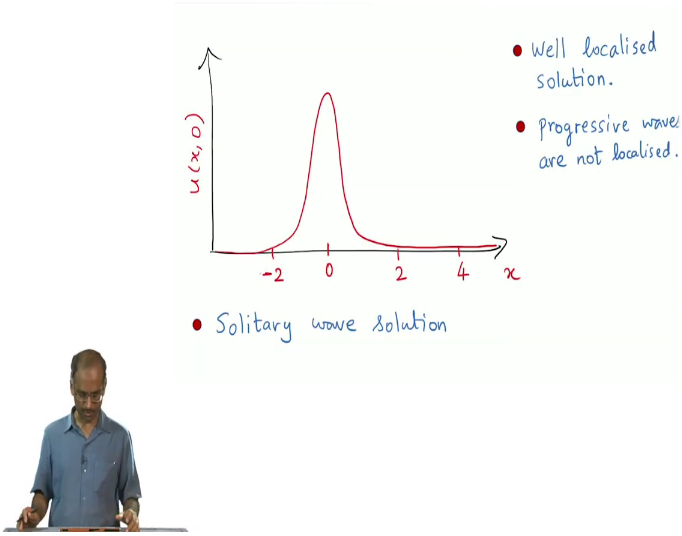
(Refer Slide Time: 19:19)



Another important theoretical development was due to Korteweg de Vries. So, it is called the KdV equation. So, here is the non-linear equation itself. It goes like $\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0$. So, you will see that the non-linearity principally comes from this term here. The product of $u\frac{\partial u}{\partial t}$. Otherwise if this term were absent it is pretty much a linear partial differential equation.

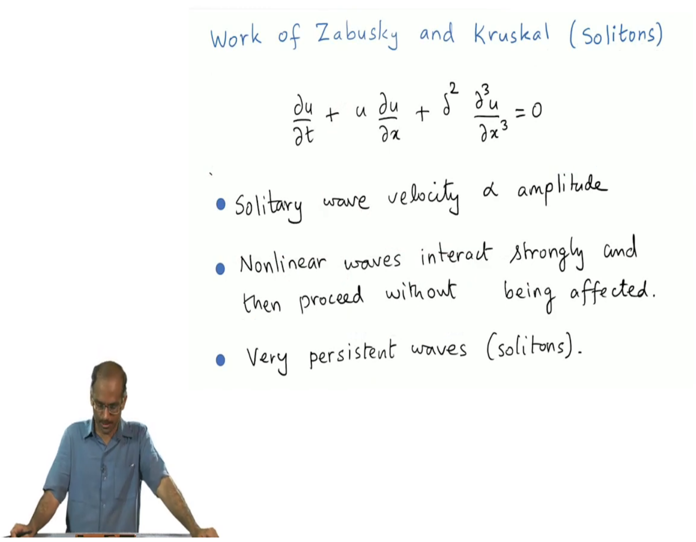
So, one possible solution of this non-linear equation is again a sec and hyperbolic square function. So, it is written down here in the slide here explicitly. So, again the important thing to note is that the velocity c is proportional to amplitude. This is a solitary wave solution.

(Refer Slide Time: 20:28)



So, this is your secant hyperbolic square solution as a function of x at some particular time. So, you will notice the features. First is it is a very well localized solution unlike the progressive wave which extends from minus infinity to plus infinity.

(Refer Slide Time: 20:50)

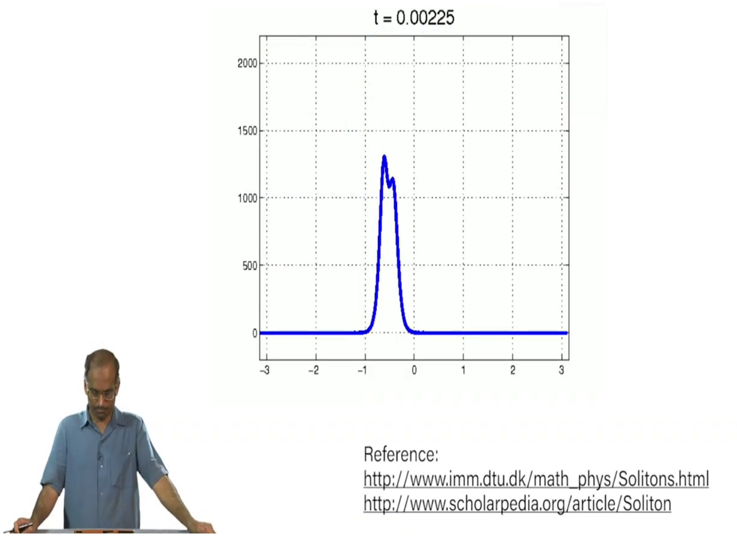


In 50's and 60's Zabusky and Martin Kruskal they launched a program of solving some of these equations numerically to understand their effects and ramifications. Solitary waves, which are the solutions of these equations? $u(x,t)$. In the case of solitary waves the velocity of these waves is proportional to amplitude again a result that is known since the work of John Scott Russell. And non-linear waves they interact strongly and proceed without being affected.

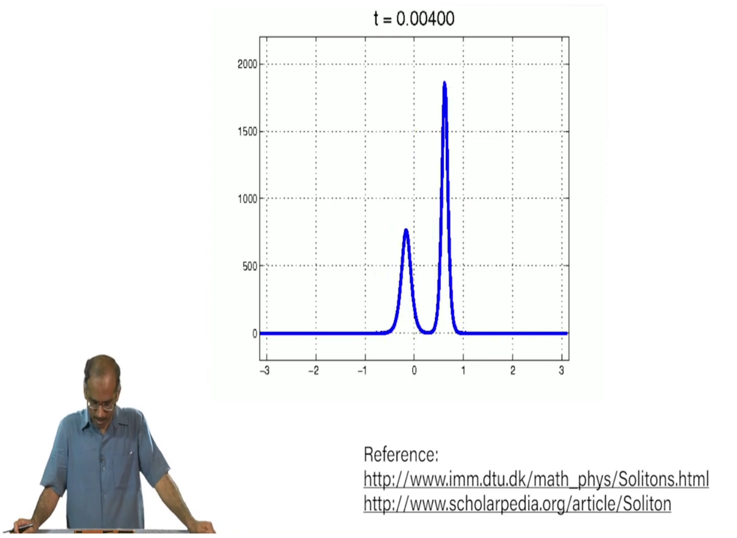
So, which means that if I have one non-linear waves say which is coming in this direction. And another non-linear wave which is coming in this direction. They would interact when they come closer. So, many things happen when they interact. And after the interaction this wave would move as though nothing happened, and this wave would move as though nothing happened. So, that is again something very unusual that these non-linear waves are able to preserve their shapes even in the phase of interactions. So, two waves colliding interacting and then going past to one another as though nothing has happened. So, if you see after their collision both the waves would have recovered their forms.

So, that is an example of how differently non-linearity behaves. So, essentially in all these cases non-linearity effectively tends to cancel the dispersion, which is why they are able to maintain their shapes. In fact, that is what leads to strong persistence; meaning that they can preserve their shape and move along for very long times, which is why Zabusky and Kruskal gave the name Solitons to them. Because, if they were such strongly localized wave packets moving without any dispersion you can almost ascribe particle nature to them.

(Refer Slide Time: 23:08)



(Refer Slide Time: 23:09)

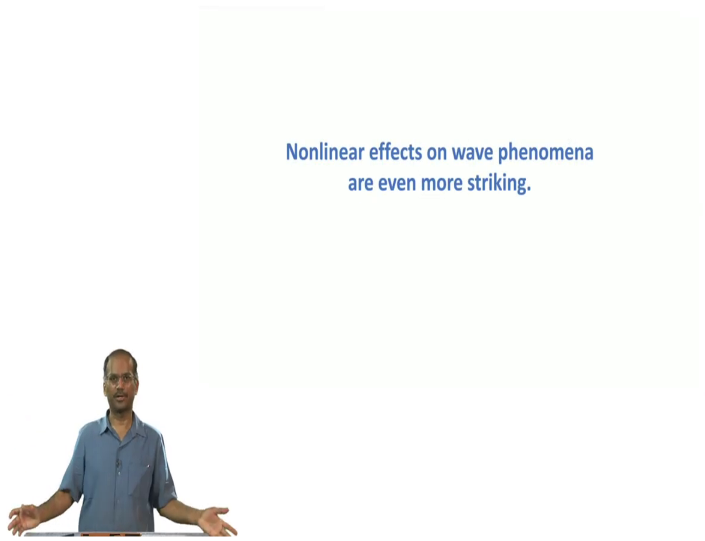


So, here is a simulation you can see that actually there are two waves, which are going in the same direction. So, one wave has a smaller amplitude and the other wave has a very large amplitude. The one with larger amplitude has amplitude of about 1800 in let us say some arbitrary units and the one with smaller amplitude is about 750 or so less than half of that.

Now, as we discussed while before, which of these would go faster. The one with the larger amplitude, because we saw that for these kinds of non-linear solitary waves or Solitons, the velocity of the wave is proportional to the amplitude. So, which means that even though at initial time we start following it from initial time from the left you will notice that the one that is going ahead is the one with the smaller wave, but at some point the bigger one comes and catches up with the smaller one.

And at some point they actually merge together and interact and after the interaction and you can see that the bigger one goes much faster and in fact, it has retained its shape even after the interaction.

(Refer Slide Time: 24:32)



So, effectively the lesson that we have learnt is that the non-linear effects on wave phenomena are far more striking.