Waves and Oscillations
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Lecture – 54
Beyond Linear Oscillators: Non-linear Pendulum

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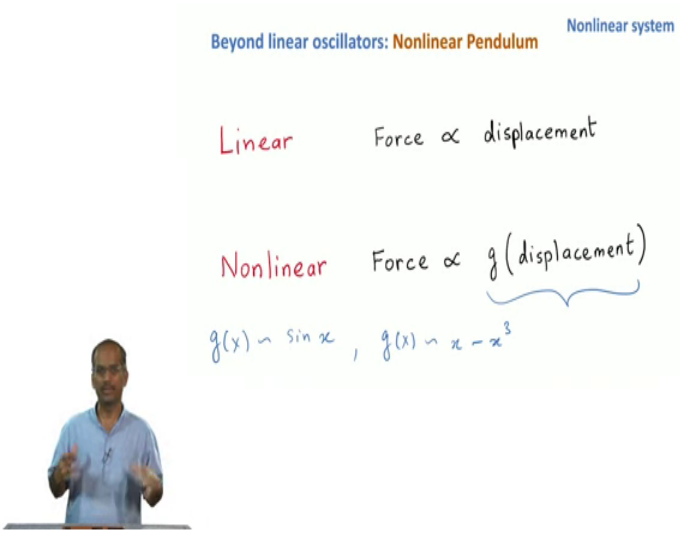

Welcome to the 12th week of this course on Waves and Oscillations, this is the last week and congratulations to all of you who have been following this course without a break until this point. So, this is a point to take stock of what we have done all these 11 weeks, but that I will do in my last class of this course.

But in this class I am going to look at what is it that we can do beyond what we have studied that is the broad question. In the sense that pretty much everything that we have done throughout this course has to do with an important assumption that the oscillations are small enough. So, essentially you are sticking to linear systems much like the harmonic oscillator that we began the course with.

Now, the broad question is what happens if you go beyond linearity, beyond linear oscillators what is it that we can do? So, much of this is at some level for information purpose in the sense that this will not be taken into account for the exam. I wish that you take time off from say exam pressure in a sense and look at the new physics that comes out when you leave linearity and get into non-linear systems.

So, as it is stated here in the first slide for all linear systems everything that we had seen force or the restoring force more correctly is proportional to displacement. So, when you want oscillations you are restoring forces proportional to displacement and you had a minus sign sitting in front of it. So, that provided the necessary ingredient for oscillatory behaviour.

So, that is our template for linear systems. What happens with non-linear systems? So, in this case the restoring forces some function of displacement like this. So, here I have indicated it as g of displacement meaning that it is some function g which is a function of displacement and in general this function could be a fairly non-linear function. So, all these are non-linear functions.

So, there is infinite variety of non-linear function that you can put in there and ask for how does the dynamics work out in such a case. But we will restrict ourselves to physically interesting situations and since we are going to discuss it pretty much in the fair end of this course. We will also restrict it to one or two simple cases, but otherwise there is a huge variety of possibilities once you leave the linear regime and get into non-linear systems.

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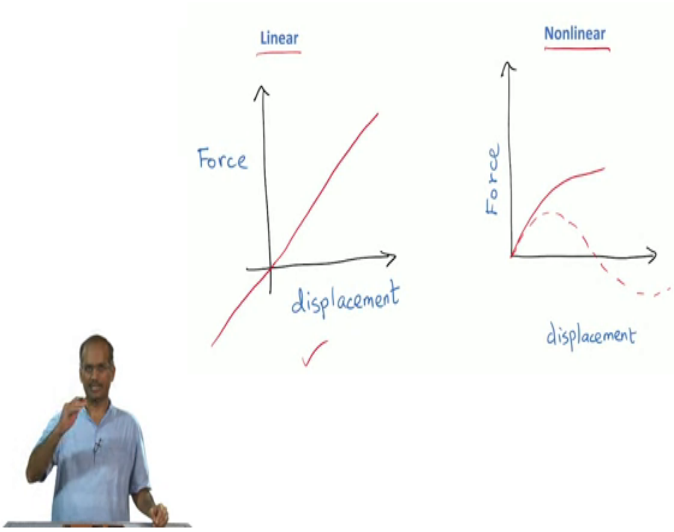

So here I have pictorially depicted what is linear and non-linear. So for example in the case of linear as I said the restoring force is proportional to displacement. So, there is a straight line relation between force and displacement. On the other hand if you look at the case of non-linear systems in general. So, the restoring force is no more linearly related to displacement like I have shown here. So, here I have shown two different non-linear functions one that seems to rise and appears to saturate the continuous red curve and the other that seems to oscillate the dashed red curve.

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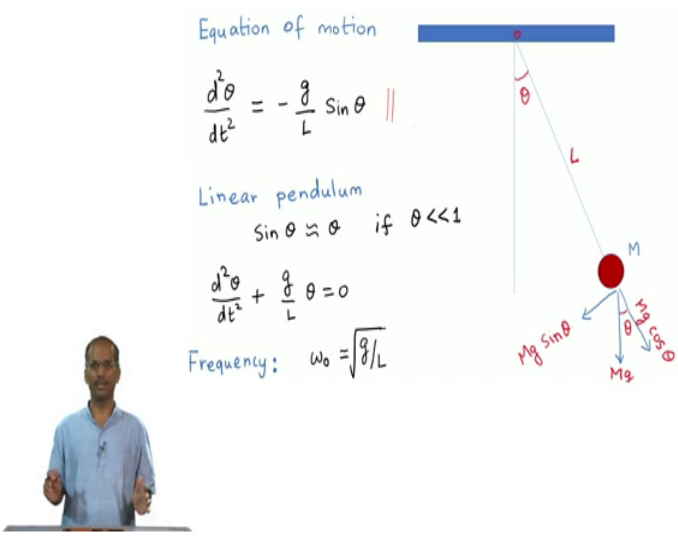

So, a simple prototypical example of non-linear system would be the case of pendulum ok, you might tell me I have already studied the pendulum. But if you remember quite well we studied the pendulum in the linear approximation, when the $\sin\theta$ was approximately equal to θ.

So, here I have the set up for the pendulum. So, there is a bob of mass M which is hanging from a tout string with uniform tension in it, the length of the string is capital L. And so again to write down the equations of motion you just need to look at the free body diagram which is shown here at the bottom of the bob of mass M. So now, when I finally, write the equation of motion here is what I have, $\frac{\partial^2\theta}{\partial t^2} = -\frac{g}{L}$, g is the acceleration due to gravity L is the length of the string so it is $-\frac{g}{L}\sin\theta$.

Now, when we did the same problem in the regime of linear approximation, we converted this what is essentially a non-linear problem into a linear problem by doing this. By saying that $\sin\theta$ is approximately equal to θ provided θ is much less than 1. So, this is one way of stating that θ is small, but typically angle of about few degrees like 3, 4, 5 degrees would still give you a reasonable approximation. But if you take your let us say you have the bob of your pendulum and you leave it here the angle that you have made is really large, close to 90 degrees in which case this approximation would not work at all.

Suppose we go ahead with this approximation then you could write the equation of motion, the way it is done here and immediately identify what is the frequency of motion and from the frequency of motion it is possible to directly write the time period. So, the frequency of motion in this case is $\sqrt{\frac{g}{L}}$ and once you realize that ω_0 is $\frac{2\pi}{T}$, where T is the time period you can write an expression for time period of oscillation of the pendulum.

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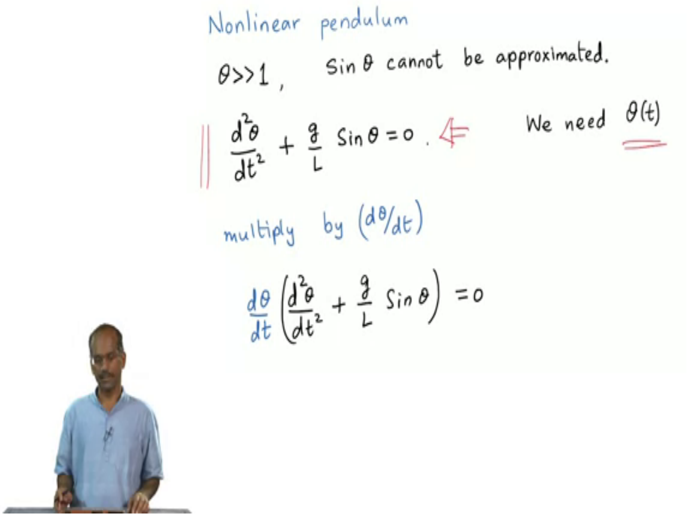

Now, I want to work with non-linear pendulum in the sense that I do not want to make this simplifying assumption that $\sin\theta$ can be approximated by θ. So, here angle is arbitrarily large that you cannot make that approximation. Hence I need to work with this full equation of motion and as usual the solution that I want is $\theta(t)$ we will see if we can even get to that, but that ideally is what I want to obtain. But more importantly we want to find out the time period even that is going to tell us quite a bit about the nature of the system itself.

So, let us keep in mind that this ω_0 is $\sqrt{\frac{g}{L}}$, in particular keep in mind that the time period is also going to be a function of g and L and both are constants. So, now to solve this what I need to do is to go to this equation of motion, multiply throughout by $\frac{d\theta}{dt}$. So, that is what I have done in the next line here. So, simply multiply the equation of motion by $\frac{d\theta}{dt}$.

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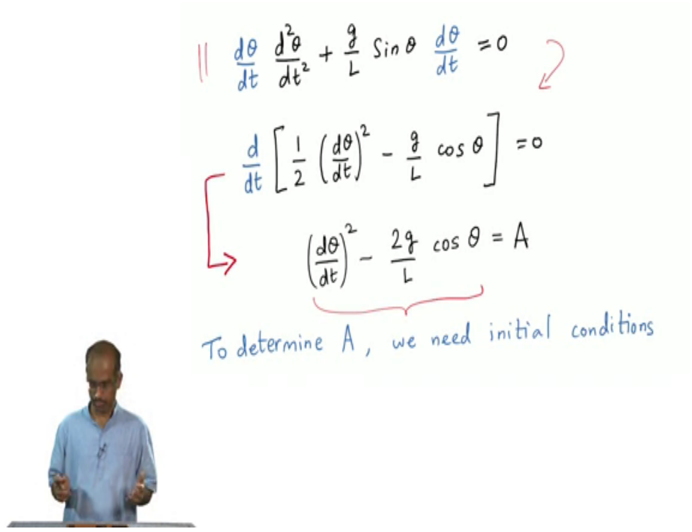

If I do that this is what I am going to get, so you can see that there is a $\frac{d\theta}{dt}$ term in both the terms of this equation. Now, if you look at it closely you can easily realize that it can be rewritten in the following way. So, I can take $\frac{d}{dt}$ outside and put together it put rest of the terms together like this. So for example, take this term the first one half $\frac{d\theta}{dt}$ whole square which I have here. Now, if I apply this operator $\frac{d}{dt}$ on this, it will give me $2\frac{d\theta}{dt}$ into $\frac{d^2\theta}{dt^2}$.

So, 2 and 2 will cancel and I will be left with this. So, this seems to be correct and you can also check that the second term also gives what we expect. So, I have $\frac{d}{dt}$ of some quantity is equal to 0 that would imply that the quantity inside the bracket it should be equal to a constant, only then $\frac{d}{dt}$ of that constant will be equal to 0. So, I have identified one constant of motion, remember that this θ itself is a function of time and $\frac{d\theta}{dt}$ is also a function of time. But this combination here is independent of time that is a constant. Now, to determine this constant we need to get the initial conditions.

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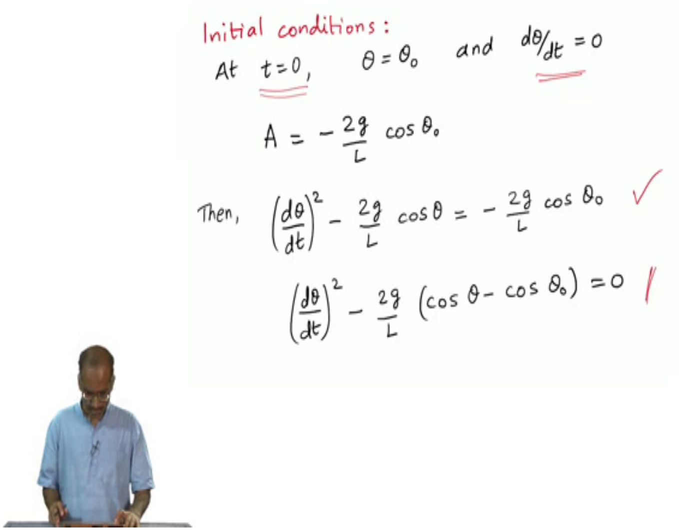

So, I have my initial conditions as follows at $t = 0$. So, you identify some arbitrary time as your 0 th time and say that the initial angle is θ_0 and the initial angular velocity or $\frac{d\theta}{dt} = 0$. Now, put these two conditions back in this equation. So, $\frac{d\theta}{dt}$ is 0 which means that this term will go to 0 and $\frac{2g}{L}\cos\theta$ will become theta 0. So, I am going to have A to be equal to $-\frac{2g}{L}\cos\theta_0$.

So, now I can put this value of a back in my previous equation which is this. So, if I do that this is what I am going to get and then a slight rearrangement will give you this equation, normally to be able to find $\theta(t)$ I need to integrate it.

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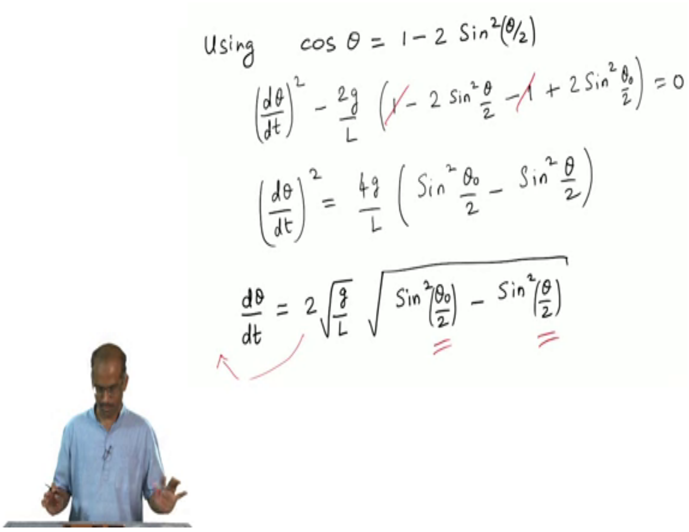

So, let us go towards that goal even though it is not a easy task, but before that this is what I will get if I use this trigonometric identity. Then finally, rewriting everything I will have $\left(\frac{d\theta}{dt}\right)^2$ whole square will be equal to $\frac{4g}{L}$ and rest of it would be simply $\left(\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}\right)$. And simply take the square root on the right hand side you will get an expression for $\frac{d\theta}{dt}$ and again now rest of the work is you need to integrate it.

You will notice that on the right hand side the variables are of the form $\frac{\theta}{2}$ and θ is $\frac{\theta_0}{2}$. Now, to make it common I can bring this 2 here and make the variable here also $\frac{\theta}{2}$ if I do that and bring all the terms involving θ on one side this is what I will get.

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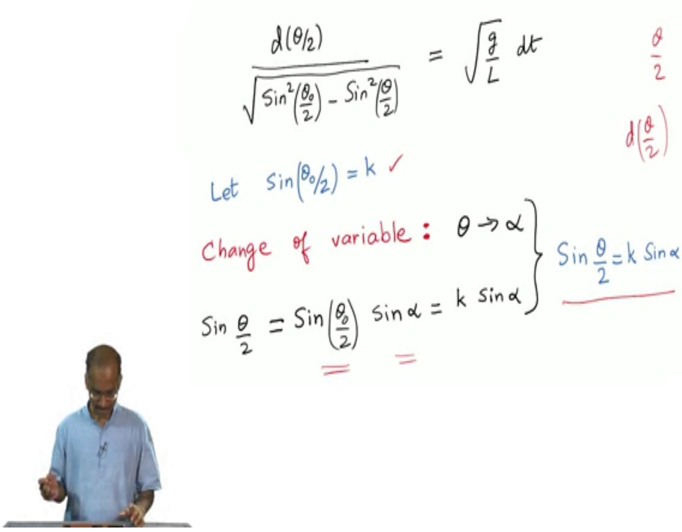

Now, you see that the variable of integration is $\frac{\theta}{2}$. So, before that let me identify some constants. So, I am going to designate this $\sin\frac{\theta_0}{2}$ as some constant k after all θ_0 is a constant. So, $\sin\frac{\theta_0}{2}$ will be a constant.

And again for ease of analysis I am going to do this change of variable go from θ to a new variable which I call α and the relation is this that $\sin\frac{\theta}{2} = k\sin\alpha$. Now, since I have made a change of variable from θ to α I need an expression for $d\left(\frac{\theta}{2}\right)$. dt will not change. But $d\left(\frac{\theta}{2}\right)$ I need to replace it by $d\alpha$. So, I need to find out how $d\left(\frac{\theta}{2}\right)$is related to $d\alpha$ and here is how we do that.

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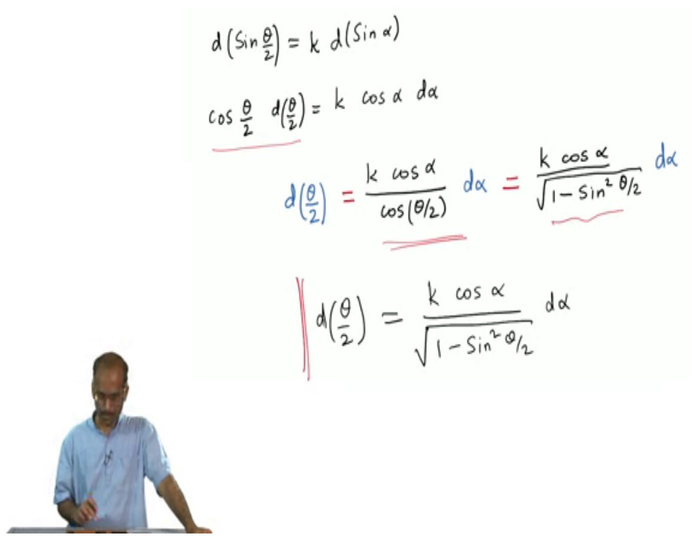

Now, we are going to use this relation substitute it here and integrate. So, I take this the right hand side of sorry the left hand side of this equation here and substitute for $d\left(\frac{\theta}{2}\right)$.

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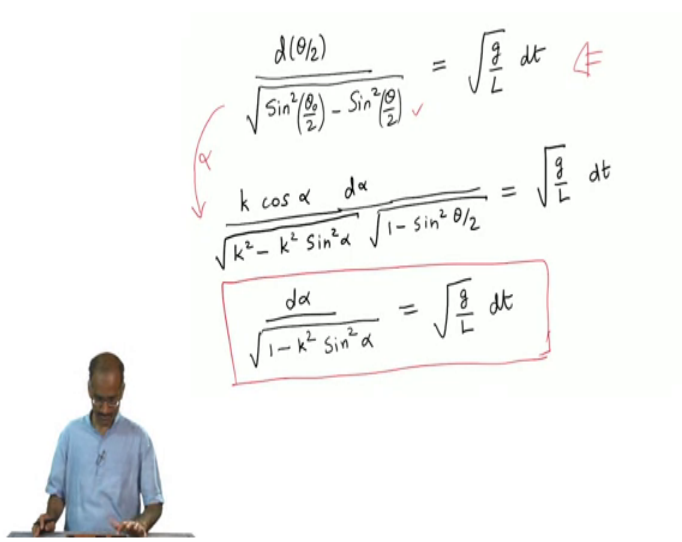

So, I have this full equation written here as it is, now I replace everything by α. So, after you do all these manipulations finally you get this relation. Now, this can be integrated.

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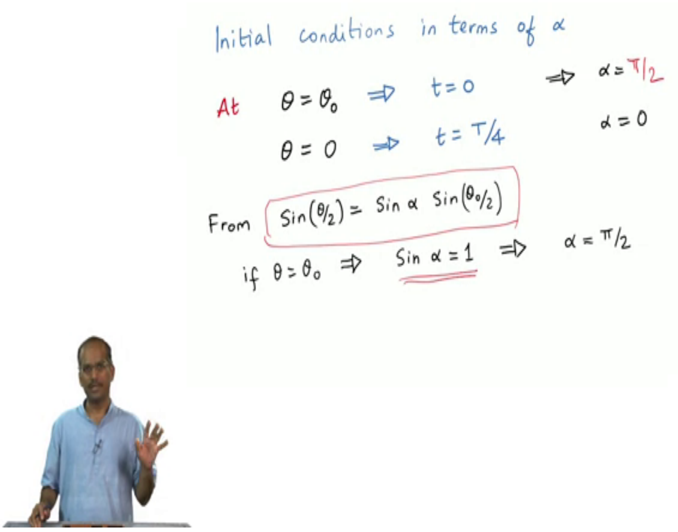

Now, even though I want to find solution you will realize that it is not an easy equation to integrate and solve, so we will first settle for finding the time period. So, I knew the initial conditions in terms of θ which is how the problem was specified. Now, we will write the same initial condition in terms of α, we said that at $t = 0$ at initial time θ is equal to θ_0. So, that corresponds to some angle of the pendulum and if $\theta = \theta_0$, so you note that we have this relation here which we just discussed two slides back.

Here if I take θ to be θ_0 then I have a sin $\frac{\theta_0}{2}$ on the left hand side and another sin $\frac{\theta_0}{2}$ on the right hand side. They will cancel one another and I will be left with the condition that $\sin\alpha$ should be equal to 1, this implies that α has to be $\frac{\pi}{2}$. So, here the α should be equal to $\frac{\pi}{2}$. So, the correct initial condition written in terms of α is that had $t = 0$, α is $\frac{\pi}{2}$, this corresponds to the case when θ is equal to some arbitrary value θ_0.

Now, what happens if $\theta = 0$ at $t = \frac{T}{4}$, $\theta = 0$. So, I am considering a case where there was this initial angle θ_0 that is the amplitude and it comes to this point that is when $\theta = 0$. At this point the time it has taken is one fourth of the time period. Because it starts from here it goes here goes back by the time it comes here that is one full time period and it has basically moved one fourth of the total distance to be covered and it has taken one-fourth of the total time. So, in this case at $t = \frac{T}{4}$ $\alpha = 0$, so that gives us the limits for integral.

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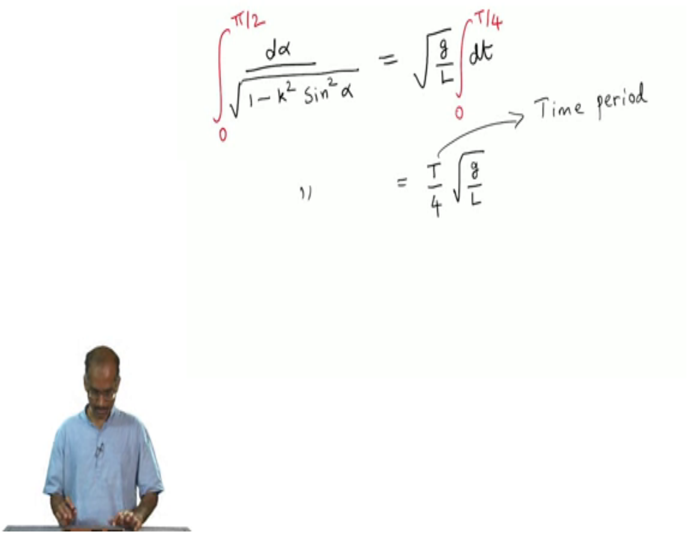

So, t will go from 0 to $\frac{T}{4}$ and at the same time alpha will go from 0 to $\frac{\pi}{2}$ this is the integral that needs to be done. The right hand side of this equation is very easy to do this will simply be $\frac{T}{4}\sqrt{\frac{g}{L}}$. I have everything to write an expression for time period in terms of this integral.

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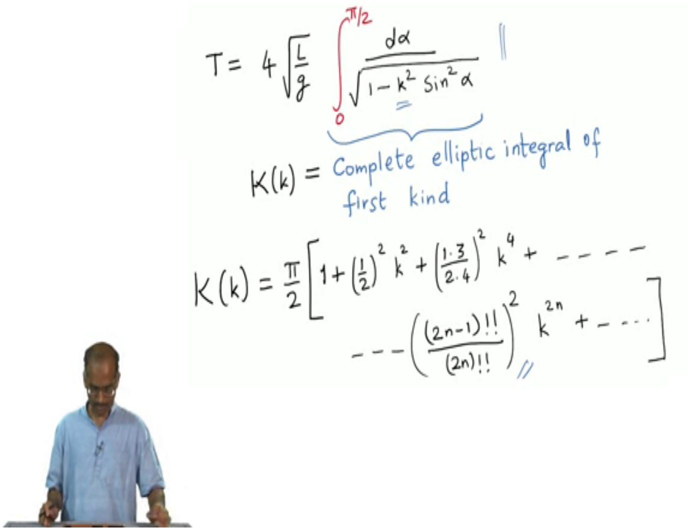

So, here is what I have now for the time period, unfortunately this integral cannot be written in terms of simple functions. So, we give it a name this function is called the Complete elliptic integral of first kind. So, this function which is which has a standard symbol capital K is a function of this value which is small k. So, given a given a value a small k the integral can be calculated.

As I said it is called the complete elliptic integral of first kind. But what is useful for us is the fact that actually there is a infinite series expansion for this integral, it goes like this I have written that we will take this infinite series expansion and put it in the place of this integral. And you will notice that this expansion has this double factorial notation $(2n-1)!!$ and $2n$ plus $2n$ double factorial.

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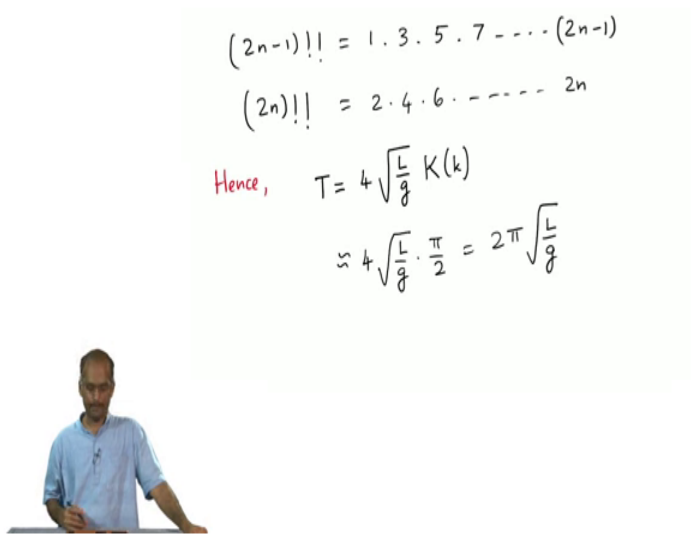

So, this is what a double factorial notation means. So, in general if you see something like $(2n-1)!!$ where n itself is an integer, whatever be n, $2n-1$ is always going to be an odd integer. So, it basically means the following that $(2n-1)!!$ is equal to 1 into 3 into 5 into 7 and so on until 2 n minus 1.

So, it is actually a product of all the odd numbers until $2n-1$ and similarly $2n!!$. So, n is an integer $2n$ will always be an even number. So, $2n!!$ is a product of all the even numbers, so it starts from 2 4 6 8 and so on until $2n$.

If I keep only the first term of this expansion, so this is an infinite series expansion for the elliptic integral. If I keep only the first term which corresponds to saying that all the k s are 0 for me, so I will cut down every term here which is a function of k, so in that case $K(k)$ is simply equal to $\frac{\pi}{2}$. So, I just put in that term $\frac{\pi}{2}$ here and you will see that it gives me the time period of the standard pendulum or the standard linear pendulum or the time period of the harmonic oscillator limit of the pendulum.

So, what we see is that in the limit where you do not include the non-linear effects by neglecting all these values of k in this expansion, you are able to get the correct limiting behaviour corresponding to the harmonic oscillator limit. And as you keep adding higher order terms in k you are going to take into account more and more of non-linearity and hopefully your answer would be closer to the exact result. So, to see how close it will be here is a picture that I have calculated.

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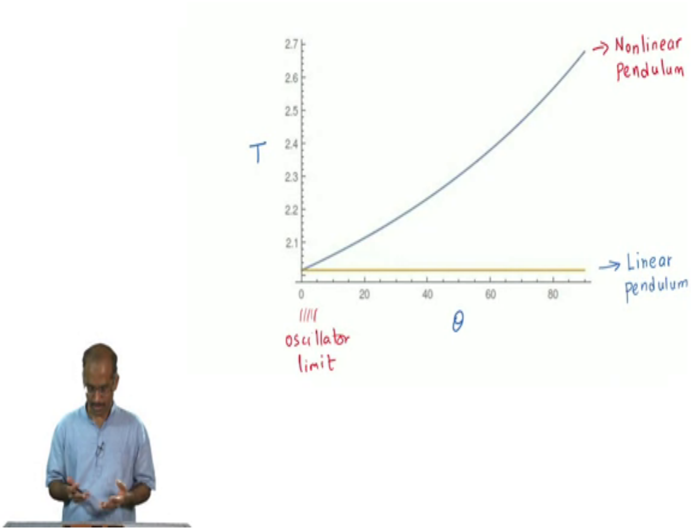

So, what is plotted on y axis is the time period and the x axis is θ, what we know is for small values of θ it should behave like a simple pendulum and that is the harmonic oscillator limit. As this θ increases more and more it is equivalent to saying that I am pulling my pendulum to a very large amplitude and then leaving it. So, in that case the small θ approximation breaks down. So, this there should not be any agreement between the time period that you calculated in the oscillator limit and the time period of the full non-linear pendulum.

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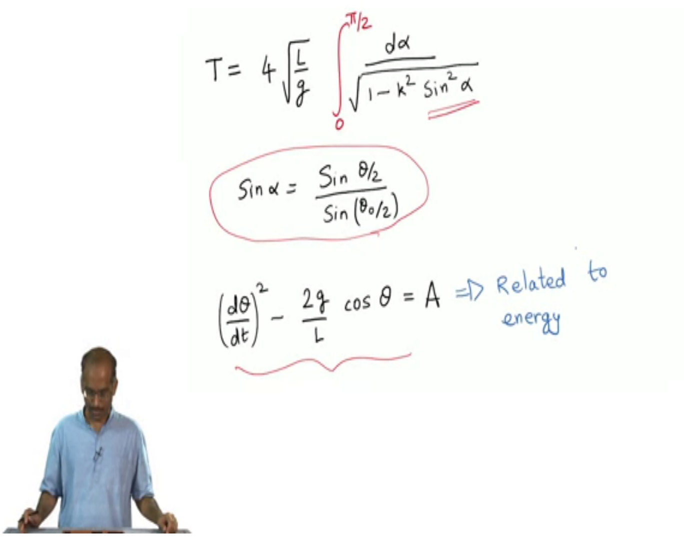

So, what you see here is precisely that let us see why is there something in what we derived which will tell us that the system is non-linear. So, I have this expression for T that we just derived time period is $4\sqrt{\frac{L}{g}}$ and then you have this elliptic integral of first kind.

So, you see this term $\sin\alpha$ here inside the integral you are integrating over α, but you have this term $\sin^2\alpha$. And from what we studied earlier on you can write this relation, $\sin\alpha$ depends on θ_0 which is that initial amplitude initial angle with which by which you move the pendulum and leave it to set it to oscillation.

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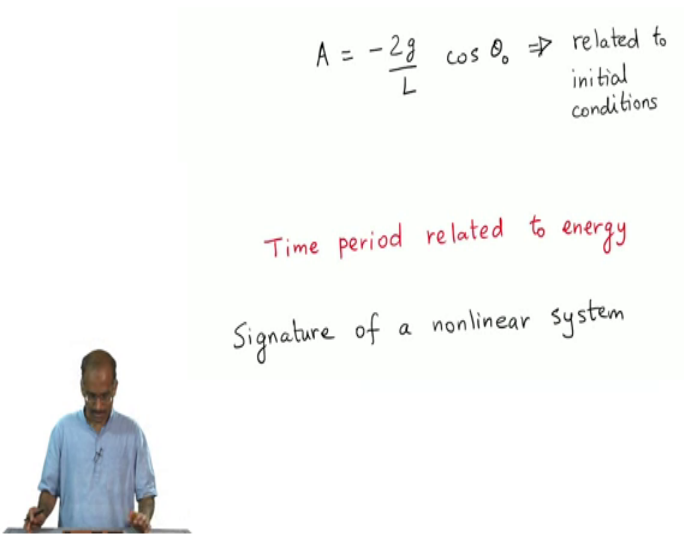

So, which means that the time period depends on this θ_0 your initial condition and this quantity A here is related to energy. So, which means that A if I at $t = 0$ would correspond to $-\frac{2g}{L}\cos\theta_0$. So, energy is also related to the initial conditions, so here is the point that I am trying to drive. That this time period that we have is a function of energy of the system, it is also a function of the initial conditions. So, that is the signature of a non-linear system. So, if you compare with what we had for the simple pendulum that the time period is dependent only on $\frac{g}{L}$ both are constants, g is the acceleration due to gravity, L is the length of the pendulum.

So, time period depends only on these two quantities not on initial condition it is in particular independent of energy. So, the simple linear pendulum time period is independent of energy. But when you analyze a non-linear system it turns out that time period depends on energy or in other words it also depends on initial conditions. So, this is the signature of a non-linear system. So, it is something that is a common attribute for all non-linear systems.