Waves and Oscillations

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Lecture – 55
Beyond Linear Oscillators: Forced Oscillator

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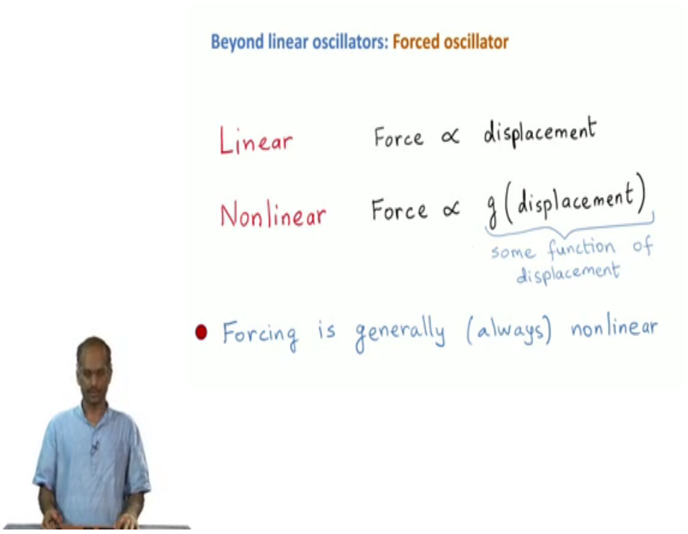

Welcome to the second module of this 12th week. We were looking at general questions about what would happen if you go beyond the linear regime. The motivation being that up until the 11th week everything that we did about waves and oscillations we stuck to the linear regime. In other words somewhere down the line the response has taken to be proportional to displacement or the restoring force is proportional to displacement and of course, there was a minus sign to take care of to ensure that there will be oscillations.

So, we are looking at general questions about what does non-linearity do to the oscillations. So, in the first lecture this week what we studied was one of the simplest problems of that type, which is the simple pendulum. The only point is that you should not make the approximation that sin θ is approximately equal to θ.

So, then you go ahead and finally, look at the consequence of adding non-linearity to the system. The first thing that you see is that of course, the problem gets harder to solve, you cannot even solve it explicitly in terms of simple functions that we know like for example, it cant be written in terms of simple sin and cosine functions.

So, the solutions are not that simple, they come out in terms of some more complicated functions. But somehow we were still able to obtain something useful to discuss for the time period of the pendulum. What we saw was that the time period depends on energy or equivalently the initial conditions are the amplitude.

So, that is a signature of a non-linear system that time period is now dependent on the amplitude of the system or equivalently depends on the energy of the system and if you want to compare it, go back and look at the time period of a simple pendulum. It is independent of all these quantities it does not depend on energy, it does not depend on the amplitude it depends only on the length of the pendulum and the acceleration due to gravity.

So, clearly non-linearity is having major effect on oscillating systems. Now in this class we will go a little beyond, we saw simple pendulum which was not subjected to any dissipation not subjected to any external forcing. So, in this case let us try and apply a forcing.

So, as I said just to recall once again linear systems are those for which restoring force is proportional to displacement, non-linearity a non-linear systems are ones for which the restoring forces proportional to some non-linear function of displacement and forcing is typically non-linear, typically an oscillatory function, for the reason that even in the case that we had studied up until the 11th week, when we studied the forced oscillator several weeks back we took the forcing to be some $F_0 \sin \omega t$ or $F_0 \cos \omega t$.

So, it is a sinusoidal function. So, generally we take forcing to be a non-linear function; if you take it to be a linear function it is a little bit unphysical because suppose you take it to be $F_0 \times \omega t$. It is a linear in time all it says is that, the integrated force is going to increase with time which is a bit unphysical and cannot be sustained.

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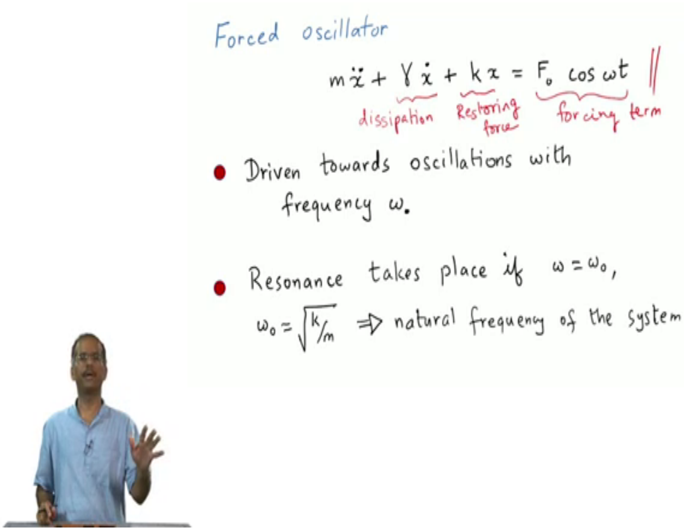

So, if you remember what we had studied in the forced oscillator several weeks back, the equation of motion is given here, $m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos \omega t$. So, this is the forcing term and this one is the dissipation term represents viscous dissipation and this of course, is the restoring force term.

In this case physically we know what happens, the forcing term forces your system at a with the frequency ω because you have a $\cos \omega t$ there. So, that is the frequency of external forcing. Now one of the possibilities that can happen is if somehow your external forcing frequency ω and natural frequency of the system which is ω_0 they coincide then resonance takes place, but the single most important lesson here is that if I am driving a system with frequency ω, the system is going to respond by oscillating with frequency ω.

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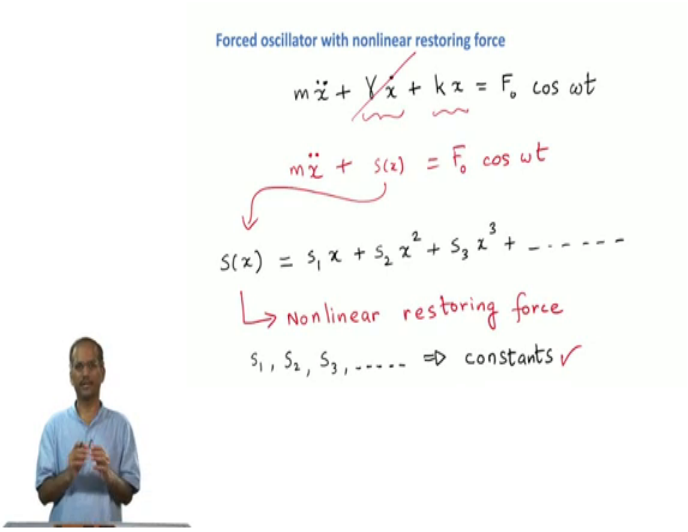

So, since we are looking at the effects of non-linearity here, the system would still be non-linear even if I do not take into account this term which is the dissipation term. So, even without that the system is sufficiently non-linear provided the restoring force is non-linear. So, the non-linearity really comes from the restoring force term. So, to simplify our work since we are really interested in finding the effects of non-linearity, right now we can actually set $\gamma = 0$.

So, I will take this term out of consideration. So, which means that now I take this linear system and it has this restoring force which is kx and replace it by a non-linear restoring force. So, which means that I will have to replace this kx by something like this $m\ddot{x} + S(x) = F_0 \cos \omega t$, where this $S(x)$ is a non-linear restoring force. So, in general it could be an infinite series with $S_1 x + S_2 x^2$ and so on and all these numbers S_1, S_2, S_3 and so on they are all constants.

So, in general you can have a problem as how does this it is a quintessential non-linear second order differential equation.

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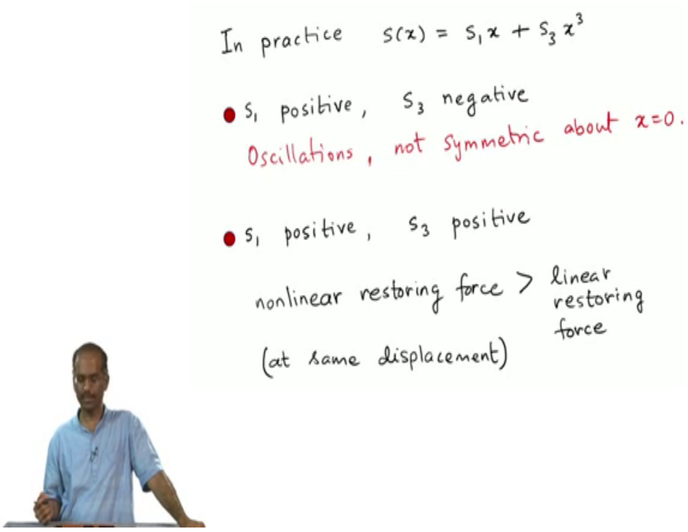

So, in practice this non-linear restoring force which I introduced really as an infinite series, in practice it is never really an infinite series, but in many cases you can make do with just two terms like this S_1x and S_3x^3. First let us consider the case when S_1 is positive and S_3 is negative. So, in this case you are restoring force is going to depend on x and since you have this x and x^3 is going to also depend on the sin x and x^3. So, in the positive side x and x^3 will be positive, on the negative displacement side it is going to be negative.

So, if I set S_1 to be positive and S_3 to be negative what is going to happen is that, you are going to have asymmetric oscillations about S_x equal to 0. Suppose I make this choice S_1 and S_3 both are positive. So, in this case what happens is that the non-linear restoring force is greater than the linear restoring force at any value of x are at any displacement to realize this we need to plot this function.

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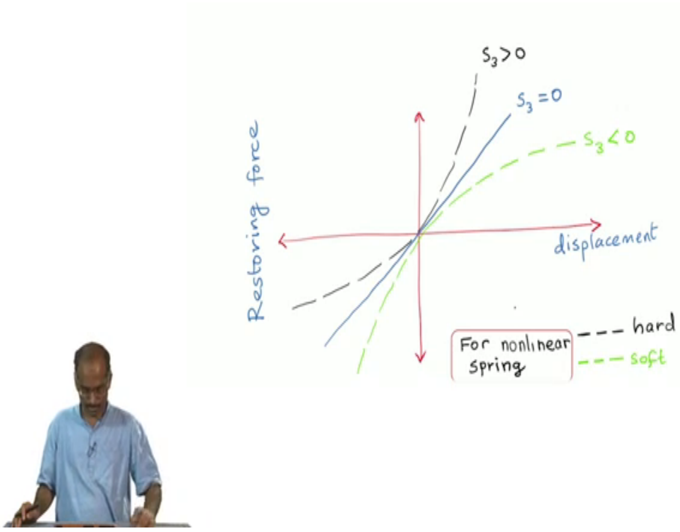

So, here I have plotted this non-linear restoring force S as a function of displacement x. So, you can see this blue curve here which corresponds to $S_3 = 0$. So, if I put $S_3 = 0$ that would simply mean that I am going back to linear regime. So, the restoring forces in our straight line linearly related to displacement, that is our benchmark a linear benchmark.

On the other hand when you have $S_3 > 0$ like this case for example, so, in this case the restoring force and displacement they are related by this curve which is shown as dashed black line. So, it curves upward. On the other hand if S_3 is negative less than 0 then you get this green curve. So, generally if you think of this as a system of mass and spring, the case where your $S_3 > 0$ is often called the hard spring and the case where $S_3 < 0$ is often called a soft spring.

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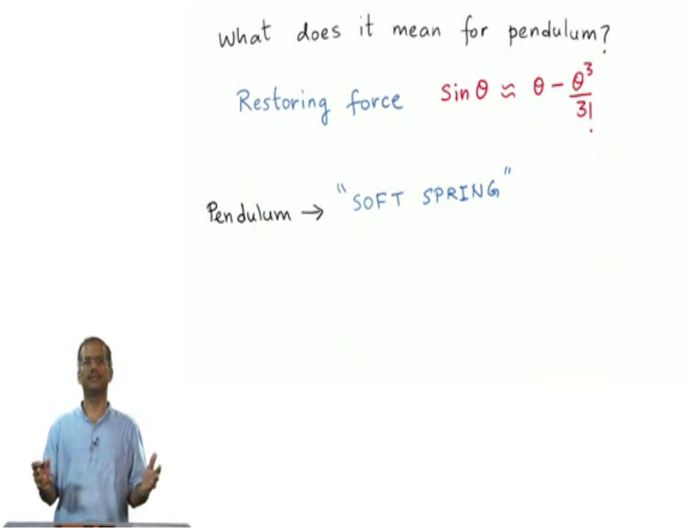

So, you can ask the question where does the pendulum case lie. So, in the case of pendulum, $\sin\theta$ is $\theta - \frac{\theta^3}{3!}$ and then there are higher order terms, but if you look at only these two terms, then this would correspond to the case when the pendulum can be thought of as a soft spring.

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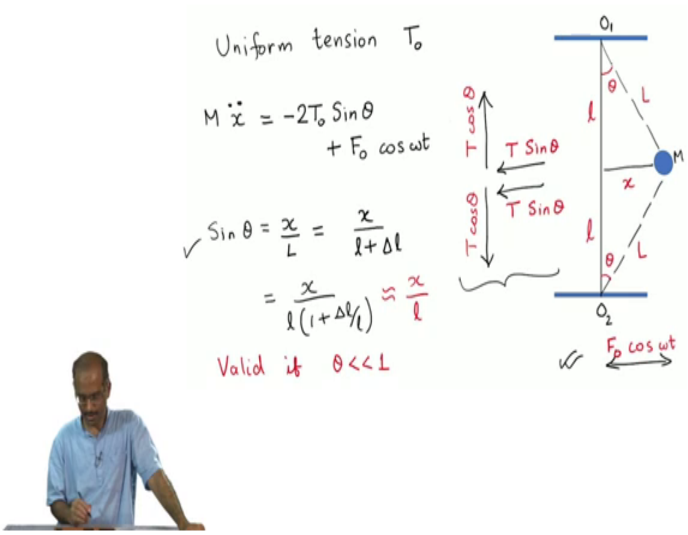

So, now with this background let us do one problem where there is forcing on the system and we are going to keep the non-linear terms in the restoring force. Several weeks back we looked at this problem of beaded string. So, you have a string and many beads in it and the simplest case there is when you have a string with simply one bead. So, we considered the oscillations of such a system and we even obtain the normal mode frequency for that case.

So, here I am turning that same system vertically like this. So, I have these two points $O1$ and $O2$ and there is a mass which is shown in that which is shown as blue circle here and it has mass m and it is held at the center of this string which connects $O1$ and $O2$. So, when it is not disturbed the string has length $2L$ and if you move it a little bit the string takes a new length capital L and more importantly what we are doing is, we are giving it a forcing along this direction and that forcing as shown here is $F_0 \cos \omega t$.

So, I am forcing it in this direction and x is the displacement, θ is the angular displacement from this vertical. So, initially I assume that there is uniform tension T_0 in that string and we can write the equation of motion. So, $m\ddot{x} = 2T_0 \sin \theta$. Since there is uniform tension T_0 in the string, there will be as I have shown here in this free body diagram, the vertical in the vertical directions there will be $T \cos \theta$ acting opposite direction to one another. So, that will keep the this bead in the same horizontal plane no motion in this direction.

So, this $T \cos \theta$ in this direction, $T \cos \theta$ in this direction would cancel out whereas, the component of tension in the horizontal direction they add up ok. So, you get $2T \sin \theta$. So, my equation of motion would be $m\ddot{x} + 2T_0 \sin \theta$. So, I am assuming that T_0 is the uniform tension in the string and when I pull it apart a little bit the T_0 does not change. Again these are the kind of assumption you make, when you work with the within the linear regime. So, that is equal to $F_0 \cos \omega t$.

Now, from the geometry of this figure you can see that $\sin \theta$ will be equal to $\frac{x}{L}$ and that capital L, I can write it as $l + \Delta l$. Typically when I write it like this this Δl is generally assumed to be much smaller than l. So, if I take l outside it can be written as. So, just look at the denominator, it can be written as $l(1+\frac{\Delta l}{l})$ as I said $\frac{\Delta l}{l}$ is very small. So, you just take $\sin \theta$ to be equal to $\frac{x}{l}$ and this kind of an argument is valid if θ is small.

So, that makes sense because if θ is very small; small l and capital L are nearly equal which is what we have essentially achieved. So, I plug this value of $\frac{x}{l}$ for $\sin\theta$ in the equation of motion and I have this equation of motion.

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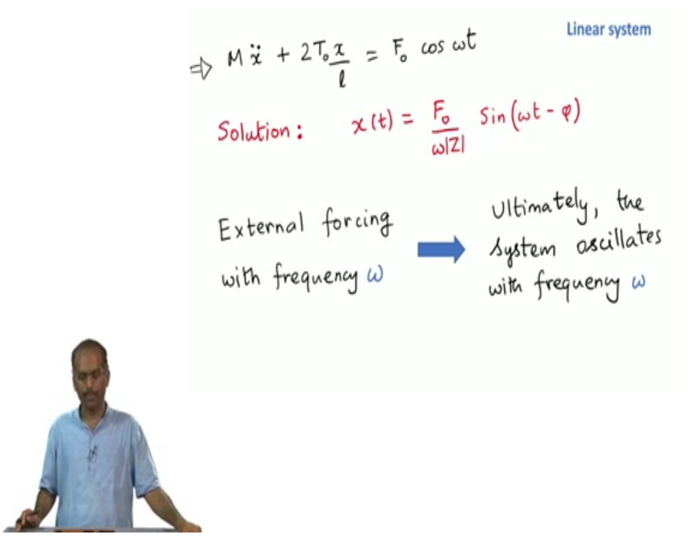

The solution is simply $\frac{F_0}{\omega|Z|}$. this capital Z is the impedance and it is multiplied $2\sin(\omega t - \phi)$. So, you can see that the displacement maintains a phase difference with respect to the forcing, but very importantly for us is that, displacement has a frequency ω which is same as the frequency of the forcing term ω. So, that is what right now we need and this is a signature of a linear system in some sense.

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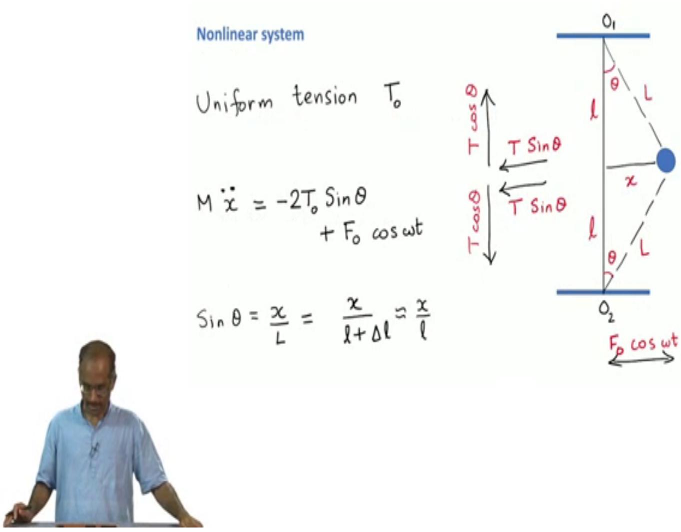

Now, let us get to a non-linear system. Now we look at the same problem, but not in the linear limit. So, I assume that the uniform tension in the string is T_0 which remains the same even now, we will see how to modify tension when the string is pulled a little bit apart and the equation of motion still remains the same.

But and $\sin\theta$ is equal to $\frac{x}{L}$ as usual, but now I can no more write this capital L as l plus small l, I cannot make the assumption that capital L is only a little bit different from this small l. In the linear system that is what we did and finally, we wrote it as $\frac{x}{l}$. So, both these are not possible, I need to keep l as it is capital L has to be kept as it is.

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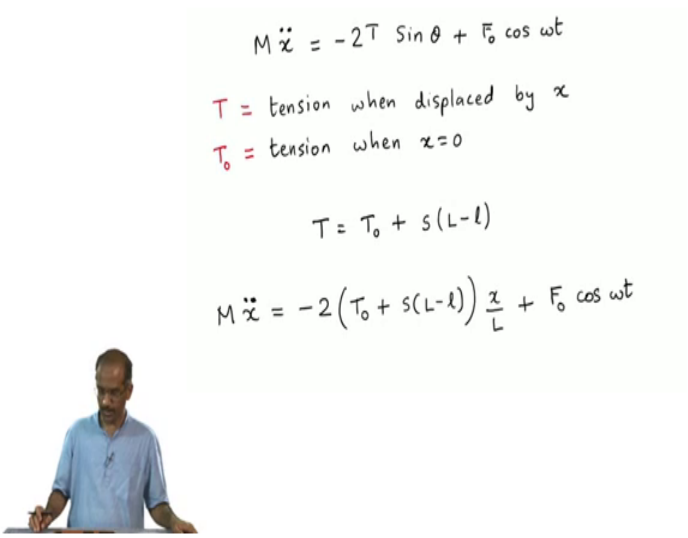

So, my equation of motion is this $M\ddot{x} = -2T\sin\theta + F_0\cos\omega t$. Now T is the tension when you displace the string by an amount x. So, the tension does not remain the same, when it is at rest position the tension is T_0 and when you pull it apart a little bit, the tension has changed and it is capital T now. You still assume that it is uniform tension, but the value of tension itself has changed.

Now, I can write an expression for this tension because what I call as s is simply the restoring force per unit length, since that is simply restoring force per unit length just like a stiffness constant. So, T will be $T_0 + s$ multiplied by the change in length. Change in length is capital L minus small l as you can see. Capital L is the new length which is slightly or substantially larger than the original length which is small l. So, this is going to be my expression for the tension when I move the bead a little bit apart in this direction.

So, now I substitute this value of T in this equation of motion and this is what I am going to get just a small rearrangement will give you this equation. Now I want an expression for L in terms of other parameters.

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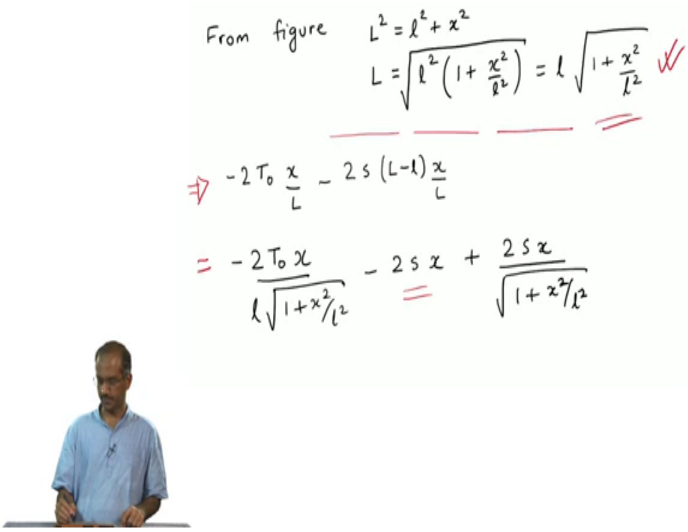

So, if you go back to this figure you will see that this triangle that I have, I can use that to write capital L in terms of small l and x. So, which is what I am going to do here. So, capital L^2 is equal to small $l^2 + x^2$. Take square root to get an expression for l you get this and finally, it boils down to this one. Now, let us get back to our equation of motion.

Here in particular I want to concentrate on this term here, this first term. So, we will assemble the whole equation back again a little later, but let us simplify this term by putting in this value of l that we got just now. So, here I have just substituted for l here. So, first term is $-2T_0x$ divided by this quantity and the next term is $-2Slx/l$ so, that is this and the last term is $2Sx$ divided by this quantity. So, I have already substituted for l here. So, I have done all these things.

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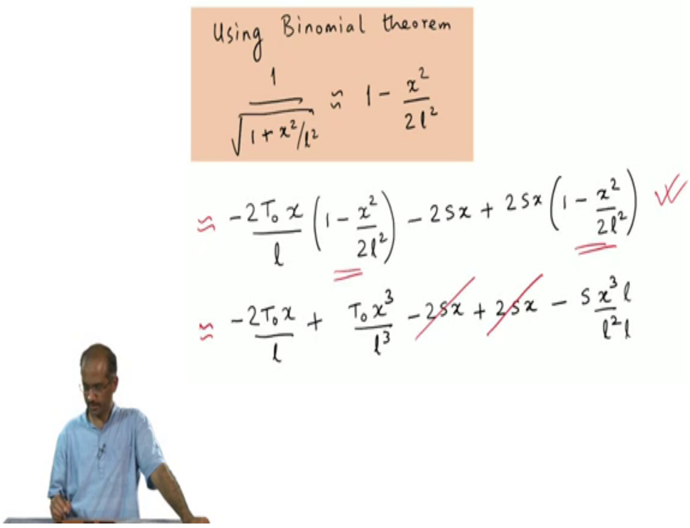

Now, you see that the denominator here, I have this square root. $\sqrt{1+\frac{x^2}{l^2}}$ here and here.

So, now, I am going to approximate this using a binomial theorem or a binomial expansion. So, if I take this and keep only the first term already that is non-linear because of x^2 term. So, that is 1 minus. So, $\frac{1}{\sqrt{1+\frac{x^2}{l^2}}} = 1-\frac{x^2}{2l^2}$. So, you see that I have substituted from the binomial theorem.

So, multiply everything and you will notice that there are two terms $+2Sx$ and $-2Sx$ both will cancel out and there are two terms with x^3 term here I will collect them together.

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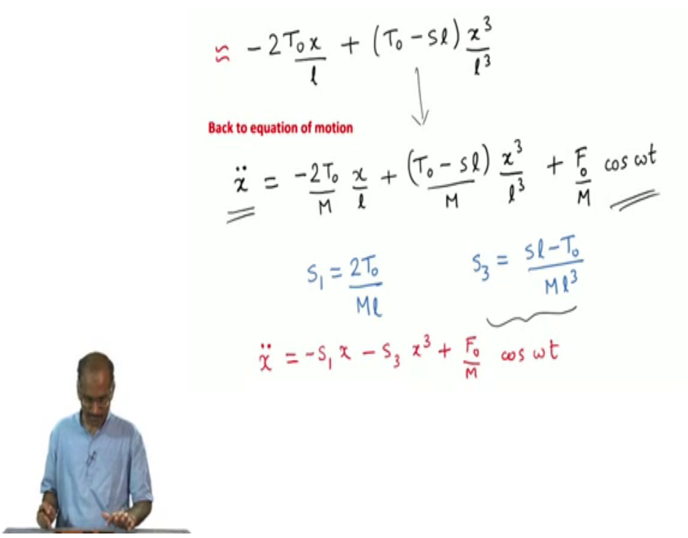

So, to do that I need to multiply this by l and divide this by l. So, if I do that I will have l^3 in the denominator, that is x^3 in the numerator just similar to this term. So, now, collecting all the terms and this is what I have it is simplified. Now, let us go back to our equation of motion. So, which means that I am going to go back to this and now I have obtained a very simple expression for this term, which I will use here, but two other terms which are this and this they will remain the same as before.

So, you can see that when I go back to the equation of motion this term remains as before, this is the forcing term remains the same as before and I have simply substituted this here. With this my equation will simplify to $\ddot{x} = -S_1 x - S_3 x^3 - \frac{F_0}{m}\cos\omega t$.

So, this is the simple equation of motion that I have for this system and you can see that it is non-linear. So, the non-linearity comes from this term.

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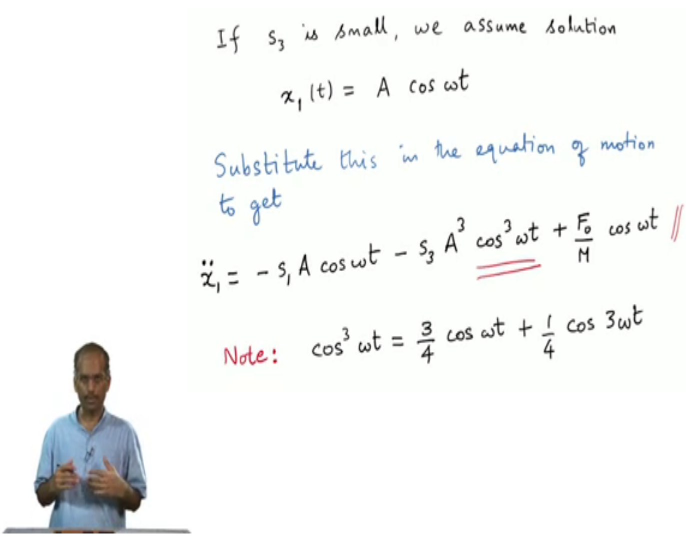

Now, if S_3 is small we can assume that the solution is of the type $A \cos \omega t$. So, A is again an amplitude we will have to determine later if you want the full solution, but will not do that here. So, now, if I substitute this in the equation of motion, I will get this following equation, all I need to do is to calculate starting from $x_1 = A \cos \omega t$ calculate $\dot{x}$ and $\ddot{x}$ and substitute it in this equation and that is going to give me this.

Now, having done that, you see that I get this term $\cos^3 \omega t$, the other two terms involve only $\cos \omega t$. So, I can actually put them all together, but this $\cos^3 \omega t$ I want to write it in a sort of linear way using ωt itself. So, just use the identity, trigonometric identity for for $\cos^3 \omega t$ that is $\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t$.

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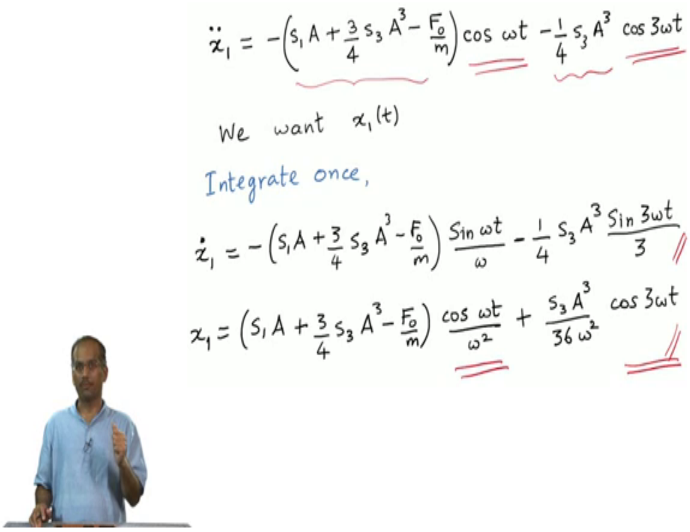

So, once you put that in and collect terms together you will see that that is one $\cos \omega t$ term and other is $\cos 3\omega t$ term and then rest are all constants, this entire thing this is a constant and this is also a constant. So, here the solution would mean that find $x_1(t)$, you cannot do that if you integrate this equation twice, it is very easy to do it.

So, when I integrate it first time, I will get this equation and of course, there will be a constant of motion and when I integrate it second time, I will get this equation of motion I have said those two constants of integral to 0 which means that it will correspond to some kind of initial conditions, where those two constants of integrals are 0.

Now, look at this solution that I have. So, there is the first term which has this $\cos \omega t$ and there is a second term which has $\cos 3\omega t$. So, if you remember there was no $\cos 3\omega t$ in our original equation of motion. So, the forcing term was forcing the oscillator at a frequency of ω not 3ω. So, somehow the non-linearity of the system has produced in a harmonic of the frequency of forcing.

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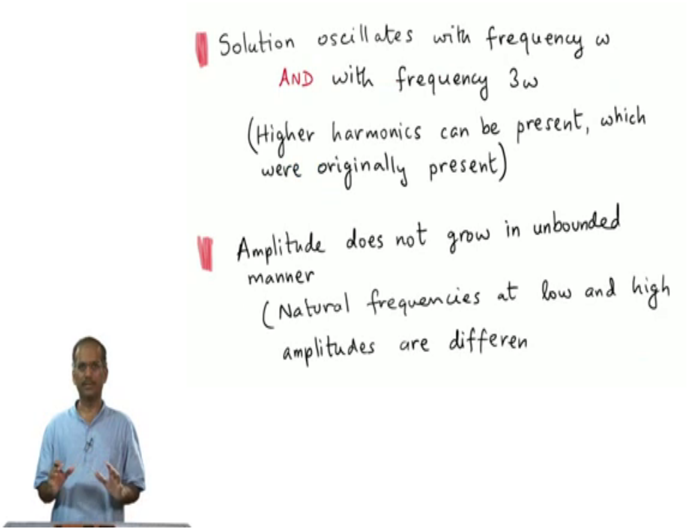

So, let us see what are the salient features of this solution. So, one is that the solution oscillates with frequency ω and also with frequency 3ω. That is unusual because if it is a linear system and you force it with frequency ω the solution also oscillates with frequency ω, it does not produce additional frequencies or additional harmonics, but somehow that is not true for a non-linear system.

And secondly, the amplitude does not grow in a unbounded manner. The reason for that is the natural frequency of the system which we identified as being one quantity in the case of linear system. So, remember that if you take any linear system there is something that you could call as your natural frequency of the system, which was not amplitude dependent. It is basically like calculating time period of a oscillator no amplitude dependence there.

On the other hand in a non-linear system, the natural frequencies are amplitude dependent. So, the net effect is that the natural frequencies at low and high amplitudes they are very different. So, what would this do? Even if you did not have dissipative term present in your equation of motion, the amplitude is not going to grow in an unbounded manner simply because there will never be a proper matching with the natural frequency of the system.

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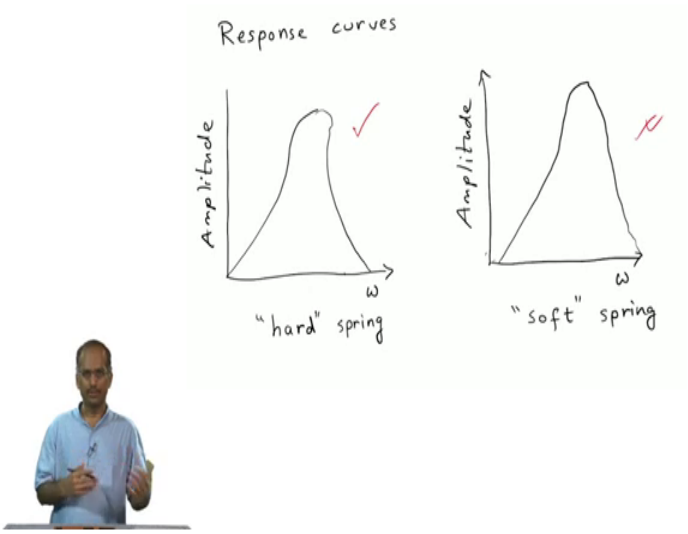

In the case when S_3 is positive to remember this term here in the solution, you will get the response curve to be something like this. So, in particular this is what you would call the hard spring limit and if S_3 is negative, you will end up with the soft spring limit this is the response curve. So, again to emphasize the point is that you are not going to have resonances at least not that easily in non-linear system.

And in fact, some very unusual things can happen, for instance there are because this response curve is a bit unusual you can have jumps in the curve because this function is not quite single value for a given value of ω you can have two different amplitudes. So, there can be fast jumps between the two amplitudes.

So, all these are essential effects of non-linearity. So, in the in these two classes taken together we have seen the effect of non-linearity in a pendulum which was not driven, no dissipation and in this class what we have seen is the effects of non-linearity in the case of a forced pendulum. We did not really quite worry about putting in dissipation which you can do, but it is not going to alter pretty much in most of these results.