Waves and Oscillations
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Lecture – 53
Diffraction

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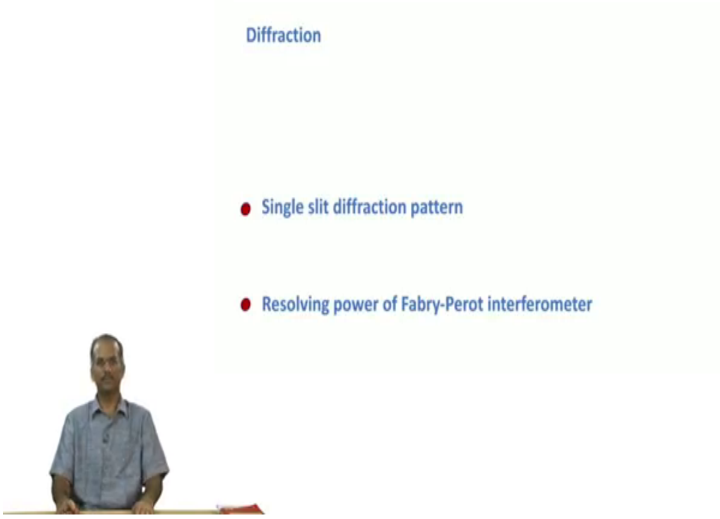

Welcome to the 5th module of 11th week, this entire week we are looking at Waves in optical systems. In particular in the last four classes we mostly concentrated on interference patterns. In this class we look at Diffraction and there is one problem that we had left incomplete in one of our previous classes, especially the one of calculating the resolving power of Fabry-Perot interferometer.

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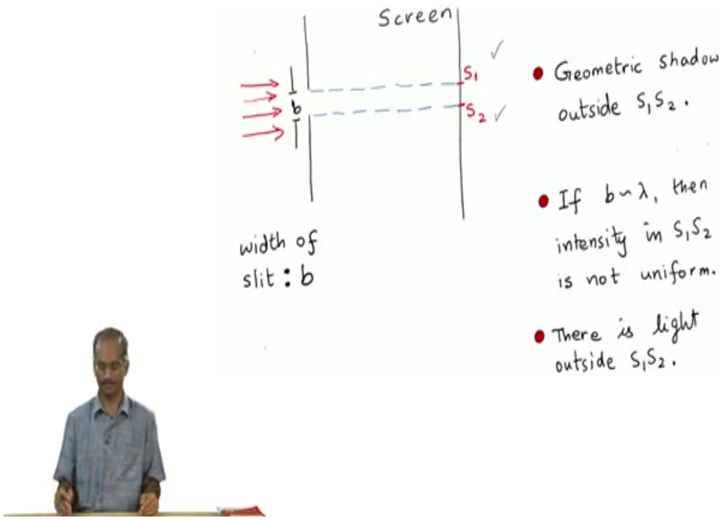

So, let me explain diffraction with the figure that I have here. So, let us assume that there is a light wave front that is coming from the left hand side and it is a plane wave front and it is falling on the slit whose width is b. If light were exactly travelling in straight lines what you would expect is that if I map the slit to a screen a little bit away from the position of the slit, I can mark these two positions S_1 and S_2 as you will see here S_1 and S_2.

So, if light were actually travelling in straight lines only that portion between S_1 and S_2 would be uniformly lit up and outside of S_1 and S_2 the screen would be completely dark, this is what you would expect. On the other hand what is generally seen is that under some conditions that is in fact light falling outside of S_1 and S_2 that is one thing. And secondly, if your wavelength of light is approximately equal to the size of the slit which we have taken to be b, then it is also noticed that the intensity of light which we expect it to be uniform earlier on, it is not really uniform within S_1 and S_2.

So, there are some bright and dark fringes it is not uniformly distributed inside S_1 and S_2 and diffraction effect cannot be explained with ray approach alone. So, if you remember we used ray approach to explain reflection and refraction and that is basically the limit of a wavelength of light tending to 0 and that approach would not be suitable here because we need to work with again waves interfering with one another. So, we really need the full wave phenomena. So, we need to consider nonzero wavelength of light.

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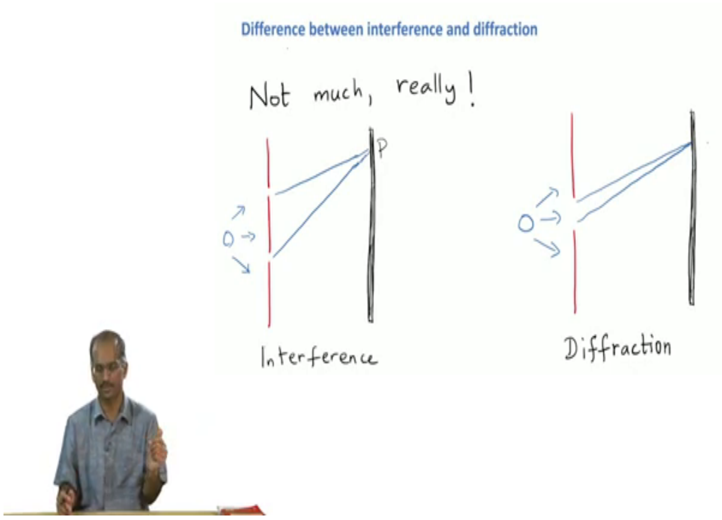

What is the difference between interference and diffraction? So, I have tried to show that with these two figures here, one common way of looking at interference is the two slit inks experiment kind of interference pattern. So, on the left hand side I have this double slit interference setup. So, you have light coming from the left hand side there is a wave front and there are two slits as you can see and the wave front part of the wave front passes through the two slits and maybe at some point P at the screen we want to know what is the intensity of light that is received at point P on the screen.

So, the way one would work is you take into account the light ray coming from one of the top slit and the bottom slit. Find out what is the path difference convert it into a phase difference and then you can decide whether what you are going to see would be a bright or a dark fringe at position P, this is interference for us. So, there are these two beams from different slits, two different slits it is very important.

So, there are these two beams coming from two different slits which interact with one another and produce the interference pattern at P. On the other hand you see this diffraction setup on the right hand side, I have this single slit setup. So, again it is the same thing light comes from the left hand side you can assume if you like that it is a plane wave the light passes through the single slit and again the question is what would I see it some position P and obviously, this position P is exaggerated.

So, ideally I would like to look at what happens maybe in this region. But in any case the point here is that here we are again looking at two wavelets or two beams which come not from 2 different slits, but from different parts of the same slit. So, look at the diagram that I have drawn here.

So, there are these two lines which I emerge from two parts of the same slit. Then we ask the question if these are the two beams how would they interact and would they produce a bright or a dark fringe at position P ? So, that is diffraction. So, if you think about it, it is not so much of a difference between interference and diffraction.

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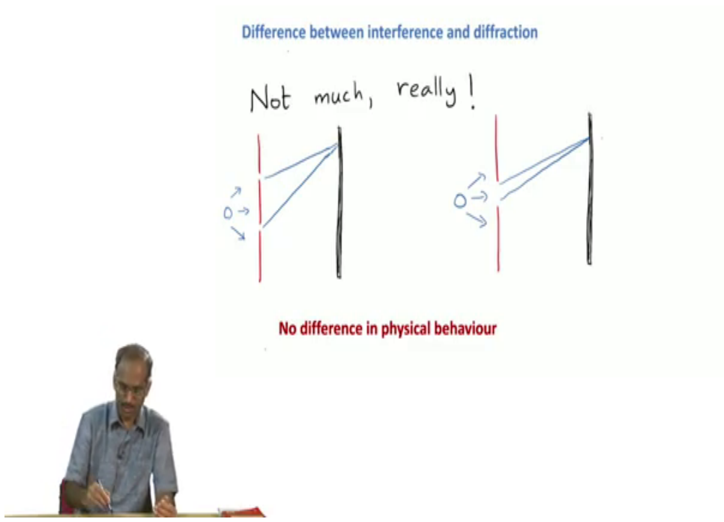

In particular you would notice that there is really no difference in physical behaviour.

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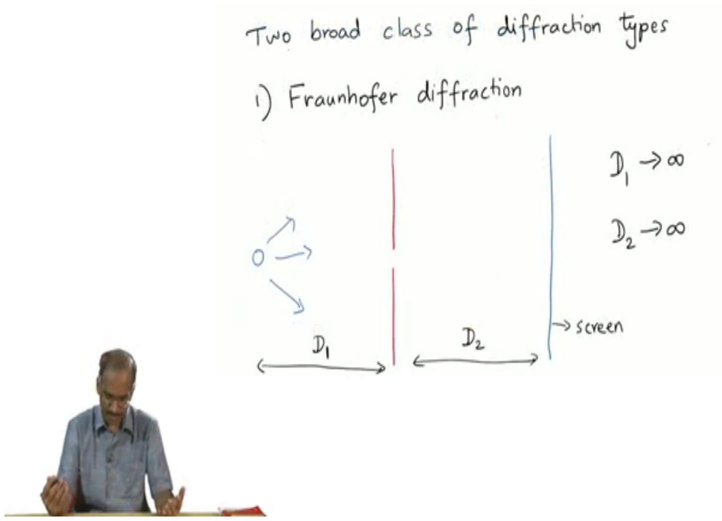

Let me say that there are two broad classes of diffraction problems. So, the first of them is the Fraunhofer diffraction. So, in this case let us say that diffraction happens at this position of single slit which is shown in red colour here this one.

Now, you have a source of light which comes really from far distance in principle infinity, but you should imagine that the source of light is really far from us that I can almost assume that when this light reaches the slit it is a plane wave. But you do not necessarily need long distance to produce a plane wave you could also have this effect happening at very short distance by using a lens for example, and if you place your object at the focus of the lens you will render it in parallel on the other side. So, for example, you can keep the source of light at the focus of the lens and that would give you parallel rays or essentially plane wave front.

And now, once the waves pass out through the slit the screen is again assumed to be really far enough, in principle at a distance which is infinity, but in practice it is fairly large distance. And, the reason as you will see is that these help in making some angles small and certain approximations work very well and help us in computing analytical forms for the diffraction pattern easily which is one reason. But more important than just the mathematical convenience is the fact that this is something that is also most useful in practice.

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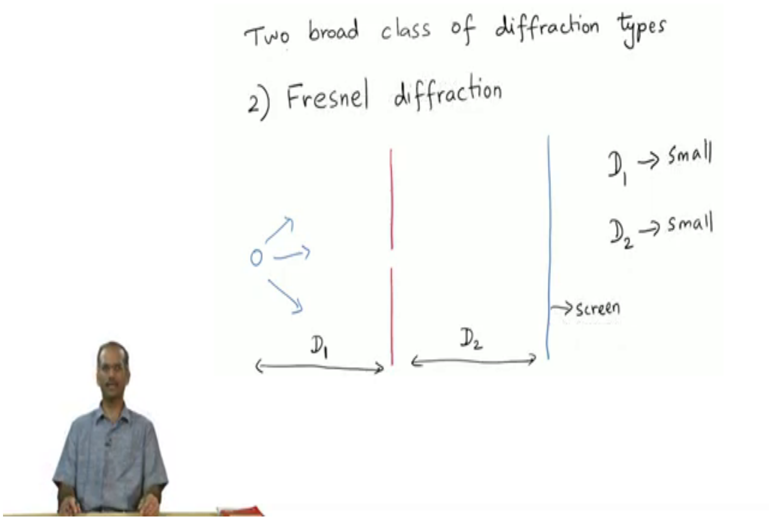

The second class of diffraction pattern is the Fresnel diffraction pattern. So, here as opposed to Fraunhofer case this D_1 and D_2 which represent the distance from the source and distance to the screen they are not infinite, they are kept really reasonably close to the place where diffraction takes place. In this case, the waves cannot be assumed to be plane waves. So, hence dealing with Fresnel diffraction at a theoretical level is somewhat more difficult than dealing with Fraunhofer diffraction.

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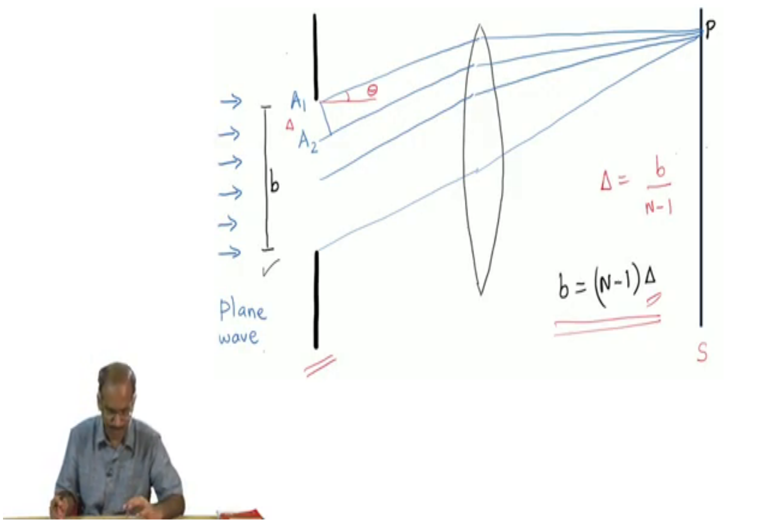

I am going to particularly concentrate on Fraunhofer diffraction of a diffraction pattern arising from a single slit. So, here is a sort of enlarged diagram of the same thing, I assume that the incoming waves are plane and the slit I have has a width b this is the width b which is marked in the figure and this thick dark lines that you see these ones here that is your slit.

The open region is the slit black region will block off the light from going ahead, and I have this screen S at the far right and the question I ask is often what is it that I will see at position P on the screen S, to bring all the waves into focus I use a lens like the one that is shown here. So, what we will do is begin by dividing the width of the slit. So, width of the slit is b we will divide it into N equal smaller regions and size of each of the smaller regions is Δ. So, the entire width b can be written as Δ multiplied by $N-1$.

So, let us say that A_1A_2 that I have here is one such small width Δ. So, what I am going to do is to consider array which is leaving from the point A_1 and another ray which is leaving from the point A_2, and the one that is travelling from A_1 makes an angle θ with the horizontal at that part. Now, let us look at this region in some detail to understand the geometry.

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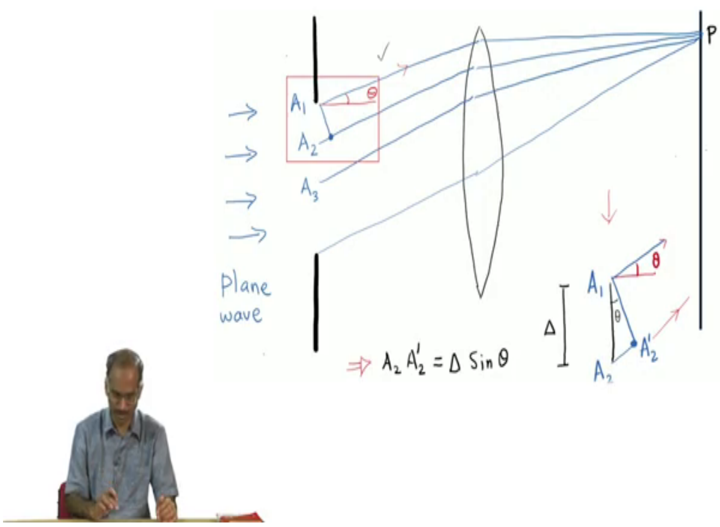

So, I have enlarged this part and shown it here. So, A_1 and A_2 you can see here black line is drawn and that has a width Δ the way that we identify a while before. And this is of course, the ray that travels here like this and there is a second ray from A_2 which is originating and travelling in this direction.

And now if I drop a perpendicular from the point A_1 on to this ray which is going from A_2, it will make an angle θ, the same θ that you see here. So, the path difference which I am interested in is given by this quantity A_2A_2' is $\Delta \sin\theta$ as simple as that.

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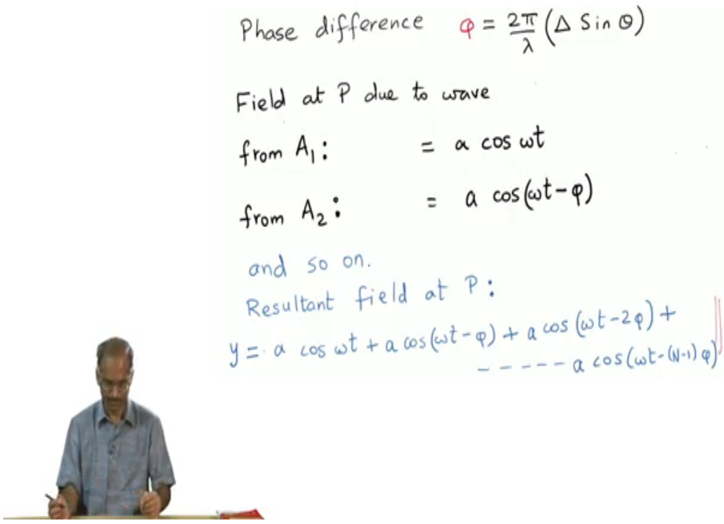

Now, that I know the path difference it is easy and straightforward to calculate the phase difference. So, the phase difference ϕ is equal to $\frac{2\pi}{\lambda}$ multiplied by the path difference which in this case is $\Delta \sin\theta$. So, now, what I want to find this what is the field at P due to wave that is emanating from position A_1 and from position A_2.

So, let me assume that since it is a wave, it should be solution of a wave equation and let me assume the solution to be $a\sin\omega t$. So, it is something that is propagating towards the screen and for the one that is propagating from position A_2 it is the same $a\sin\omega t$, but there is a phase lag because of the path difference; because of the path difference A_2A_2' there is a phase lag and we have already calculated that phase ϕ.

So, I can take that to be $a\cos(\omega t-\phi)$. And now you can actually keep doing this because if you go back and look at this figure you will see that I have defined a point A_3 which is at a distance Δ from A_2 and there will be an A_4 which is at a distance Δ from A_3 and so on. So, as we have defined there would be N of these $N-1$ gaps basically. So, each one would maintain a phase difference of ϕ with the predecessor.

For example from A_3; A_3 will maintain a phase difference of 2ϕ with respect to the wave emanating from A_1. So, in which case I could have written the field at P due to wave starting from position A_3 as $a\cos(\omega t-2\phi)$ and so on. So, now, I need to calculate the resultant at P because there are these waves coming from A_1, A_2, A_3, A_4 and so on up to A_N or A_{N-1} and they all are converging at P. So, I would like to know what is the resultant field at P. So, that would be simply be a vector sum of all these $a\cos\phi$, $a\cos(\omega t-\phi)$ and so on so, which is what I have written down here.

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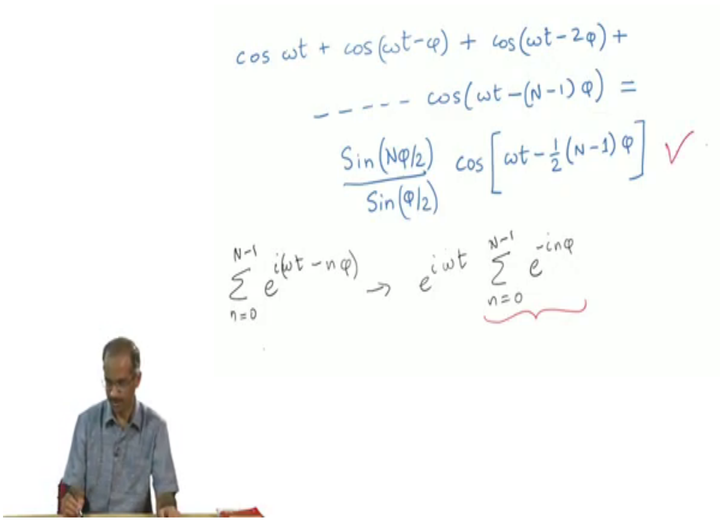

So, now what I need to do is to do this sum. So, I have rewritten that sum here, so, it is $a\cos\omega t$ plus $a\cos(\omega t-\phi)$ and so on up to $a\cos(\omega t-(N-1)\phi)$. You can do this sum by considering a geometric series for example, what you could do is to write something like this $e^{i\omega t-n\phi}$ and sum it over n small n going from 0 to capital $N-1$.

But, what we want to sum over these cross terms, but this $e^{i\omega t}$ has cos plus; it will be $\cos \omega t + i \sin \omega t$. So, it has the real part which is the cos term and the imaginary part which is the sine term. So, finally, after you do the sum you just take the real part of the answer that will give you this summation. So, I mean just to indicate one more step of how it should be done, you can take $e^{i\omega t}$ outside because it is not part of the summation $n = 0$ to $N-1$, $e^{in\phi}$

So, this quantity here this summation here is a geometric series. So, sum to n terms of a geometric series is a well known quantity, just apply that sum you will get the final result and after that take the real part of that you should get this answer.

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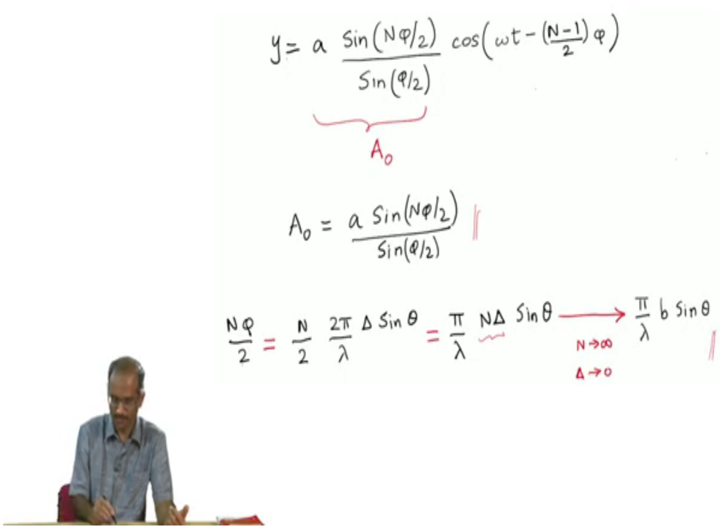

And as you can see when I want to calculate the intensity all I am interested in is this quantity A_0 which is the amplitude and I have extracted this quantity A_0. So, now I am going to assume that ϕ is small enough. So, I want to know what is $\frac{N\phi}{2}$ in the first place.

So, $\frac{N\phi}{2}$ is just put in we know what is ϕ just put in everything and if you assume that or take the limit that $N \to \infty$ which is like saying that the divisions are getting smaller and smaller or in other words the Δ that we define here gets smaller and smaller as the $N \to \infty$. So, after all Δ will be equal to $\frac{b}{N-1}$ So, as $N \to \infty$, b will be getting Δ will be getting to 0. So, which is the idea that we use here and in this case $N\Delta$ is simply equal to b itself.

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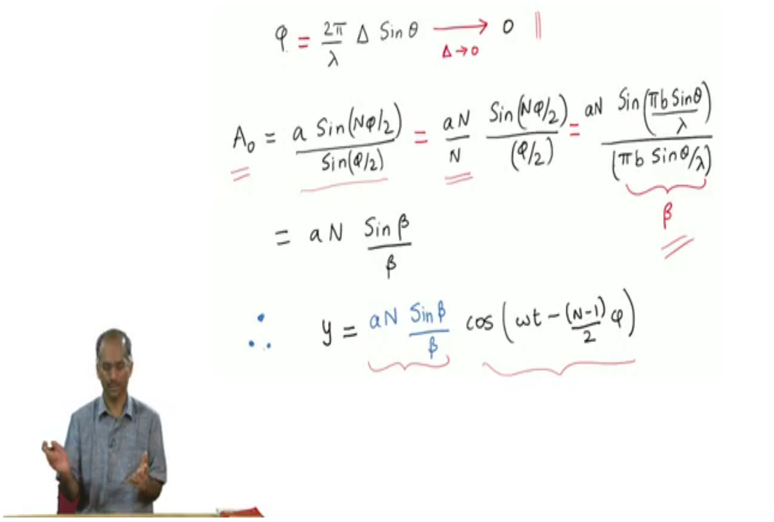

So, we are just putting back b in the formula, and now I have $\frac{N\phi}{2}$ to be equal to $\frac{\pi}{\lambda} b \sin\theta$. Now, if you take the limit of $\delta \to 0$ for the phase difference ϕ, then it turns out that phase ϕ goes to 0, after all ϕ is $\frac{2\pi}{\lambda}$ into δ multiplied by $\sin\theta$. So, if δ is going to 0, ϕ will have to go to 0.

So, now, let us assemble all these information together in A_0. So, A_0 is this formula here which came from what we wrote down earlier and what I have done here is to multiply and divide by N. So, a is multiplied and divided by N and then you have that $\sin\frac{N\phi}{2}$ and since we are going to look at the limit $\Delta \to 0$, then in that case ϕ also tends to 0. So, $\sin\frac{\phi}{2}$ will be just $\frac{\phi}{2}$ that is approximating $\sin\theta$ as θ.

But in the numerator we cannot write $\sin\frac{N\phi}{2}$ to be $\frac{N\phi}{2}$, because while $\phi \to 0$, $\frac{N\phi}{2}$ does not tend to 0. So, what we do is to use this relation that we just calculated for $\frac{N\phi}{2}$. So, you plug that in this equation and then many things will cancel out and now you can write this expression for a 0 as small $aN\frac{\sin\beta}{\beta}$ where β is this quantity which is $\frac{\pi b}{\lambda}\sin\theta$. Now, I can write the expression for y, so, it is equal to $aN\frac{\sin\beta}{\beta}$. So, this is my amplitude A 0 times the usual other term which is the cos term.

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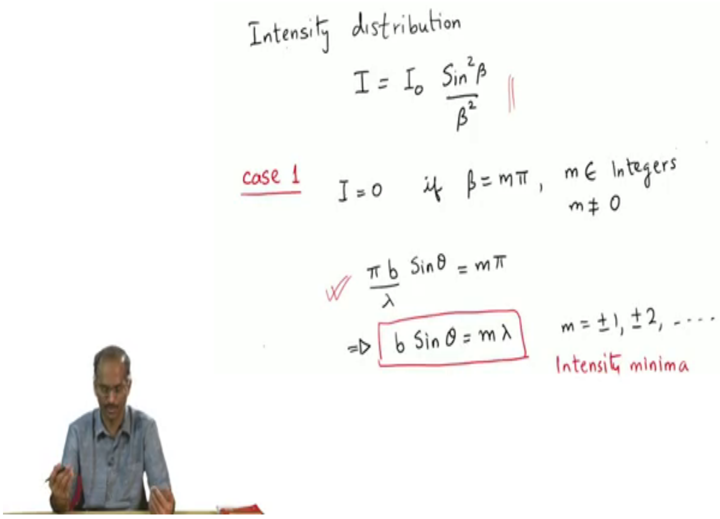

So, the intensity distribution is just the square of this quantity. So, I can write it as $I_0\frac{\sin^2\beta}{\beta^2}$. So, now, you can ask when will it give me a dark fringe, when will it give me a bright fringe. So, first case is when will intensity be 0 or dark fring? So, that will happen if as you can see from this formula if β is equal to $m \times \pi$ m is an integer.

But m should not be equal to 0 because if m where equal to 0 the denominator also would get to 0 in this case that gives me the condition that $b\sin\theta$ should be equal to $m\lambda$ for intensity minimum. And here m is again integer plus or minus 1 plus or minus 2 and so on. Now, what happens if $m = 0$?

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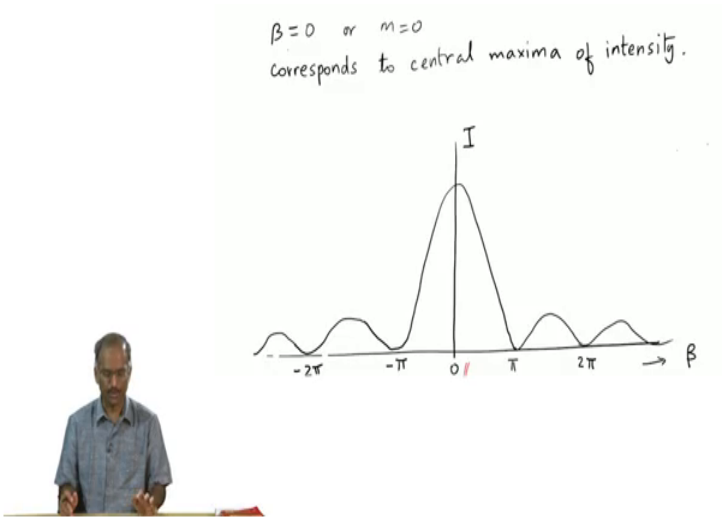

So, in that case $\beta = 0$. So, you can put $m = 0$ in this formula β will be equal to 0 and in that limit this quantity $\frac{\sin^2 \beta}{\beta^2}$ goes to 1 and what I will have is a maximum; a maxima at the centre $m = 0$.

So, if I plot intensity as a function of β there is going to be a maxima at $m = 0$ that is this point as you can see and then there is going to be a decay and the decay is of the form of square of $\frac{\sin \beta}{\beta}$. So, it looks like the kind of intensity patterns that we have seen before it is of the form $\frac{\sin \theta}{\theta}$, but its square of that quantity.

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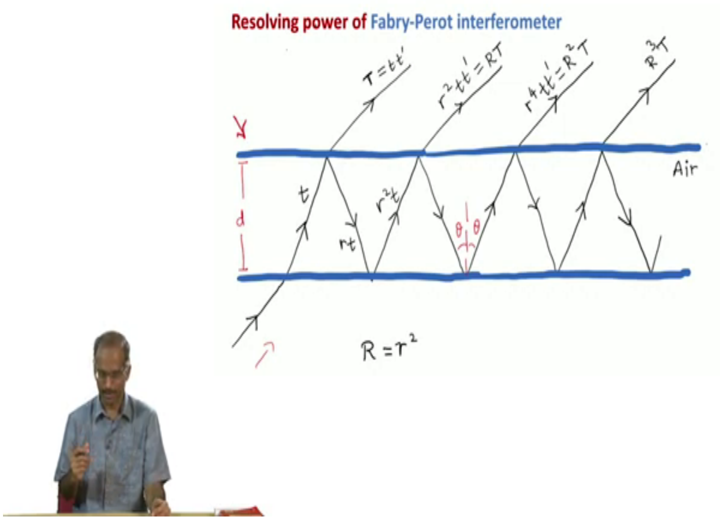

Next we will go to the problem that we decided to take up in this last module that is the resolving power of Fabry-Perot interferometer. So, let me very quickly go through the Fabry Perot interferometer.

So, it consists of this etalon. So, there is light coming from one direction and it undergoes multiple reflection and also partial transmission on the other side. So, you are asking the question what would I see on the other side due to multiple beams that are emerging out of the etalon. So, that is the question. So, we worked out the result for the Fabry-Perot interferometer we saw in particular that it has very high sensitivity.

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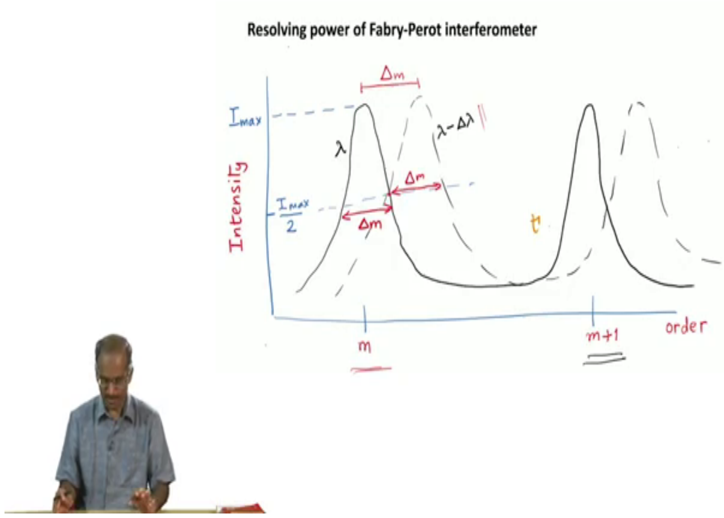

In the sense that when you plot intensity as a function of the interference order the peaks are very sharp for the cases when the reflection coefficient capital R is sufficiently large. But the question that we are now trying to address is what is the resolving power?

In other words the question is if you go back to this interferometer itself suppose I had 2 wavelengths of light that enter the etalon and individually both of them will create separate interference pattern like the one that is shown in this figure. And if you assume that the 2 wavelengths are very close to each other, the question is can I resolve them can I see them as separate peaks in the interference pattern due to Fabry-Perot interferometer.

So, here I have drawn the intensity versus order figure. So, this continuous curve is for the wavelength of light λ or interference produced by wavelength λ and the other one the dashed line is for interference produced by wavelength $\lambda - \Delta\lambda$. So, they are offset by a small distance Δm in the space of order m.

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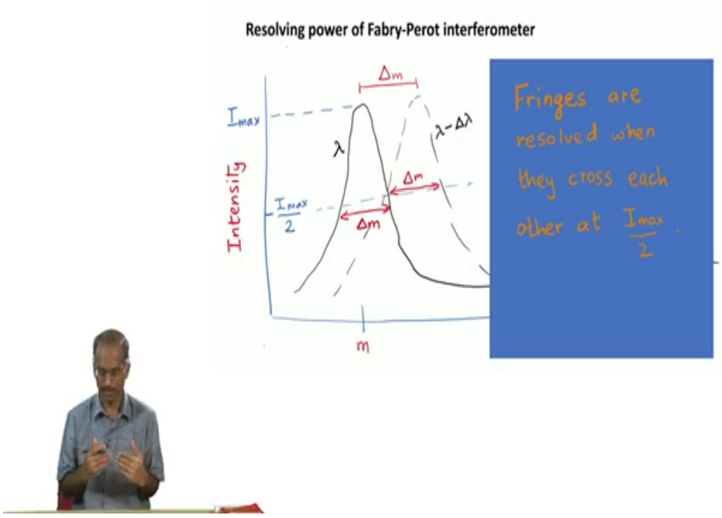

So, we will say that for our purposes the fringes are well resolved when they cross each other at $I = I_{\max}/2$. So, I is the intensity on the *y* axis. So, I have these two fringes the one with continuous curve and one with the dashed curve. So, when they move apart half the values of the peak intensity. So, peak intensity is $I_{\max}$ for both of them and half their values they cross at this point. So, when they do that then we will say that that is the criteria for us for having resolved to close wavelengths.

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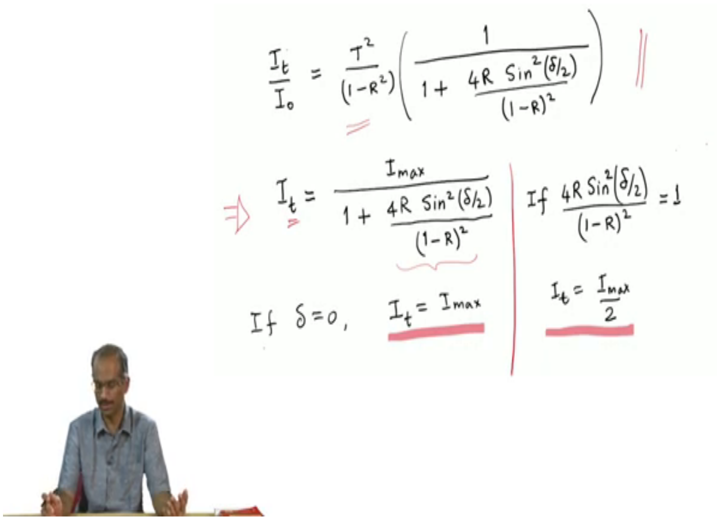

So, the starting point here is the intensity formula or the transmitted intensity which we derived entirely in the last module. So, if I identify $\frac{T^2}{1-R^2}$ as I max, then this entire formula that I have I_t can be written like this; this underscore t the small t here is to imply that intensity of the transmitted beam.

So, you can easily see that if $\delta = 0$, if the phase difference is 0 then I_t is equal to $I_{\max}$ because if $\Delta = 0$ this entire term in the denominator would go to 0 and there that would leave it with only 1 in the denominator hence, $I_t = I_{\max}$. On the other hand if this entire term is equal to 1 in that case the transmitted intensity is equal to $\frac{I_{\max}}{2}$.

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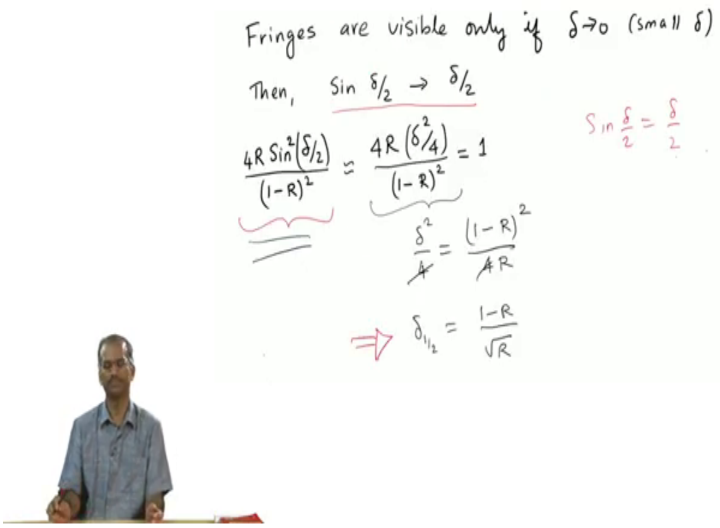

And what we know is that this kind of fringes are visible only if the phase difference is sufficiently small or in other words we need to work in the limit of Δ going to 0 in which case I can always approximate $\sin\frac{\Delta}{2}$ to be equal to $\frac{\Delta}{2}$. Then let me look at this term which is $\frac{4R\sin^2\frac{\delta}{2}}{(1-R)^2}$.

So, all I have done is to simply approximate $\sin^2\frac{\delta}{2}$ by $\frac{\delta^2}{4}$ simply because $\sin\frac{\delta}{2}$ is equal to $\frac{\delta}{2}$ and square of this quantity is $\left(\frac{\delta}{2}\right)^2$ whole square that is $\frac{\delta^2}{4}$. Now, I want to find out what is the value of δ if this has to be equal to 1; it is very easy to do that. So, $\frac{\delta^2}{4}$ in that case would be equal to $(1-R)^2$ divided by $4R$. So, you can cancel off 4 and 4.

So, $\delta/2$ if we call it that way would be equal to square root of this and that would be $\frac{1-R}{\sqrt{R}}$. This is the phase change when this quantity $\frac{4R\sin^2\frac{\delta}{2}}{(1-R)^2} = 1$ and that is the condition when I_t is $I_{\max}/2$, basically it is the condition when we would say that 2 wavelengths are well resolved.

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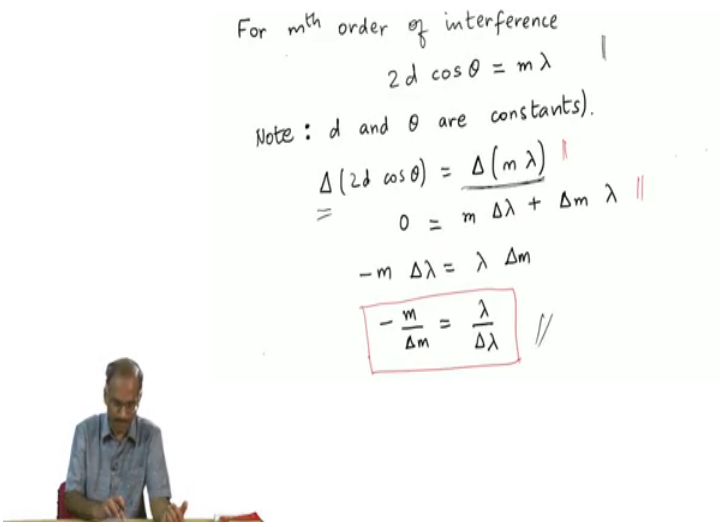

So, what we know is for mth order of interference $2d\cos\theta$ is $m\lambda$ again, refer back to the previous lecture we have seen this; $2d\cos\theta$ is $m\lambda$ we derived it in its full detail.

So, here you should note that both d and θ are constants. So, in that case now if I ask for variation on both side or in other words I take a Δ. So, this delta is basically like taking a variation of quantities which are variable. So, $\Delta(2d\cos\theta)$ would be 0 because d is a constant, θ is a constant, so they cannot be varied. So, left hand side is 0, but if I take $\Delta(m\lambda)$ that I have here, in that case that you can calculate because both m and λ are variables.

So, that will be $m\Delta\lambda + \lambda\Delta m$ and left hand side is 0, so that quantity is entirely equal to 0. So, by simple manipulation you will get a relation between $m\Delta m$ and $\Delta\lambda$. So, in other words now we know that there is this relation between how much of change in wavelength corresponds to how much of change in order.

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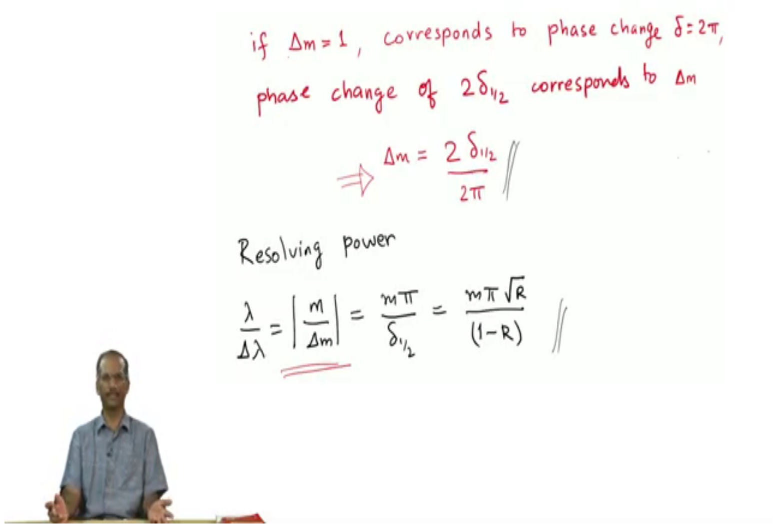

But what we know is if Δm is equal to 1 the phase change is 2π. So, go back to this figure. So, I have m here and I have $m + 1$ here. So, if m changes by unit value the change in phase is 2π. Now, if the phase change is 2 into Δ half some number that we know, then to what Δm would it correspond to? That is a question and that is very easy Δm is 2 times Δ half divided by 2π. And the definition of resolving power standard definition is $\frac{\lambda}{\Delta\lambda}$ when I say resolving power is large it means that I can differentiate between two wavelengths which are which differ by $\Delta\lambda$.

So, smaller the $\Delta\lambda$ is or smaller the $\Delta\lambda$ that I can differentiate better is the resolution. You can see that $\frac{\Delta\lambda}{\lambda}$ will be equal to $\frac{\Delta m}{m}$ and since we are interested in the magnitude we can ignore the negative sign take the modulus of m by Δ, then we just need to put in the values. So, m is the order and Δm can be now written in terms of this relation.

So, that will be 2Δ half in to divided by 2π and $\frac{\Delta}{2}$ itself was just now calculated to be this one, $\frac{1-R}{\sqrt{R}}$. So, if you put all these things here it will turn out that your; it will turn out that your resolving power is $m\pi\sqrt{R}/(1-R)$. So, it depends on reflection coefficient. If R is sufficiently small then the resolving power is going to be fairly, high.

So, let me close this session by saying that we studied the diffraction pattern. Diffraction is not too distinct from interference, it is physically not different, it is only a question of scale. And we looked at the diffraction pattern due to a single slit and we went through the usual calculation by dividing the slit into N parts and looking at waves coming from each of those and finding out the resultant due to all of them.

Finally, it turns out that you will get an diffraction pattern that is of this type, and then we went ahead to calculate the resolving power of a Fabry-Perot interferometer a problem that we had left incomplete in one of the earlier classes, and it turns out that the resolving power depends on R which is the intensity of reflection in the Fabry-Perot interferometer.