Waves and Oscillations
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Lecture – 56
Beyond Linear Oscillator: Chaotic Oscillator

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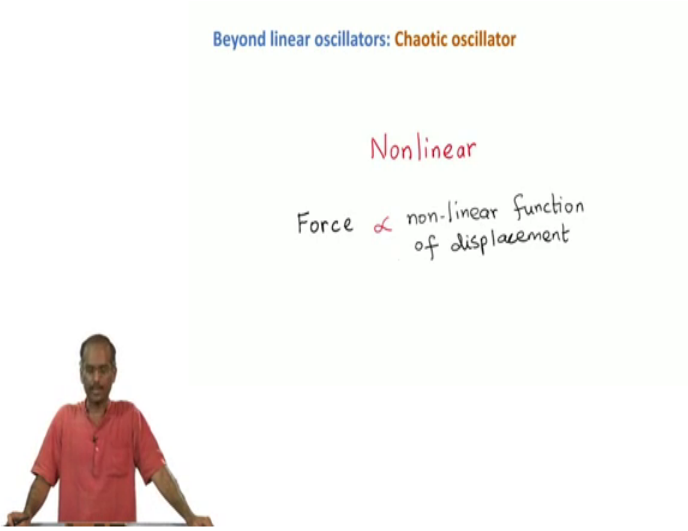

Welcome to the third class of this week, we are going to primarily look at what happens if you go beyond linearity and as I also told you in the very first class of this week, all this would not be part of the exams that we will be doing. So, this is more for curiosity value and also to see where a sort of the field itself is headed. What is it that we will come next if you go beyond what is there in the textbooks and at some level some of these connect up to many of the current researches that is going on in very many areas of physics and also some outside of physics. So, you should take it as a sort of lesson towards going beyond what is there in textbooks more towards real life research that goes on in physics and outside.

In this class we are going to look at Chaotic oscillator or more specifically one particular model of chaotic oscillator. So, again the broad theme of this week is non-linearity in the sense that, we are going to really come out of the linear regime. By non-linearity I mean that the force or the restoring force is some non-linear function of displacement. We somehow came back to putting in those approximations which essentially would make the system linear or in other words equivalent to saying that the restoring force is proportional to displacement.

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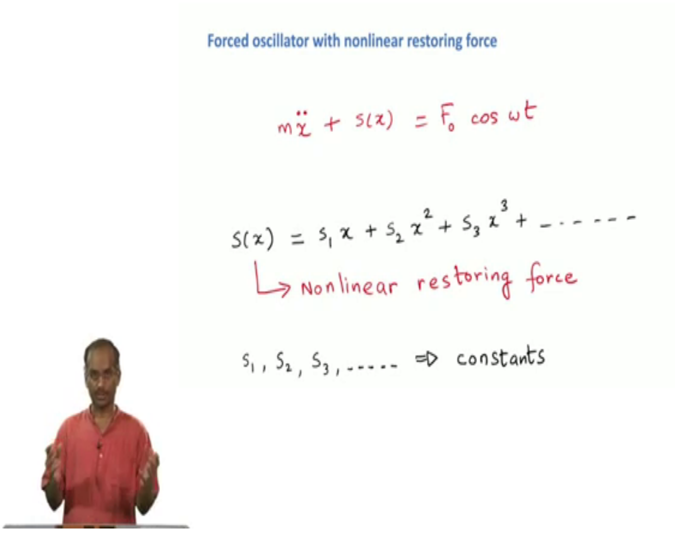

So, this is my idea of non-linearity and we had already seen in the last two classes what it does to linear system when you add non-linearity. Like for instance in the last class we wrote down equation of motion of this form $m\ddot{x} + S(x)$ is equal to some driving which is $F_0 \cos \omega t$ ω is the frequency of driving. And here this a $S(x)$ is the non-linear restoring force and it is given in terms of an infinite series like $S_1 x + S_2 x^2$ square and so on and so, forth and these numbers S_1, S_2, S_3 and so, on are all constants.

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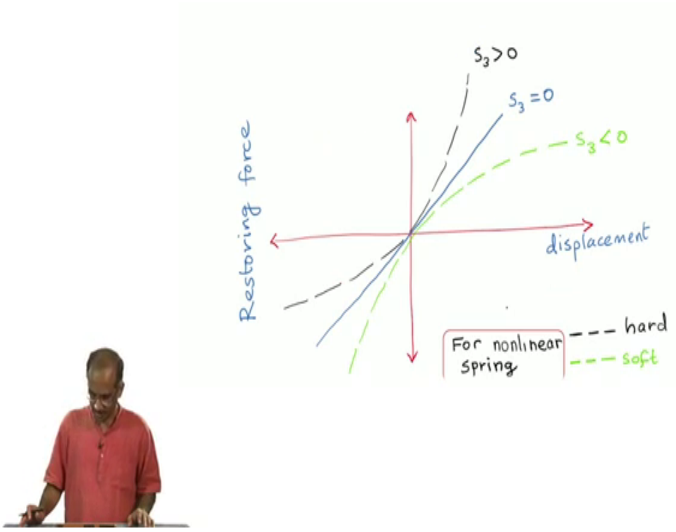

So, broadly speaking if I only restrict myself to let us say the first two terms $S_1x + S_2x^2 + S_3x^3$. Let us say these 3 terms and for the moment let me say that I will get rid of the square term. In that case what we will see is that broadly there is this curve the blue curve that you get when $S_3 = 0$ that is what we had seen as a standard for pretty much the 11 weeks that we had gone through that is the case of linearity.

And non-linearity comes whenever S_3 is non-zero is not equal to 0 in particular when $S_3 > 0$, the curve looks like this the black dashed curve that is there and when $S_3 < 0$ it is the green curve. You might want to call the case of $S_3 > 0$ as being hard spring like and $S_3 < 0$ as soft spring like case.

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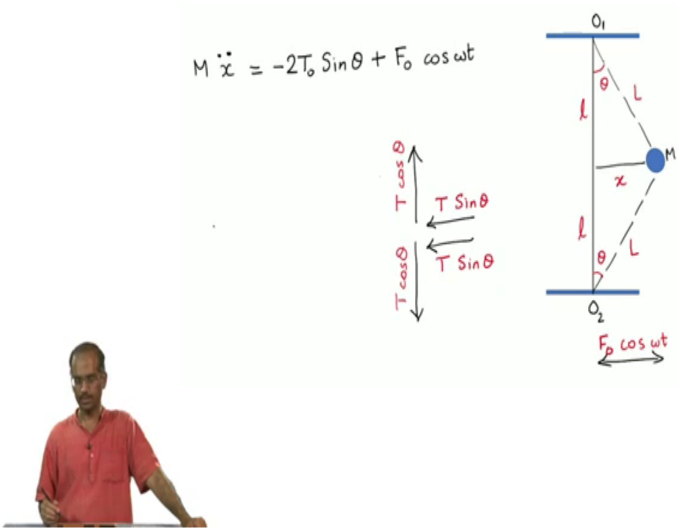

So, again this is something that we had seen and in particular we tried to look at this problem of a bead which is strung between through two wires, to top and bottom or some rigid support. So, it is there between two rigid supports and you are trying to pull it apart on one side and leaving it and then that will set it to oscillation and of course, you can ask for what is the time period and what are the oscillatory frequency of oscillations that it will display. The important thing is that this angle of displacement θ or the angular displacement is no more taken to be small.

So, whenever θ is small, you could get away by assuming that the restoring force is proportional to the angular displacement. So, that assumption we cannot make anymore.

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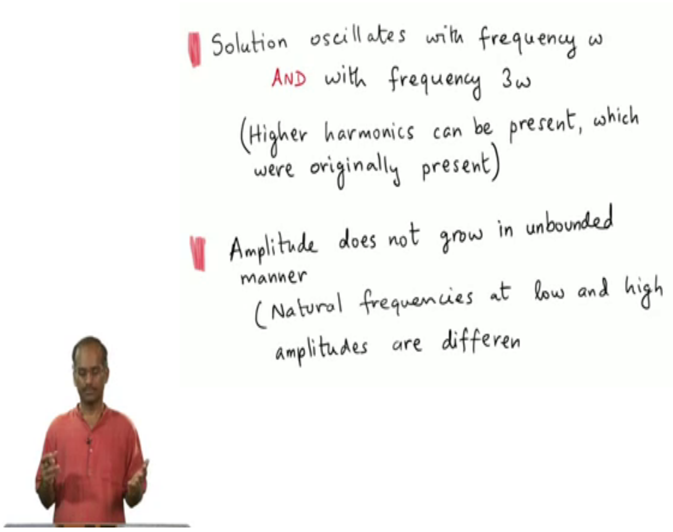

And what we saw was that the solution displays oscillations with the same frequency as driving. So, if driving has frequency ω, then the solution also has frequency ω, but in addition it also shows what are called higher harmonics.

Like for instance it will also show frequency 3ω that is one of the things that we saw and amplitude does not grow in an unbounded manner. Even if you did not have dissipation that is because natural frequencies at low and high amplitudes are essentially different so, you cannot quite match at any one frequency. So, in a sense resonance is not guaranteed to happen; it might still have large displacement at certain values, but it is not as dramatic as in the case of linear systems.

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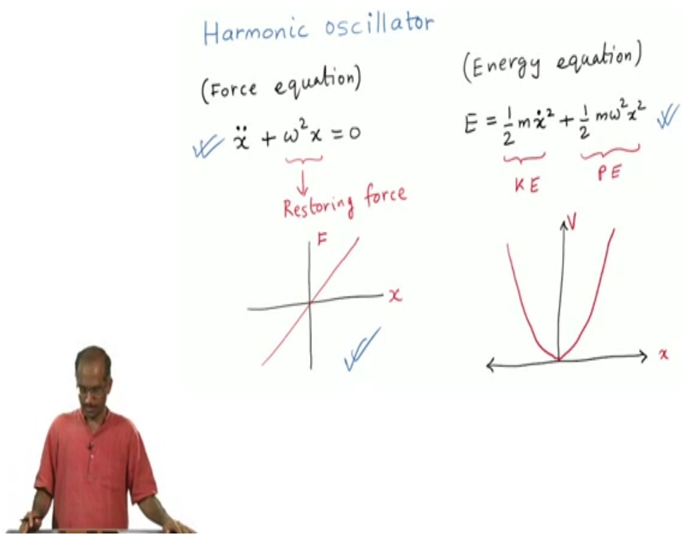

In this class as I said we are going to look at chaotic oscillators or one particular example of a chaotic oscillator. And chaos or chaotic dynamics in itself is a vast area would deserve one or two entire courses to be run to understand it. So, what I will try to give you is, from where we started with linear oscillating systems can we see some glimpse of chaos. So, I will try to give only a physical feel for what you can expect rather than complete mathematical description or even a precise description of chaotic dynamics all that would be fairly outside the scope of this course.

To take the first step towards chaotic dynamics, let us look at standard simple harmonic oscillator, but in a slightly different light. So, we have been most of the time dealing with the force equation of this type. $\ddot{x} + \omega^2 x = 0$, ω is of course, related to the time period. So, this is really a simple oscillator there is one time period and if you plot the restoring force as a function of displacement it is shown here, it is linear clearly $\omega^2 x$ is linear which is your restoring force.

If I write energy equation for the same harmonic oscillator as a system, energy would look like this. So, it is $\frac{1}{2}m\dot{x}^2$ that is the kinetic energy term plus the potential energy is $\frac{1}{2}m\omega^2 x^2$. So, the sum of kinetic and potential energy is the total energy and if you remember what we did in almost the first or the second week, even though the individual kinetic and potential energies are time dependent because this $\dot{x}$ and x are time dependent, but the total energy is independent of time.

So, the energy that you put in initially is going to remain with it forever. Now somewhat intuitive way of looking at it is to look at the potential of the system. So, here the potential is of course, this term $\frac{1}{2}m\omega^2x^2$, m is the mass, ω is the sequence all these are constants. So, I plot this potential energy which I have indicated by the variable V. So, I plot V as a function of x. So, it is a parabolic curve like you see here. So, this would be 0 here. So, that is a parabolic curve.

In principle if you imagine there is a particle inside this potential and you start it off at some point in the potential. So, you can imagine that you have a bowl of the shape and maybe you can place a marble at some point. The marble would go down go up to the same height come back to the same height and it will keep oscillating provided of course, there is absolutely no friction in it. In reality that is not the case. So, the marble would finally, come to a stop somewhere.

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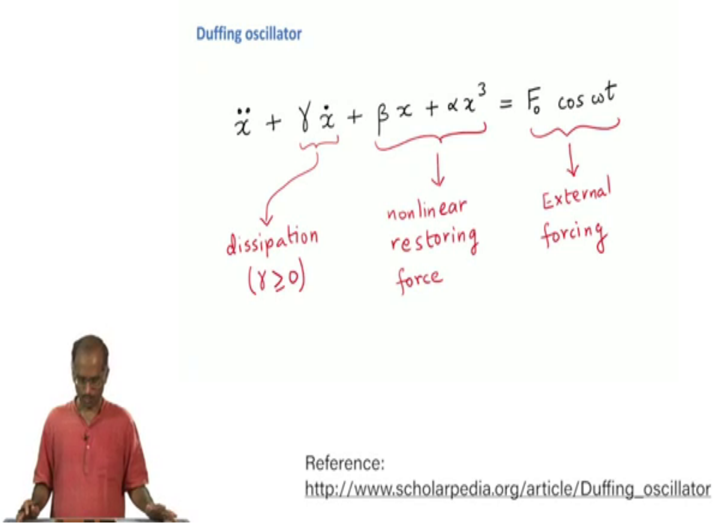

So, we already worked with a non-linear oscillator which was also forced, but it did not have the dissipation term, in today's class we will put in everything. So, we have a non-linear restoring force for our oscillator. So, I have written the force equation for what is called the Duffing oscillator. So, it has this specific non-linear restoring force which is of this form $\beta x + \alpha x^3$ and the dissipation is of course, proportional to velocity. So, it is a viscous dissipation and γ is of course, the dissipation coefficient.

So, these ingredients are there and on the right hand side, I have the external forcing which is $F_0 \cos \omega t$, ω is the frequency of forcing. So, the external forcing is periodic and F_0 is the amplitude of that forcing. So, β is part of the non-linear restoring force term, but I am taking β as that particular coefficient which appears in front of the linear term. So, you can think of it as the frequency associated with linear oscillator.

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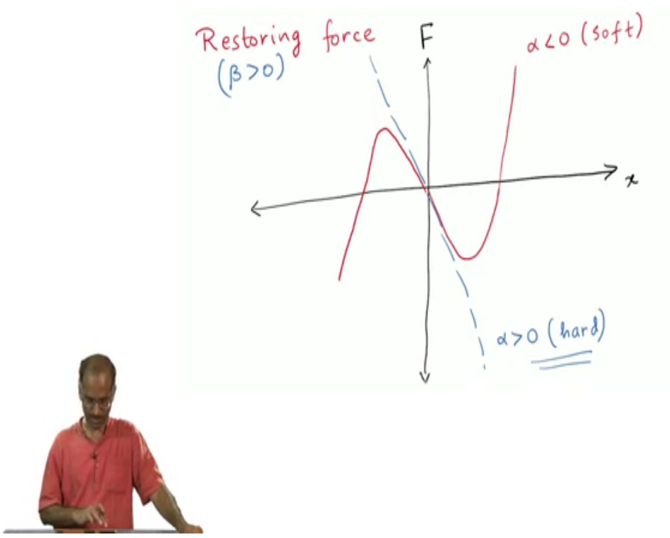

And I take $\beta > 0$ and then there are two possibilities for α. So, α is the coefficient that comes in front of the x^3 term as you can see here. So, either α could be negative or positive. If α is negative, the curve that you will get in this plane restoring force as a function of displacement is this one that is shown in red and this is of course, a soft spring limit. On the other hand if α is greater than 0 you will get something which is which looks linear, but it is not quite linear so, alpha greater than 0 would correspond to the hard spring limit.

So, the restoring force versus α greater than 0 would follow this blue dashed curve that you see here. Now, before we actually look at the complicated solutions of this oscillator, let us first understand what happens if there is no dissipation and no driving. So, in other words γ is 0, F_0 is 0. So, what I will have is oscillator which is a non-linear oscillator, but most importantly it will be an oscillator which will conserve energy.

As you can see we are removing the dissipation term. So, whatever energy you put in will remain there. So, it will be a system which will conserve energy in that sense when γ is 0 and F_0 is 0, it will be a conservative system. And when F_0 is 0 as usual there is no input of energy. So, there is neither output of energy nor input of energy. So, energy that you put in initially will stay there forever. The only difference being that with respect to the harmonic oscillator here are restoring forces non-linear.

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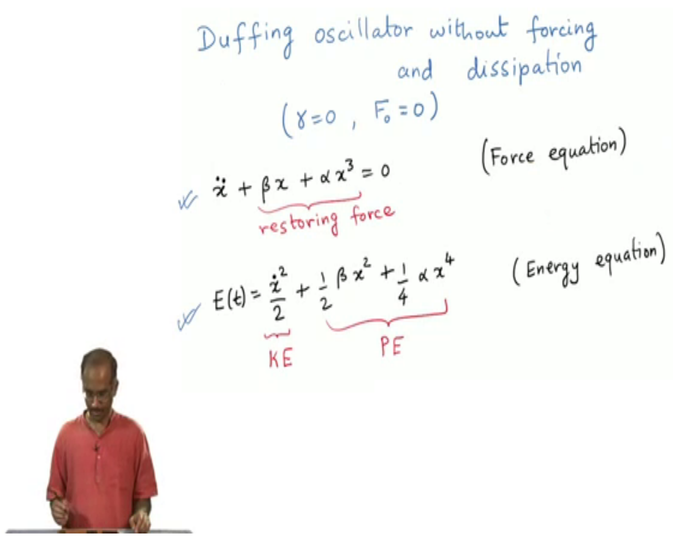

So, first let us consider this limit; without forcing and dissipation and as a force equation this is how it will look like $\ddot{x}+\beta x+\alpha x^3=0$. And now you can write the corresponding energy equation that will look like this. Again $\dot{x}^2/2$ is the kinetic energy term. So, that is like $\frac{1}{2}mv^2$ you can imagine that in this case m is set to 1. So, it is simply $\dot{x}^2/2$.

So, that is the kinetic energy and the potential energy is given by $\frac{1}{2}\beta x^2+\frac{1}{4}\alpha x^4$ and you can see that if you integrate the restoring force, you should get the potential or the potential energy. So, it is the standard relation between potential and the force which needs to be used to write the energy equation. Now, once again like we did just now for the case of harmonic oscillator, we sketch the potential. So, we will do the same thing now I have my expression for the potential which is this.

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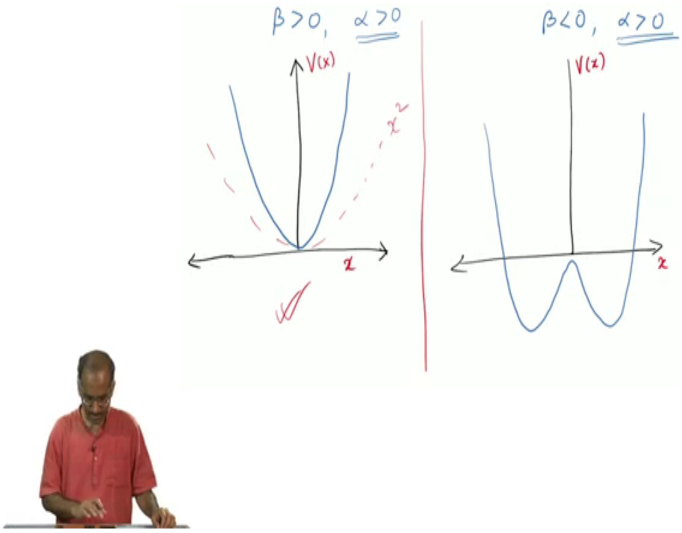

Now, let us sketch this. So, I am going to assume that $\alpha > 0$ and then sketch the potential for two different values of β. So, you can see that the potential involves α and β. So, we will take β to be negative and positive, but keep α to be positive. So, if α is positive and β is also positive, this is the potential curve that you will get. Potential is again indicated by V as a function of x and as you can see both the terms will be positive β and α if they are positive. So, it is just going to be a curve that rise faster than the standard parabolic curve.

So, in other words I can even try and plot x^2. So, this is the one that corresponds to x^2 term and our potential which is shown in blue will rise much faster than that. On the other hand if $\beta < 0$, the first term is going to take a negative sign. So, it is a competition between two terms one is positive, αx^4 term is positive, but βx^2 term is negative. So, you can see that if x has a large value. So, if you are looking at large displacements, then x^4 will dominate in this of these two terms.

So, ultimately for large displacements, it is going to go like the first case it is going to increase go towards infinity both that positive displacement and negative displacement. On the other hand if $x < 1$, then for small values of x; this x^2 term will dominate, that will be more than the x^4 term. So, that is this region here. Now like I gave you the analogy of placing a marble in a bowl which has the shape of a oscillator and looking at it is dynamics, we saw that it was it would be an oscillatory dynamics, we can do the same thing with this now that we realize what the potential is, it is easier to figure out the dynamics without doing calculations.

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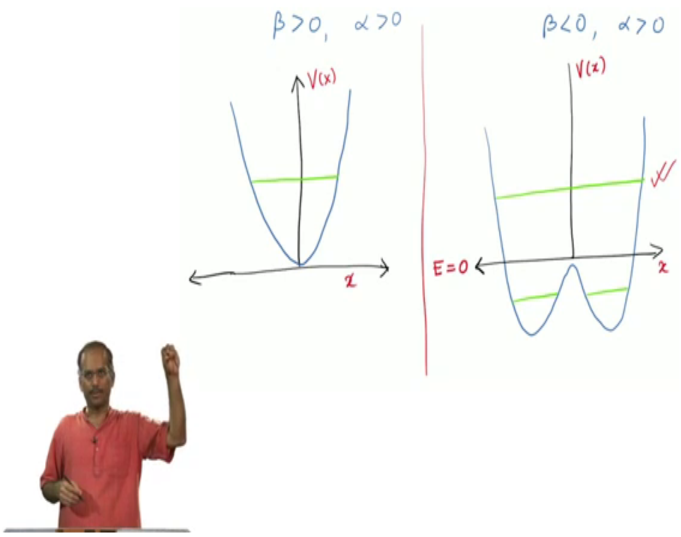

So, for example, here this one looks like a oscillator potential on the left side $\beta > 0$ and $\alpha > 0$, but we know that it rises up faster than the standard oscillator potential. So, here if I leave a marble it will keep going between these two and that effect is indicated by this green line. So, in other words that also will tell you that the energy is a constant.

In the case when $\beta < 0$, then there are different cases that can possibly arise. For instance, if you take the value of energy to be greater than 0 for large energy you will get this green curve. So, there is one periodic orbit in a sense, because again going back to the analogy of placing a marble in a bowl; when I place the marble at this point and if there is no friction it will go down go up this smaller hill, go down again and reach the top and then it will again come back and retrace it is path.

So, it is actually executing periodic motion, a little more complicated than the standard periodic curves we saw for the oscillator. So, that is one kind of what would be called a periodic orbit and there is also periodic orbit in this case as well. But now, suppose if your energy is small such that consistent with a small energy that you have, let us say that where you can place the marble is somewhere here within one of the bowls let us say the left side bowl.

So, in that case the marble will simply oscillate within that left hand side bowl and similarly if you had placed the marble on the right hand side bowl somewhere here, where the green curve is shown it is going to oscillate in that right hand side bowl, it will never have the energy to be able to cross the barrier and come to the other side.

So, in other words depending on your initial conditions or depending on your energy, you can see different kinds of dynamics possible, but in this case all of them are periodic; now, let us put in the dissipation.

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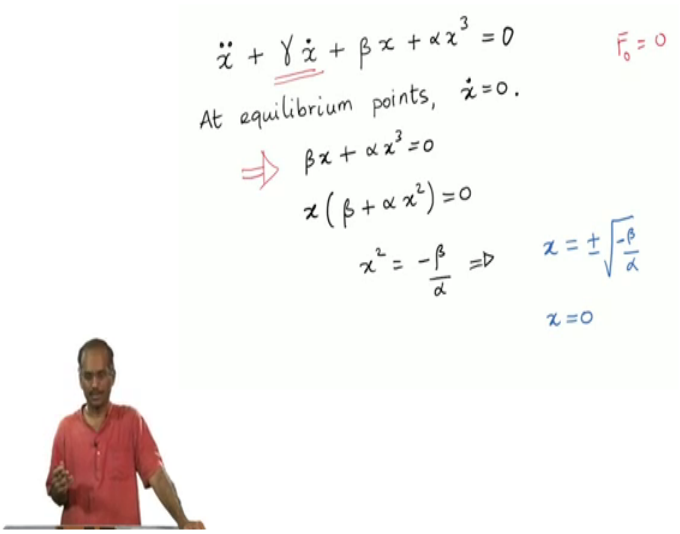

So, now I have added this term, but still there is no forcing. So, F_0 will still be 0 so, you will expect that depending on certain condition after a long enough time, you will find the marble either in this at the lowest point of the potential. It is either in the left bowl or in the right hand side bowl, because dissipation is going to continuously remove energy from the marble and it is going to finally, come and stop at the point, which is lowest in potential energy and these points are called equilibrium points.

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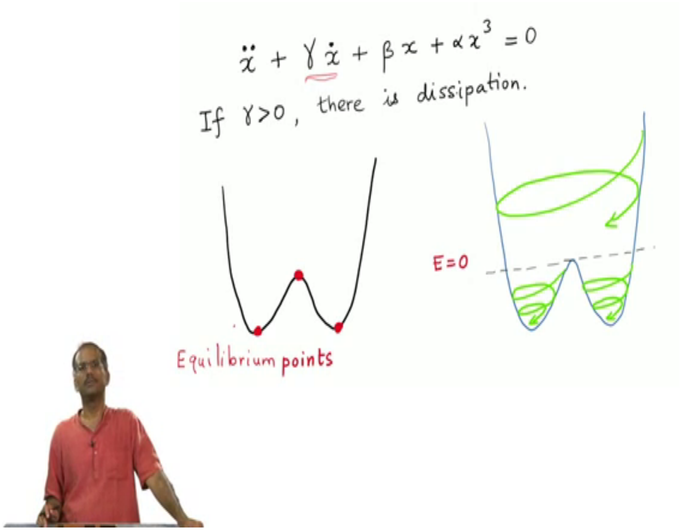

So, equilibrium points are where the net force is 0. So, the fact that the marble actually finally, comes and settles there, it means that it has found a position where there is no net force acting on it. Because if there was net force acting on it will keep moving further at that point there is no net force acting on it and after that no motion is possible, once it comes and settles to that position.

Purely from intuition and from the kind of pictorial arguments we are making, it would have been very easy to guess that the two equilibrium points would be the two lowest points in the bowl, but what comes as a surprise is that, actually there is also a equilibrium point here. As I said at equilibrium point once the marble comes and settles it is speed is 0 which velocity is 0.

So, to find equilibrium point all you need to do is to put $\dot{x} = 0$, velocity is 0. So, in which case this term $\gamma\dot{x}$ will go to 0 and $\ddot{x}$ will also go to 0. So, you are left with this condition that $\beta x + \alpha x^3 = 0$. We need to find out the value of x corresponding to this algebraic equation and you will see that this equation has three roots; one is $x = 0$ and other is $x^2 = -\frac{\beta}{\alpha}$. So, $x = \pm\sqrt{-\frac{\beta}{\alpha}}$.

So, there are three possible roots and especially for the case when beta is negative, you will actually get three possible real roots and those three real roots are exactly these three positions which are marked in red dots here. So, clearly the equilibrium points are known and physically it corresponds to saying that, that is where the particle would ultimately end up if you give dissipation, but no external forcing.

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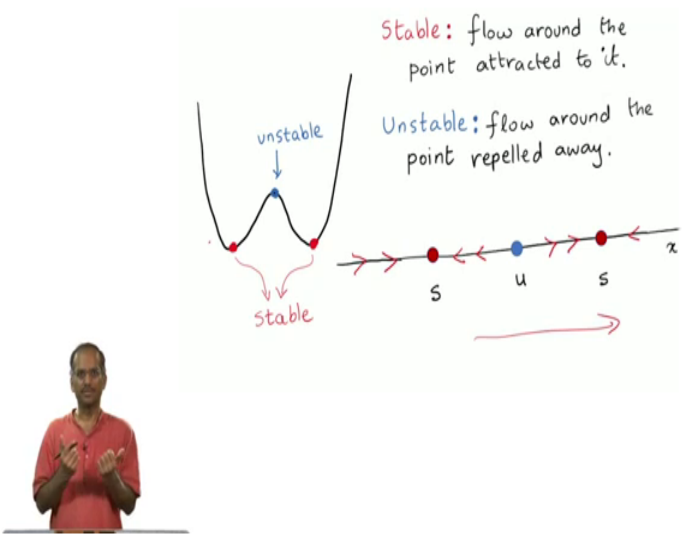

Now, let us address the question of this specific equilibrium point, which is at the top of that hill like structure here; why is that an equilibrium point. So, as we found out there are 3 equilibrium points, two are stable which we could have guessed purely from the dynamics the third one is actually unstable. So, I am not going to derive for you how to find that it is stable or unstable, but let me tell you that when something is stable the trajectories that are nearby to it are attracted to it.

So, that is what happens when you are at the bottom of a hill. So, let us say that you have a marble at the bottom of a hill, give it a little bit of push it will go around somewhere, but finally, after some time it will come and settle down there. So, that is an example of a stable equilibrium. But suppose you have this hill like structure, you have actually a ball with a hill like structure you keep a marble there; you need to be really precise otherwise the marble would just go down in one or the other side.

So, which means that a little bit of perturbation around the top of the hill is going to take you far away from that point. So, that those are called unstable points which means the trajectories around it are not being attracted to that point; so, that is what happens here. So, if I actually plot these two points on the x axis here, what I will see is that there are two stable points. So, I can indicate the fact that the flow or in other words the trajectories are attracted towards it by putting an arrow here.

So, this arrow is to indicate that trajectories will get attracted towards the stable point. So, you can see that when I put in this arrow, everything is moving towards the red points which are stable and the same thing also tells us everything is moving away from the blue point which is unstable. So, which is why your marble is never going to settle at this unstable point, it requires infinite amount of projection to be able to balance a marble on a hill like that.

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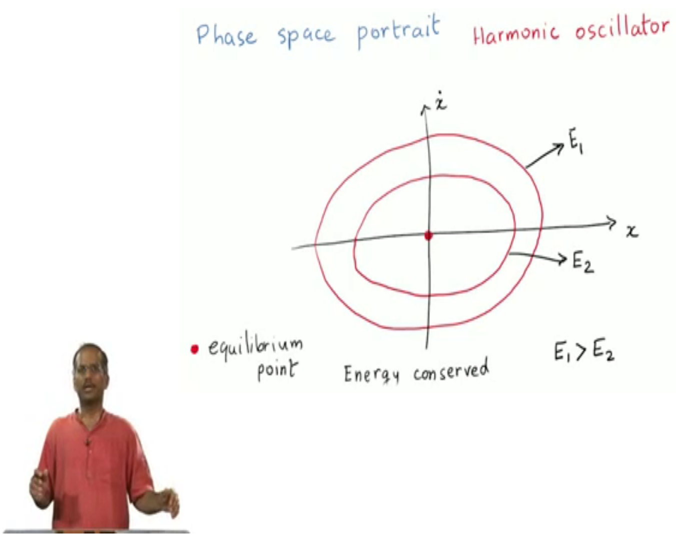

Now let us look at harmonic oscillator again from this perspective. I am going to introduce this idea of looking at it in what is called the phase space. So, on the x axis I have the displacement, on the y axis it is the velocity. So, it is position versus velocity and now I have drawn this for the case of harmonic oscillator standard simple harmonic oscillator.

You will see that I have drawn two curves, but in principle you can draw any number of curves. For every value of energy that you choose remember that in harmonic oscillator energy is a constant for every such value of energy that we choose you can draw one curve here, and all these are closed curves and reflects the fact that the system gives periodic orbits.

So, both these curves I have labelled by E_1 and E_2 corresponding to two different values of energy. So, the one that lies outside the larger the curve are larger the area that it encloses, the energy is larger and you can ask what is the equilibrium point here. So, here since you do not have any dissipation these are called technically centers, the equilibrium point is right at the center. So, if you are there nothing will happen.

So, it is like saying that I have a pendulum and it is just hanging vertically. So, it is basically in the equilibrium position so, unless you give it some energy or something nothing is ever going to happen. So, this red point here indicates that equilibrium position and for this harmonic oscillator, if you add in dissipation then these elliptic curves that you see here will spiral around and finally, go to this equilibrium point.

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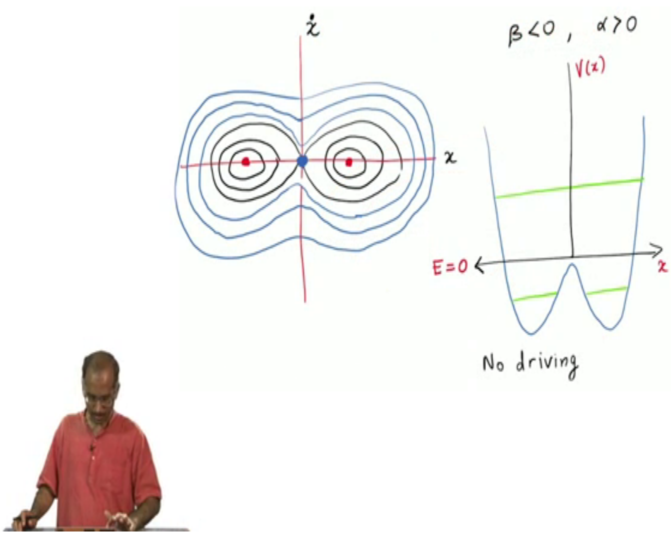

Now, let us come back to our problem which is this. Now, let us see how the phase space would look like for this problem. Now, in this case when $\alpha > 0$ and $\beta < 0$. So, here this one and now again I am not going to introduce dissipation. So, in that case the phase space curves would look like this. So, you see that corresponding to two equilibrium positions at the lowest point of the two bowls, I have these red dots and the unstable point is indicated by this blue dot at the centre.

So, within each of these bowl like structures, it is almost like a harmonic oscillator. So, the curves that you see here they are like this as though it is a harmonic oscillator. But, as energy increases you go pass this structure of two bowls and then you get much larger ellipse like structures, which also in some sense look like harmonic oscillator closed curves they also give periodic solutions.

So again, here if I had put in dissipation, but no forcing. So, these what looks like elliptic curves would become spirals, they would spiral around and finally, settle at the two one of the two stable fixed points which are indicated in red color here.

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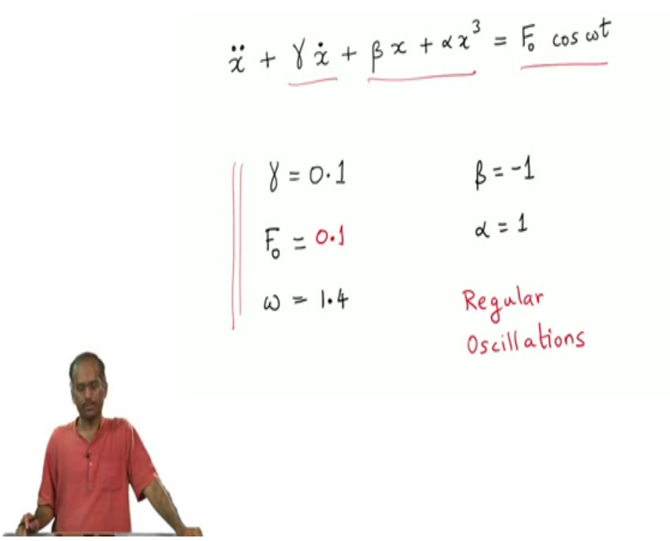

Now, we will put in all the ingredients. So, we have the non-linear restoring force which is this, will have the dissipation viscous dissipation with γ and also the external forcing. So, in this case, it is going to show what is called chaotic oscillations. So, chaos would mean very irregular orbits suppose I make this choice of parameter. So, you will see that there are several parameters here, there is this γ which is the dissipation coefficient, there is β and α, which are two parameters in the non-linear restoring force that we have introduced and F_0 is the amplitude of the external forcing.

So, now what I am going to do is to keep all parameters except F_0 constant. In the sense that I am going to put γ to be 0.1, I am going to take ω which is the frequency of the external driving to be 1.4, I am going to take β to be -1 and α,1. So, the only thing I am going to keep changing is F_0. Now, if I take F_0 to be 0.1, what happens?

So, before we do anything. So, I must tell you that when the problem is considered in it is fullest details with all the ingredients thrown in, there is no simple way to analytically solve this in it is most gentle form. In some limits there are solutions, but there is no general solution which you can write down analytically and see the chaotic oscillations, that is not possible. So, it has to be done numerically. So, what I am going to show you are numerical solutions from now on.

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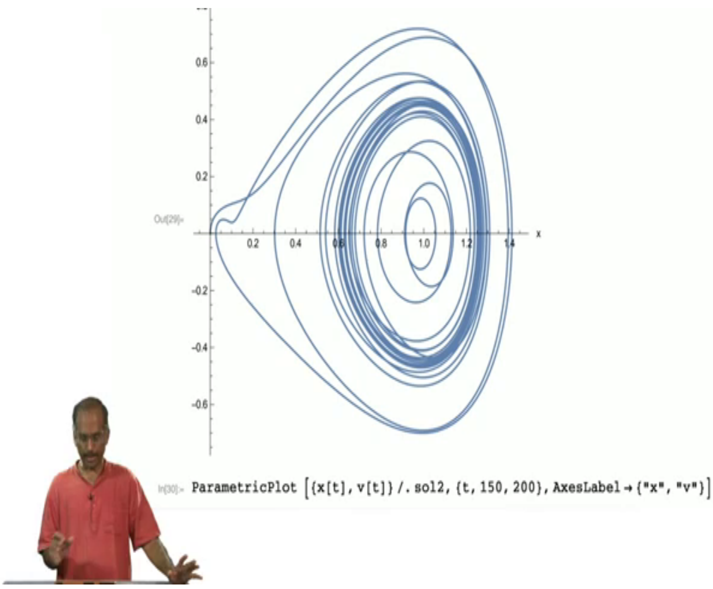

For the choice of parameters that we made, this is what we will get. As I said this is the numerical solution and what is plotted is the phase space plot. So, x axis is the position and y axis is the velocity. So, it already looks like it is very irregular, but that is not so. So initially for first 200 or first 100 time steps when I look at it in phase space, this is how it looks like something it looks like there is no regularity here, but that is not true because these are what would be called the transients.

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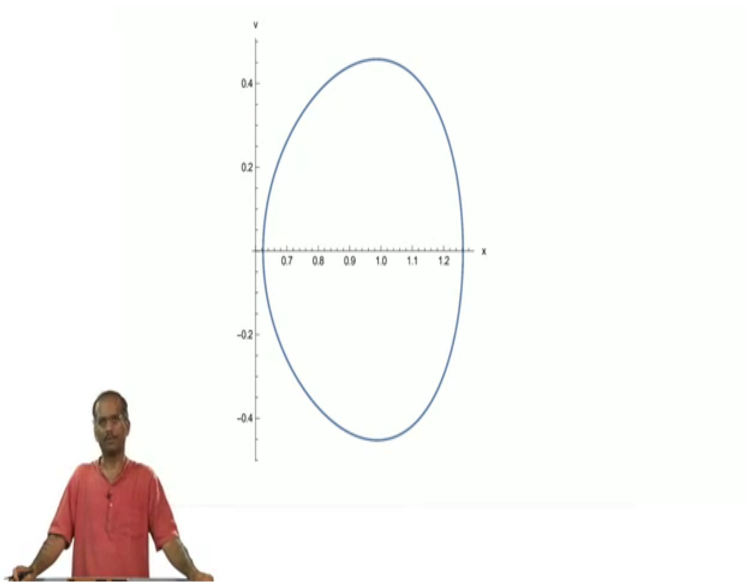

If I wait for some more time and look at the phase space after about say 300 or 400 time steps, all the transients die out and what I get is something like this. So, very nice ellipse like curve that I have, the orbit is periodic and if I plot x as a function of time, I should be able to see it as a periodic curve.

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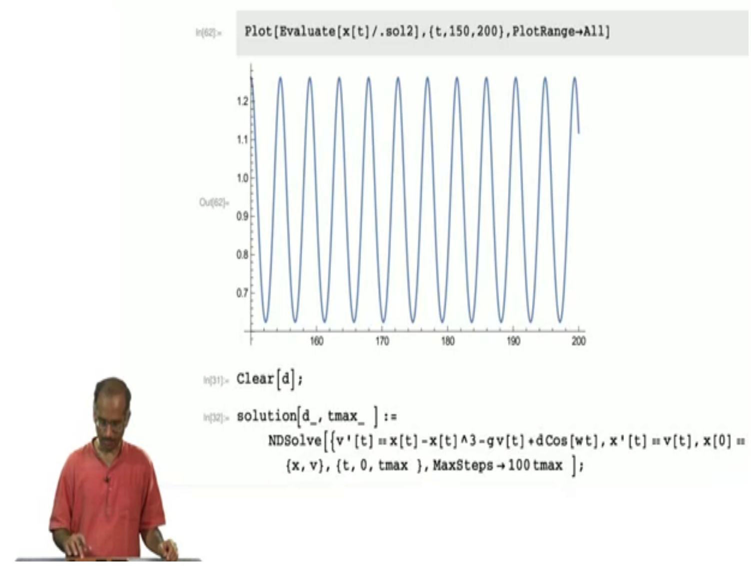

And indeed when you plot x as a function of time, which is what is shown here you see that it is a periodic curve. But, it not it need not necessarily be a periodic curve like a sin or a cosine function.

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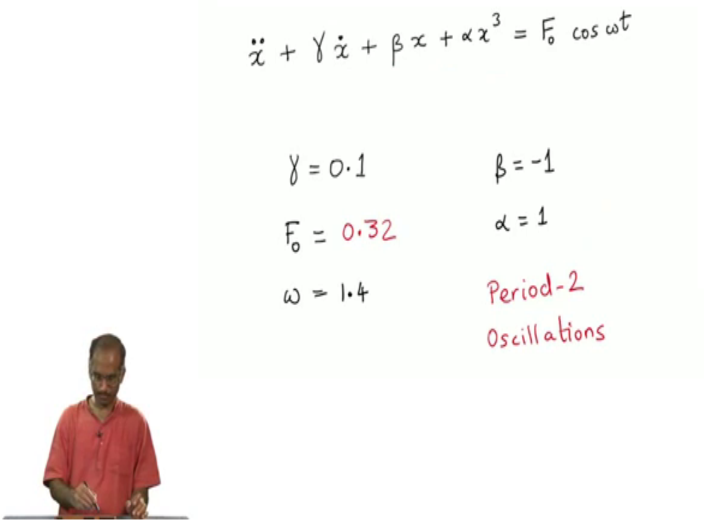

Now, from F being 0.1, I am going to put the value of F to be 0.32, let us see what happens in this case when F_0 is 0.32 all other constants being the same.

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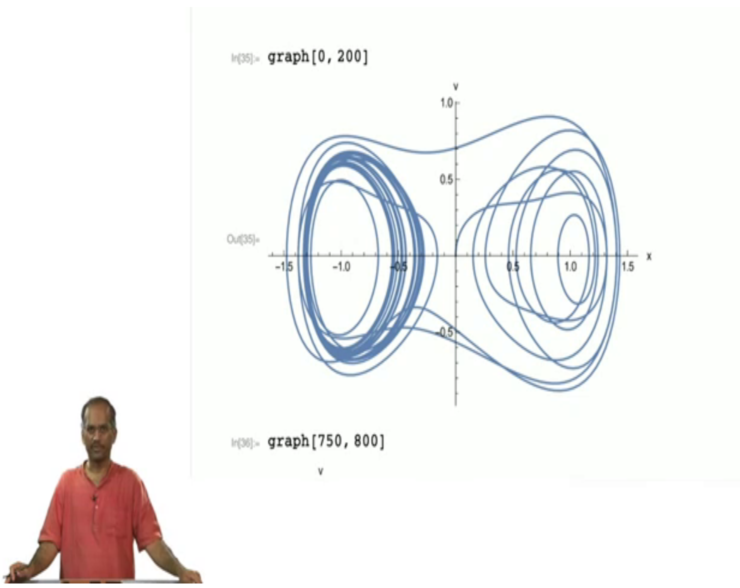

Here again we are looking at the phase space with F_0 being equal to 0.32 and you will see that again it looks like a mess in phase space the curve does not seem to show any kind of pattern, but again I should remind you that this is the initial transient.

So, if you wait for some amount of time for all the transients to die down and let us say then plot the same phase space after about say 700 time steps.

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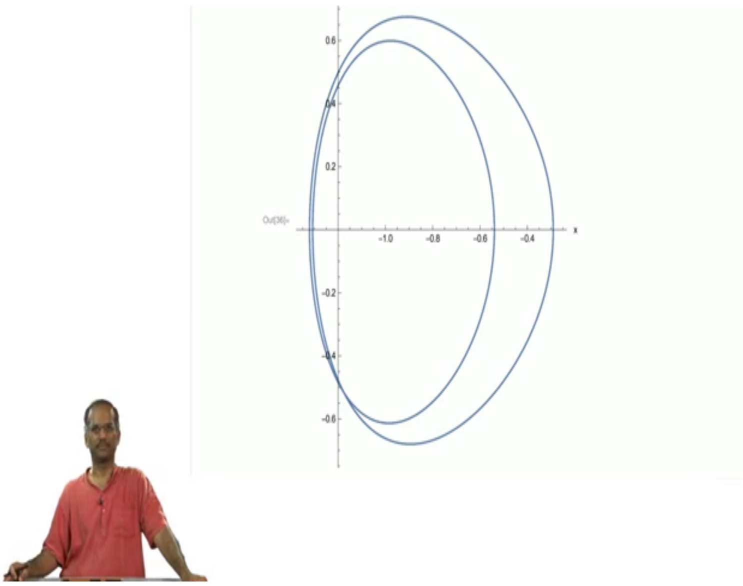

Then you notice that you get a much nicer, curve and it looks like periodic and actually it is indeed periodic, but the difference is now is that. So, if you remember the previous curve that we got, this is also a periodic curve it has one period. So, it starts from a point and comes back to the point.

So, especially when I see it in a phase space like this, for instance if I put in let us say a screen at this point, where the curve goes and comes back it will leave one dot in that screen. On the other hand here you see that it is doing two rounds. So, these are called period two orbits. In other words if I put a screen across this, it will once go and go second time and then again retrace the path. So, it will leave two dots. So, that is what would be called a period 2 orbit, it is a period 2 cycle.

By the fact that we change the parameter what has happened is, what was a period 1 solution has become a period 2 solution and this is the period 2 solution and now this is in phase space, now I can look at it as position as a function of time which is what you see here.

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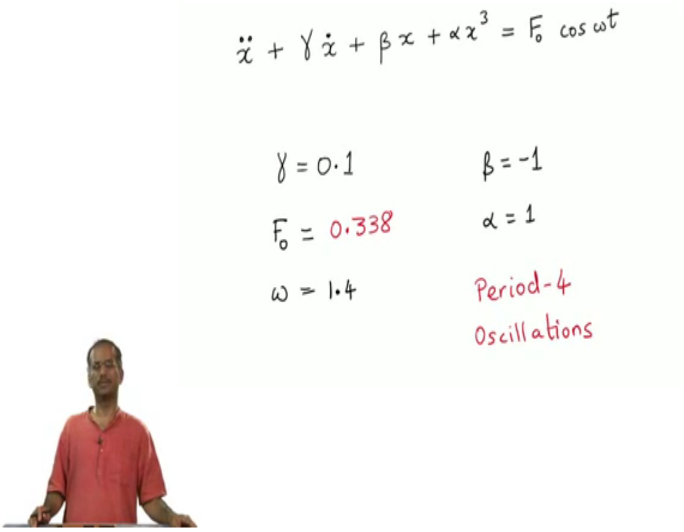

Now if I change the parameter F_0 once again let us say that I make it 0.338, again all the other parameters are kept constant. Now, what I get is what are called period 4 oscillations. So, you can see the sequence first we had a period 1 solution or a period 1 oscillation and when I change the parameter I got period 2, now I am getting period 4. So, this kind of sequence is called a period doubling sequence.

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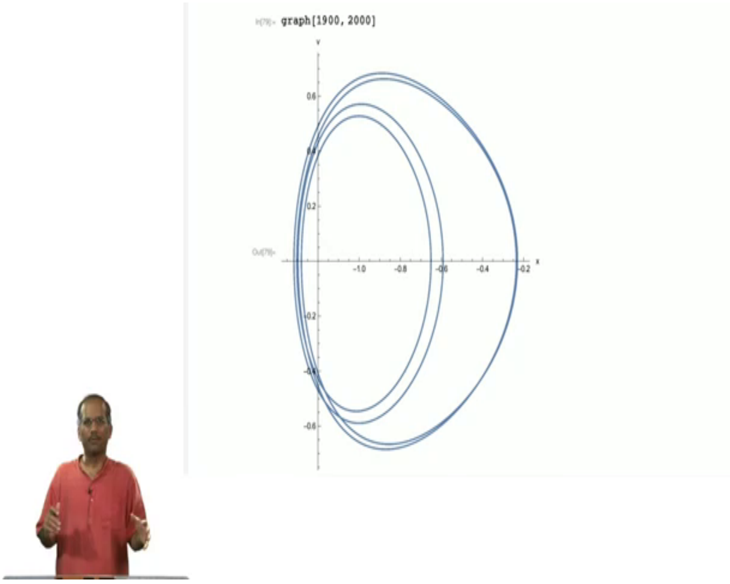

So, again now I am not even showing you what happens in the transient case; so, let us assume that all the transients have been taken care and they are out of the way. So, when I plot the asymptotic nature of phase space this is what I get. In this case what was originally used to be a period 2 solution has become a period 4 solution because, if I put a screen on top there it will leave 4 points, and this happens at the parameter value of about 0.337 or 338. And you can also check cross check this by plotting position as a function of time.

So, essentially what happens is that as you keep changing this parameter F_0, you are going to have higher and higher periodic orbits generated and ultimately. It will lead to what essentially would look like a random oscillations which is what we call chaotic oscillations, but I should emphasize here that chaos is not just randomness.

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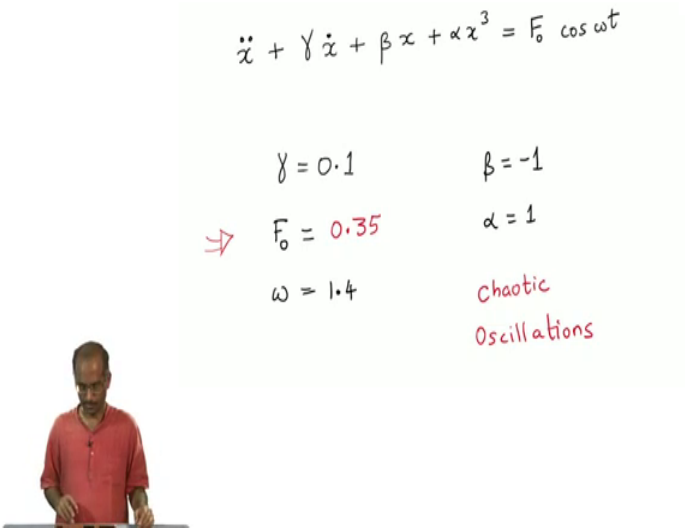

So, any set of chaotic looking oscillation is not mess, any set of random looking oscillations is not chaotic. So, chaotic as I said is much deeper property of dynamical systems and we really will not be going into the depths of what is chaos and so on. But let me give you a glimpse of what happens, in the regime where we have set let us say F_0 to be 0.35 and we expect quite irregular set of oscillations.

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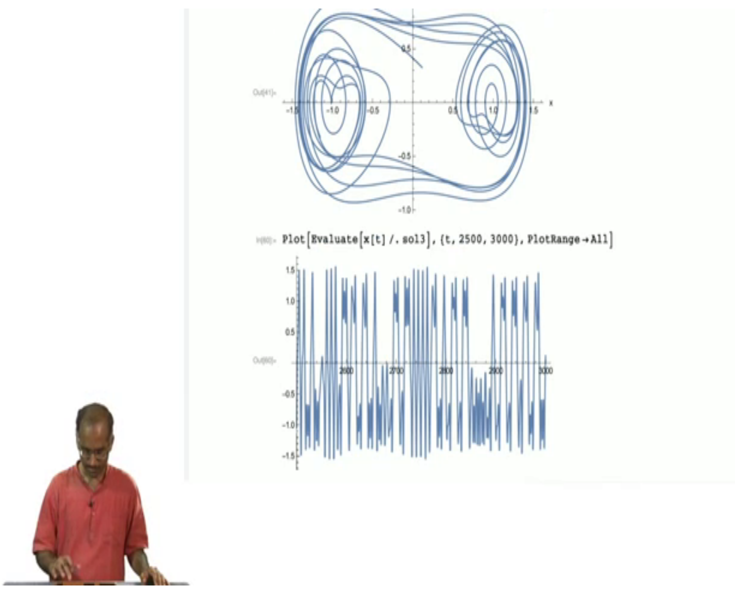

So, here when I plot the phase space now, what you will see is that it looks quite haphazard sort of miss and if you think that maybe it is a transient and if I wait for long enough maybe it will settle down to something that is meaningful no. So, in this case you are never going to have a situation where it will settle it down to something meaningful. So, the trajectory will keep bumping between these two left and right side in a very irregular manner.

And if I look at the position as a function of time, it is going to capture for me essentially some random looking function. So, as I said let me emphasize once again chaos is much more than just randomness. So, systems like these, which are which have non-linear restoring force, they have dissipation, they have external forcing they have all the ingredients to display chaos.

For example, when we had only linear restoring force we did a similar problem. We had dissipation, we had external forcing, but the restoring force was linear in that case you will not see such chaotic oscillations whatever your parameters are. So, chaotic oscillations are in general chaotic dynamics is just at some level rather random looking trajectories. But, it shows an important property that if I start two initial conditions close by, they will diverge from one another exponentially in time and that would not be displayed by any of the non-chaotic systems.

So, like for example, we saw that when the parameter F_0 was sufficiently small you had regular oscillations. You take two initial conditions which are close by in that regime where it shows regular oscillations. The only thing you will see is that now there will be two trajectories which are closely following one another, and they will keep and they will continue to closely follow one another for any amount of time that you want to integrate the equations of motion.

On the other hand, in the region where you chosen your parameters such that the system is in chaotic regime do the same thing, take two initial conditions and look at the trajectories as a function of time. Initially they will tend to go together or close by for a certain amount of time, but after some time they are completely two different trajectories. So, this is one of the ingredients of chaos that it shows extreme sensitivity to initial conditions, ok. If the initial conditions are even a little bit different from one another, the trajectories the outcomes after certain amount of time is going to be very different.