Waves and Oscillations
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Lecture – 48
Waves in Optical Systems: Problems

Welcome to the last lecture of this week. In this lecture let us do some problems using the matrix method.

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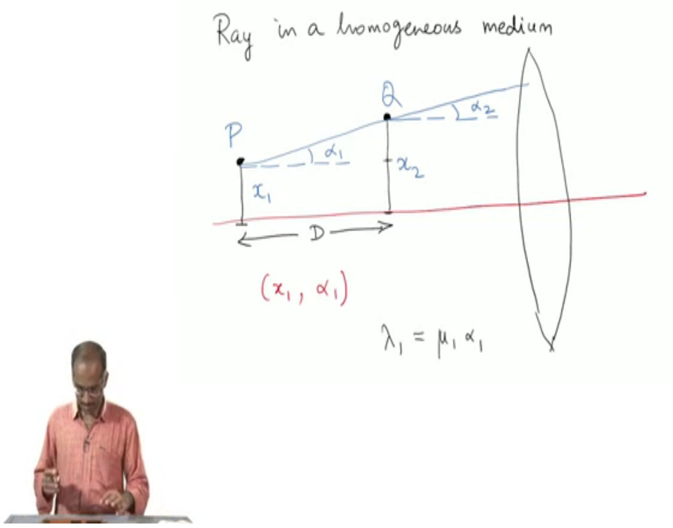

So, let me quickly remind you what we did in the last class by looking at the ray in a homogeneous medium for instance the kind that I have shown here on the screen. Point P is represented by these sort of coordinates (x_1, α_1); x_1 is the height distance measured from the axis, which is the red line here and α_1 is the angle.

So, ideally I would like to use $\tan \alpha_1$; however, because we are working in the paraxial approximation, $\tan \alpha_1$ is α_1 so, this is sufficient for us. What we actually used in practice was a quantity called λ_1 which is $\mu_1 \alpha_1$. So, you could see that this should have been $\mu_1 \sin \alpha_1$, again in the paraxial approximation it is $\mu_1 \alpha_1$, this is often called the direction cosine for a ray.

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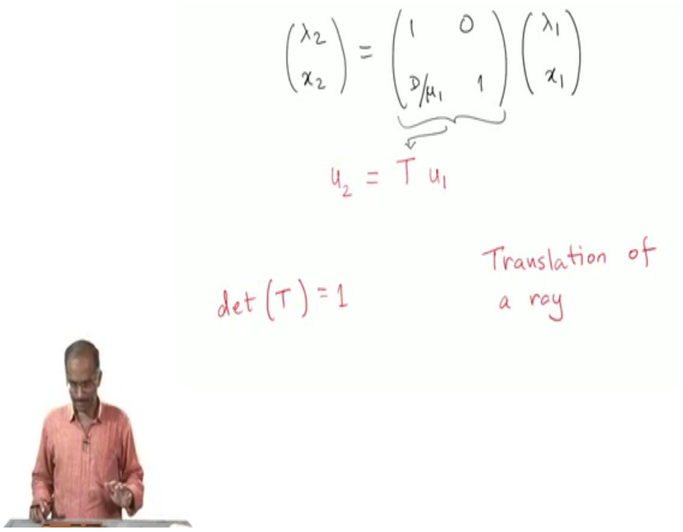

So using this, we wrote down this equation for translation of a ray from point P to point Q. At point Q, the coordinates are described by $\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix}$ and the initial point P is described by $\begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$. So, you want to know, how to go from one to other and otherwise given $\begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$; how do I go to $\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix}$ and that is provided by this solution in terms of this matrix.

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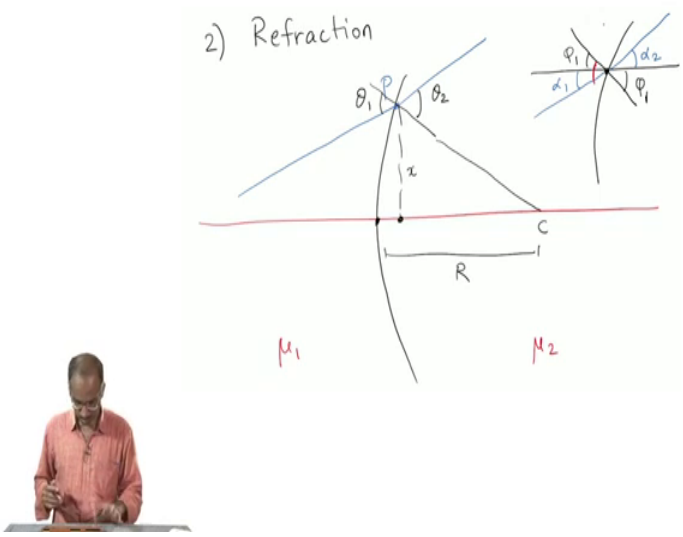

And similarly one can work out a similar matrix for the case of refraction. So, imagine that I have this refracting surface which is flashed on your screen and there is an incoming incident ray which gets refracted bends when it goes into the second medium. So, then you define all these angles of incidence, angle of refraction, use the law of refraction. So, you can do all that and finally, you end up with this matrix equation.

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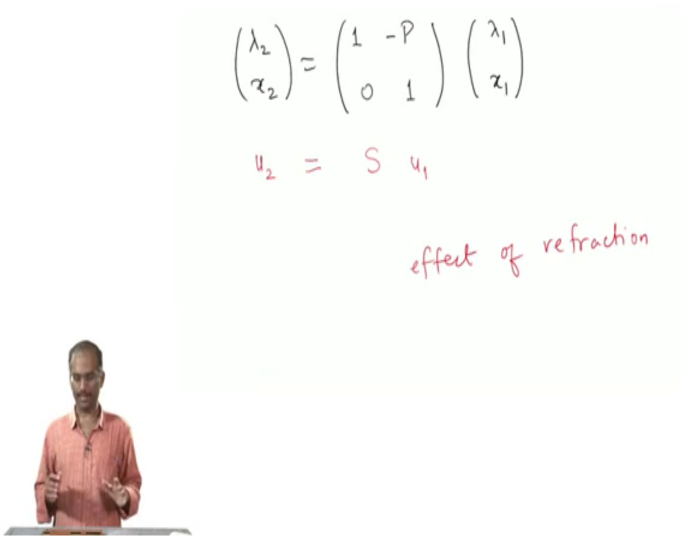

(x_2, λ_2) is the point after refraction. So, you can imagine (x_1, λ_1) to be a point before refraction takes place, even ideally just before refraction takes place. And (x_2, λ_2) is some point immediately after refraction takes place and they are related by this matrix S.

So, clearly here you will notice that since, x_1 and x_2 are two points on either side of the point where refraction takes place and they are very close to one and other that you could take x_1 and x_2, which is the height from the axis the red line here to be x itself. Basically, statement that x_1 is equal to x_2. So, this matrix S captures the effect of refraction.

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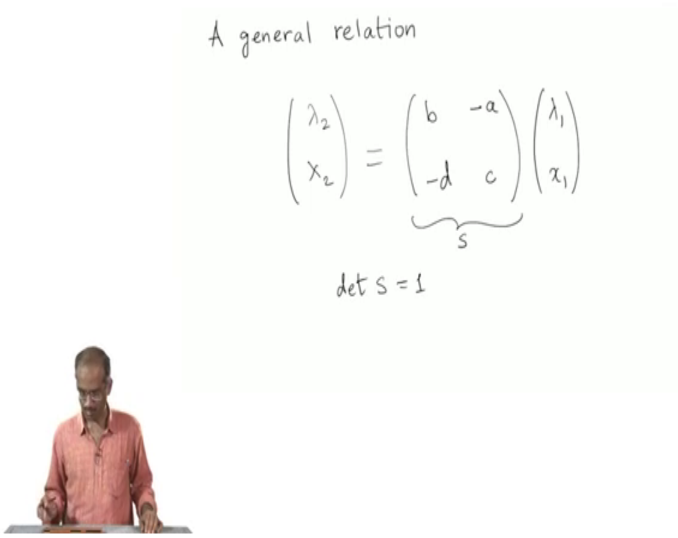

So, here $\begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$ is your initial coordinate and then the ray goes through a system of optics, where many things could take place; translations, refractions, several of them in any order and finally, the when it comes out you describe it by $\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix}$.

Now, how is $\begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$ related to $\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix}$? That is the question. So, in general without doing anything, I could simply, so everything that happens inside your optical system is sort of encoded in this matrix. The determinant of this matrix, so if I call this matrix S; the determinant of this matrix is 1.

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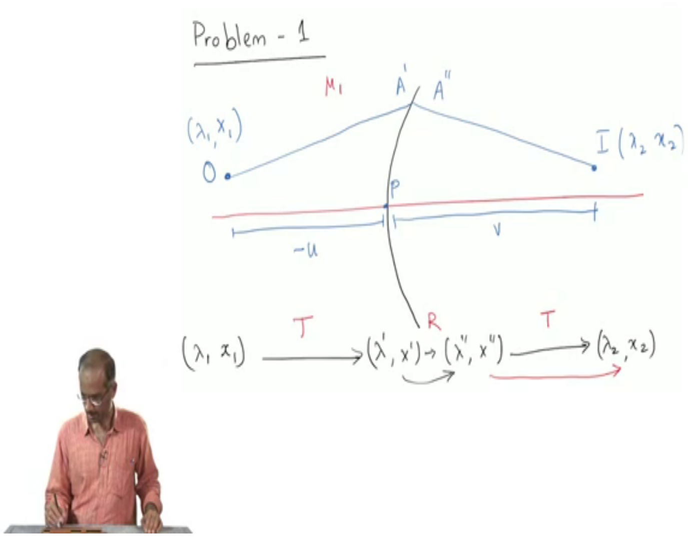

First problem is something that we are done earlier on using the standard method. So, let me restate the problem. Let us say that, I have this refracting surface and this of course, defines for me the axis. And I have an object here at O. I am going to denote this object by (λ_1, x_1). The ray goes and hits the refracting surface and gets refracted and reaches some other point which I shall call I and this is denoted by (λ_2, x_2) .

And we shall also identify two other points very close to the refracting surface on either side; one which is just before the refraction would take place, let me call it A' and other which I shall call A''. So, the object is at position O, described by coordinate (λ_1, x_1). It goes and hits the refracting surface. Let me call this distance u and because the distances to the left of this point P here, I shall give it a minus sign. And, the distance from P to this point image I we shall call it v, because x_2 is the height of the image at point I, and the height of the object is x_1 at O.

So, when you divide the height of the image by height of the object, you get magnification. So, it is simply x_2 by x_1. And once, I know how these two are related, we can obtain many such information. And from here I have the other coordinate which is (λ', x'), which is just before the refraction takes place. And just after the refraction takes place the coordinate is (λ'', x''). And from here finally, I go to the last point, where the coordinates are (λ_2, x_2). Going from (λ_1, x_1) to (λ', x'), it is a translation.

And from (λ', x') to (λ'', x'') which is this, it is a refraction and again going from (λ'', x'') to (λ_2, x_2), I have a translation. So, I have a translation, refraction under translation. And I know what kind of matrices will take me from one to the other. So, it is easier to do these calculations.

So, let us first relate (λ', x') to (λ_1, x_1). And I must also state that we are going from one medium, which is characterized by μ_1 to another which is characterized by μ_2. So, when I write the translation matrix, it involves μ_1. So, I have done the first part.

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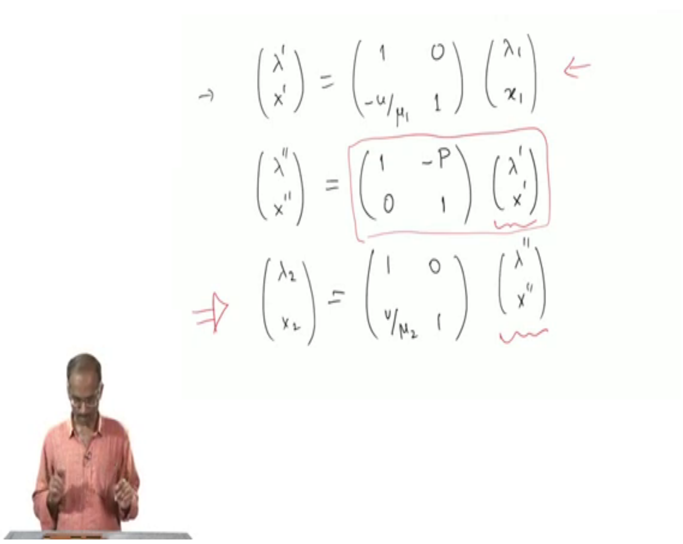

Now, going from primed to double primed coordinates involves a refraction. And this, once again it is going to be a matrix for refraction. So, it will be 1 1 minus P here and 0 here. And now, the last part is translating from double prime coordinates to (λ_2, x_2). Since it is a translation it will look like the first matrix that I have here, it will be 1, 1, 0.

Now, it is happening in a medium whose refractive index is μ_2, so that will be v by μ_2. As you can see here, you look at the last equation here which is this. Now, here I have (λ'', x'') and I can substitute for this vector from here, which is this. And here, I have (λ', x') vector, which can be substituted from this equation. So, if I do that successively I can directly relate (λ_2, x_2) to (λ_1, x_1), let me write that equation.

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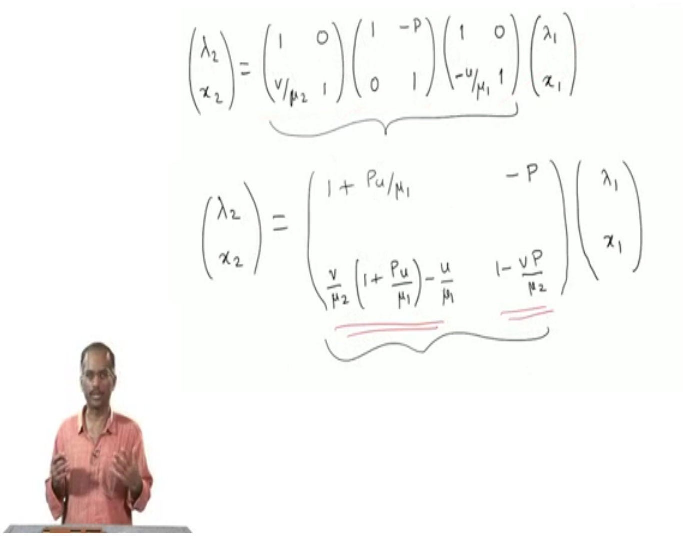

Now, to make it into more simpler equation, all I need to do is to simply multiply these three matrices together. Then I can directly relate (λ_2, x_2) to (λ_1, x_1). Let me write the result, and now this directly relates, (λ_2, x_2) to (λ_1, x_1). Now, we can analyze this and obtain the results that we want.

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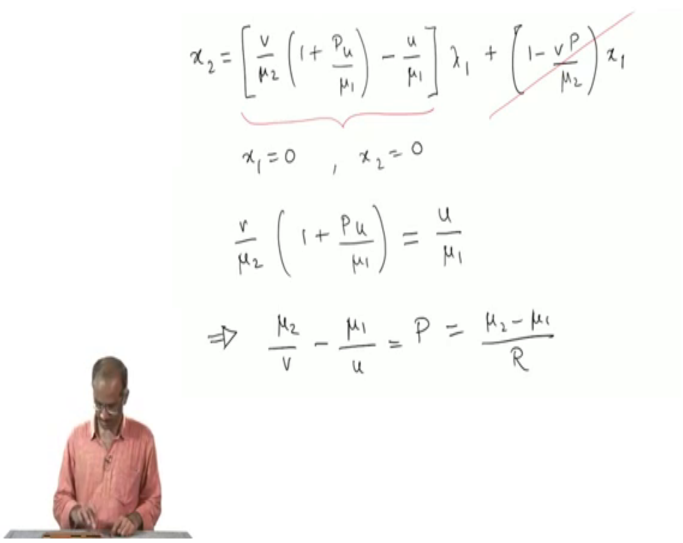

So, now I have a relation that relates x_2 to x_1 directly. Now, if I have an object which starts from the axial point which means that it starts along this red line.

The condition in that case would be that $x_1 = 0$. So, the height from the axis is 0. In that case, the image the condition for the image is that x_2 also should be equal to 0. Now, if I put in this condition; when x_1 is 0, so this term will completely go away. And if x_2 is 0, since λ_1 is not equal to 0, the only possibility is that this term should be equal to 0; which means that a small manipulation would give me the following result. So, it gives me this result which is equal to P; P is simply the power and if you remember this is inverse of the focal length..

So, that is $\mu_2 - \mu_1$ divided by the radius of curvature of the refracting surface, not surprising, this completely tallies with what we had derived before. To get a relation between x_2 and x_1 in the image plane or in the plane that is defined here, all I need to do is to substitute for P from what we had just obtained. And if you do that, I urge you to do it yourself and see what happens, you should that particular element would go to 0 in the matrix. So, let me directly write the result.

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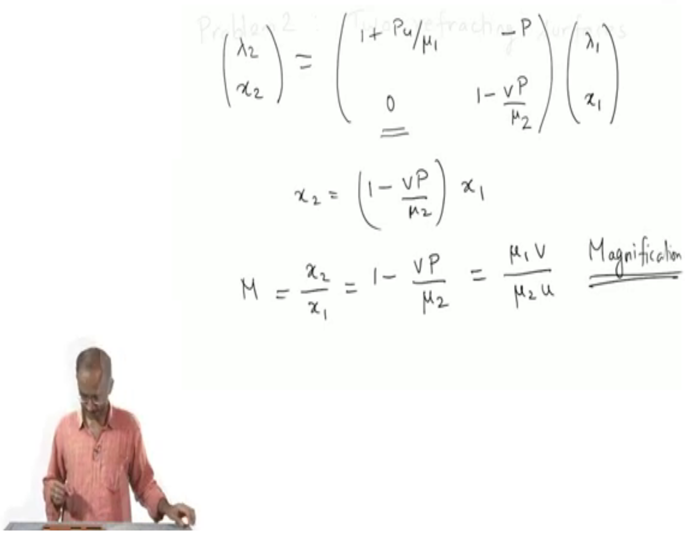

So, this would be the relation of the image plane. So, the relation that I need is x_2 divided by x_1, it is already there x_2 divided by x_1 is $1 - \frac{vP}{\mu_2}$ and if I substitute for P from here, what I have, I should get the following relation.

So, this is my magnification. So, clearly if both the media are same, μ_1 and μ_2 would be same, they would cancel off and it is simply the distance of the image divided by the distance of the object, sort of familiar formula that you would have seen earlier.

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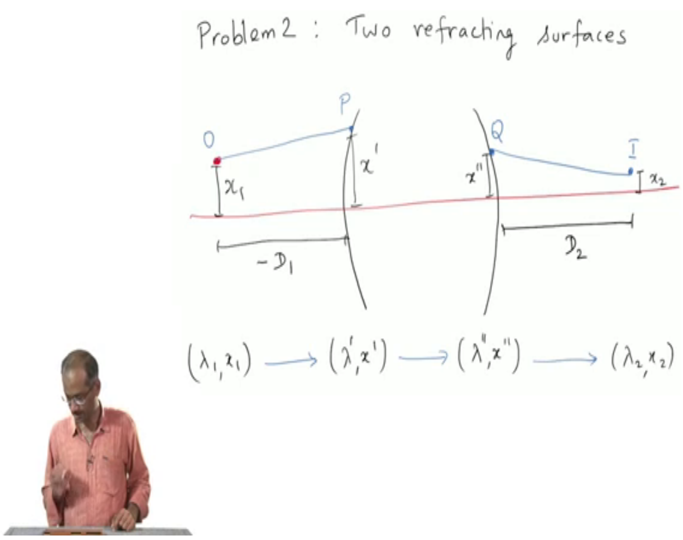

Now, let us do the problem of two refracting surfaces. Ultimately, I would like to again go back to thin lenses problem to start with I shall have the following situation. So, I have two refracting surfaces and this is my axis and I have this point O, which has an object this meets the first refracting surface at point P. And then, there is refraction happening internally in the sense that, you can imagine that there could be even other refracting surfaces in between these two. But finally, when it finishes with all that, it starts from this point and goes down like this.

So, the net effect of whatever is there inside these two systems of two refracting surfaces has given rise to ray which goes from O to P and other one which goes from Q to I. I shall call this height x_1 and this height as x_2. And we also need two other heights so, this one we shall call it as x' and this one is x''.

And let us also define the distances. So, the distance from here to here let me call it D_1, of course, I need to put a negative sign and this distance we call it D_2. So, the first point has coordinates (λ_1, x_1) and here I have (λ', x'), (λ'', x'') and (λ_2, x_2).

Now, I will not go through these calculations in detail; because it is nearly similar to what we did. So, we have one translation then there is refraction and a translation.

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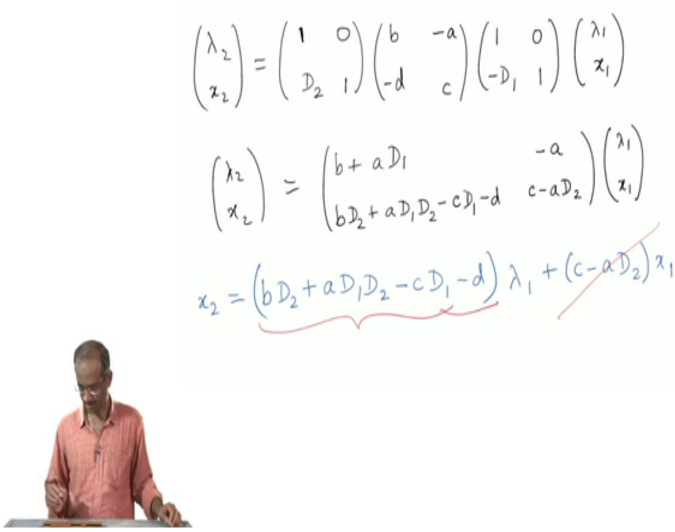

Here I have written the matrix equation. So, clearly you can see this part which is translation another one which is translation. And we are relating (λ_2, x_2) to (λ_2, x_1), but this one, I have left it empty I am going to fill it up now, but as I said that determines what happens inside the system here, between these two curved surfaces.

So, let me in general write it as; so, this you can think of as your system matrix; which encodes all the information about what happens in between these two refracting surfaces. So, after the multiplication this is what you are going to get. Once again if you remember, if your object is coming out of the object plane from the axis itself then $x_1 = 0$, in which case of course, this last term would be 0, but x_2 would also be equal to 0, in which case the condition is that; this quantity here is equal to 0.

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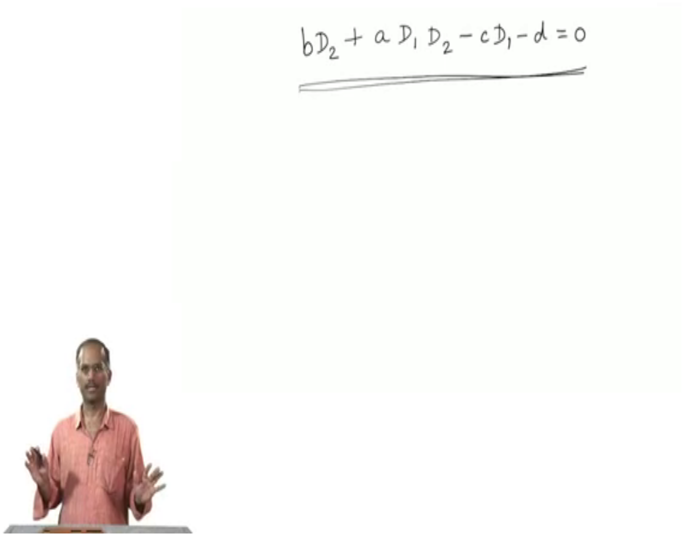

And this relation would give us would tell us how D_1 and D_2 are related. Let us now define refracting surface, work out the details and see whether we can use this particular relation to obtain the relation between D_1 and D_2.

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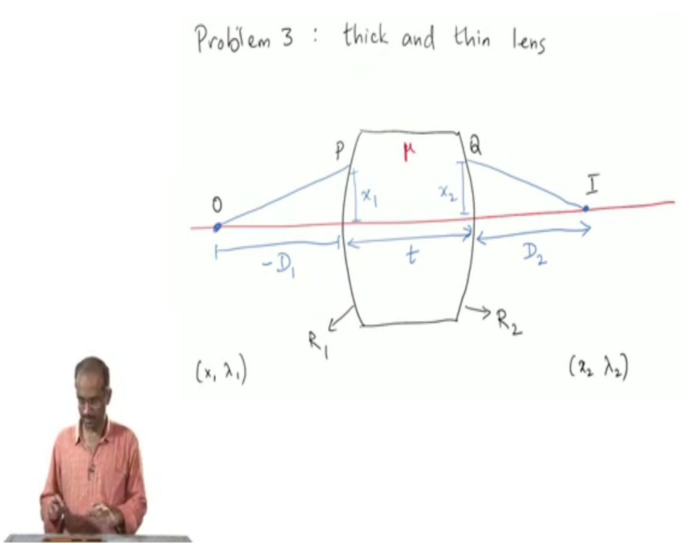

So, I have this thick lens here, object at O, the ray hits the lens at point P and then it comes out at point Q and there is the image plane at position I. x_1 and x_2 are of course, the distances of the heights with respect to the axis which is shown as red line here. And the thickness of the lengths is this quantity, which I will call as t. Further, I need to define these distances D_1 and D_2. So, this will be the distance D_1 and of course, I need to put a minus sign here and this distance is D_2 with the refractive index of the material is μ. And the radius of curvature of the left surface is R_1 and the radius of curvature of the right surface is R_2.

So, as you can see, there are three components to it, that is a translation and inside the there is refraction and then again translation. So, which means that if my image position is described by (x_2, λ_2) and object is described by x_1, λ_1, they will be related through a product of three matrices. So, let me directly write the product of the three matrices. Product of these three matrices let me use the symbol S to indicate that.

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Note:

- There are 5 transformation matrices involved in this problem, one corresponding to each arrow in the following representation of the optical process.

$$(\lambda_1, x_1) \rightarrow (\lambda', x') \rightarrow (\lambda'', x'') \rightarrow (\lambda''', x''') \rightarrow (\lambda'''', x'''') \rightarrow (\lambda_2, x_2)$$

- The matrix S involves a product of the three matrices corresponding to the second, third and fourth arrows only.

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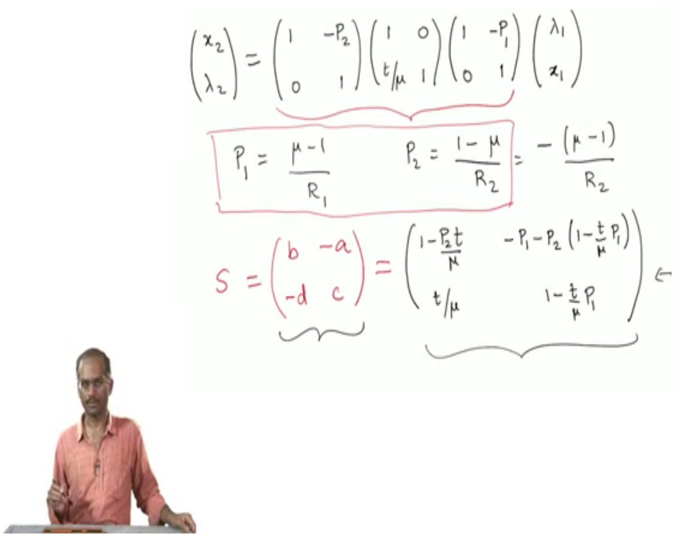

So, if I take that to be this general matrix b, $-a$, $-d$, c. This will be equal to the product of three matrices. This of course, comes from my own calculations for this problem.

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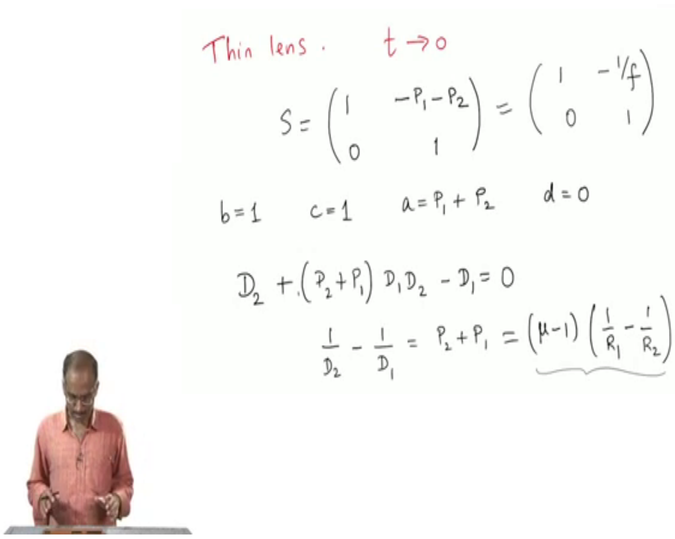

Now, if I have to specialize to the case thin lens, then I need to take the limit of thickness t going to 0. So, just put $t = 0$ in this matrix and you should get this. And now if you compare this with this matrix here, you can write what is a, b, c and d.

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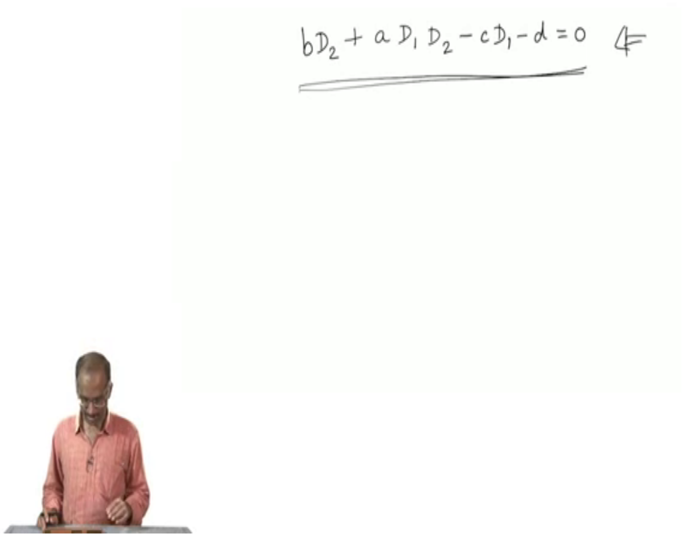

So, to relate D_1 and D_2; we said that, this is the condition we derived it from this case of two refracting surfaces. So, if I substitute these a, b, c, d value is there, I get the following equation. And now if I substitute for P_2 and P_1, remember that; we have these relations for P_1 and P_2. Now, I just need to put them in here, in which case I will get.

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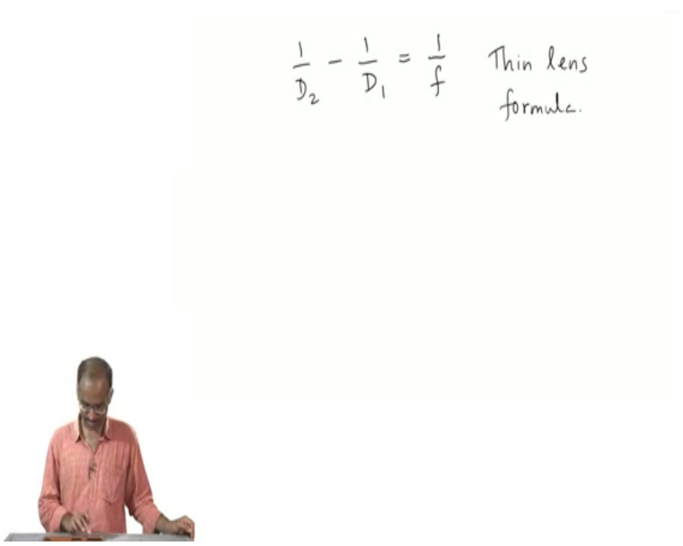

And this one can equally be written as $\frac{1}{D_2} - \frac{1}{D_1} = \frac{1}{f}$. Again it is a result that, we had derived before from other considerations; this tallies with the previous relations that we had obtained.

For the case of thick lens I will not do the problem, but you need to go back to the full equation which is this. So, you cannot take the limit t goes to 0. So, keep this as it is, identify a, b, c and d, and then substitute them in this equation and you should be able to get a relation between D_1 and D_2. This S, which is the sort of system matrix for the thin lens can be rewritten in terms of the focal length or focus. Notice that $\frac{1}{D_2} - \frac{1}{D_1} = \frac{1}{f}$ which is equal to the quantity here; hence this equation or the matrix can be written as 1 minus 1 by f_0 and 1.

So, I shall stop this discussion of matrix methods here with these examples. But the central idea here is that identify what is happening between the incoming ray and the outgoing ray. And simply relate the two depending on whatever number of refractions and translations that happen. You know the matrices corresponding to each one of them multiply them together, that will be your system matrix. And then using the conditions for where the image starts where it where the object starts and image is obtained, you can finally relate x_2 and x_1 and obtain quantities like magnification and so on.