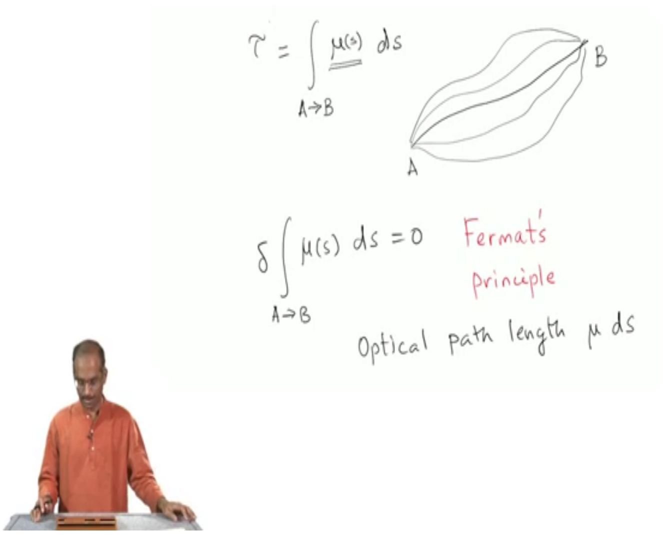
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 45
Waves in Optical Systems: Applying Fermat's Principle

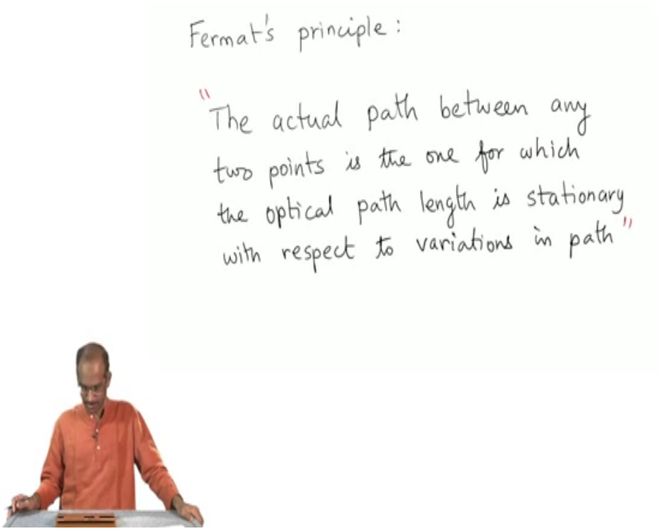
Welcome to this second part of 10th week, we will not be studying optics in detail because optics in itself is a vast subject. So, instead what we are going to do is to simply confine ourselves to few simple things which we need. So, that we can understand how Waves and Oscillations as phenomena work in Optical System.

(Refer Slide Time: 00:39)



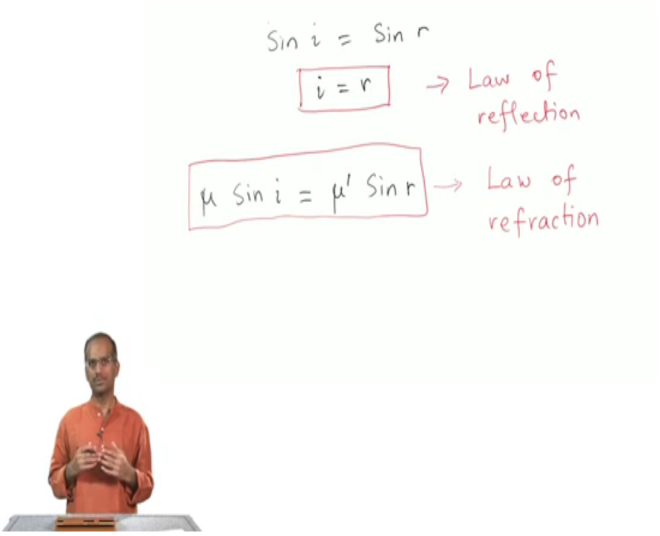
To begin with let me remind you of Fermat's principle. So, the principle simply states that the actual path taken by the light is the one for which the optical path is an extremum. In the standard simple case of a homogeneous medium and light travelling from point A to point B its equivalent to saying in a sense that light takes the shortest path between those two points in this case it would be a straight line.

(Refer Slide Time: 01:14)



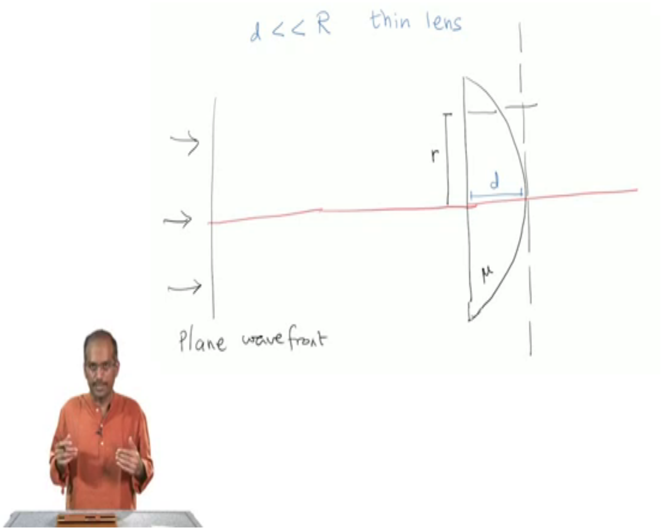
Fermat's principle can be stated as the actual path between any two points is the one for which the optical path is stationary with respect to variations in the path.

(Refer Slide Time: 01:27)



Today we will apply Fermat's principle to a little more complicated system mainly to the case of a thin lens I am going to start with the case of plano convex lens.

(Refer Slide Time: 01:38)



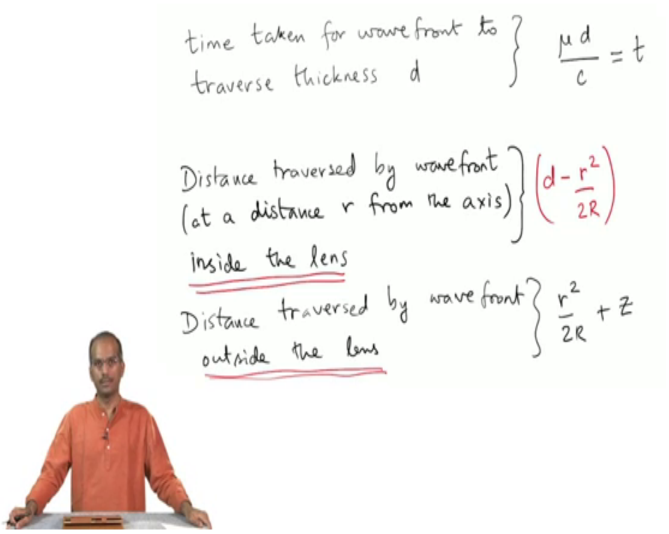
So, the lense is plane on one side and has a convex face on the other side. So, this distance here would define for me the thickness of the lens all you need is a circle of which this convex arc is a part and the radius of that circle which have denoted by capital R is the radius of curvature of this lens. Whenever this d which is the thickness of lens is much smaller than the radius of curvature then you call it thin lens.

I am going to assume that a plane wave comes from the left side which will be our convention to assume this pretty much in most of the problems that there is an incident plane wave coming from the left side. And I am also going to assume that the medium of this Plano convex lens has a refractive index μ and in general we can assume that the refractive index of the medium outside of this lens is equal to 1.

Now, let us focus on a ray it should be at a distance r from the optical axis. So, the optical axis here in this case is indicated by this red line the origin of y axis coincides with this red line. So, this small r would define for me some arbitrary distance from the optical axis. What happens to this incoming incident plane wave after it passes through this plano convex lens? So, if your along the optic axis along this red line the ray which comes along this red line would travel and entire distance d inside the lens as you can see.

On the other hand the ray which is hitting the lens at a distant r from this red line would only travel part of the distance inside the lens there is this one part which would be traversed inside the lens and there is the other part which should be travelled outside the lens.

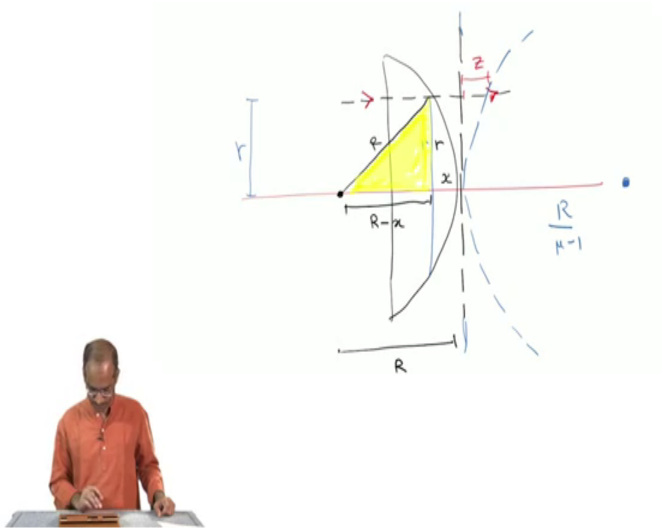
(Refer Slide Time: 04:09)



So, let us call this time t, the time taken for the wavefront to travel this distance d inside the lens along the optic axis or the central axis and that time would be given by $\mu \times d$ divided by c; c is the velocity of light; d is of course, the thickness of lens and μ is the refractive index of the medium of lens.

Now, what happens to the wave front that hits the lens at a distance r from this axis. So, what we want to know is how much distance does the wave front travel in the same time t. You can take a look at this diagram which is nearly the same diagram that I have here, but I have try to focus on this particular ray which comes and hits the lens at a distance r from the optic axis.

(Refer Slide Time: 05:09)

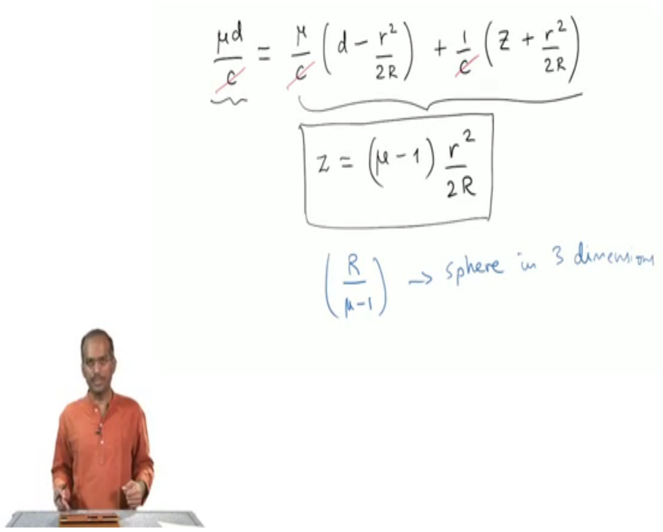


Now, from the geometry of this figure you can use Pythagoras theorem to write the following equation $(R-x)^2 + r^2 = R^2$. So, that is the solution for x it cannot be a value that is larger than R, if I keep the plus sign in the solution it would tell me that x will have a value that is larger than R. Hence I will take x to be $R - \sqrt{R^2 - r^2}$.

Now, we make the binomial approximation. So, the binomial approximation is of the following types. So, if I have function of this type, $1 + x$ to the power half or in general I could consider 1 plus or minus x to the power half this will approximately be equal to 1 plus or minus half into x. It actually an infinite series which we are terminating at this point and this approximation would work only if x is much smaller than 1. this is the result that we want.

So, the distance traversed by the wave front at a distance R from the axis inside the lens that is very important. So, this is the distance travelled by the wave front, but inside the lens its given by $d - \frac{r^2}{2R}$. In addition to this it also travels a certain distance outside the lens if we indicate this distance by, Z So, the distance traverse by wave front outside the lens would be given by $\frac{r^2}{2R}$ plus some Z.

(Refer Slide Time: 07:08)

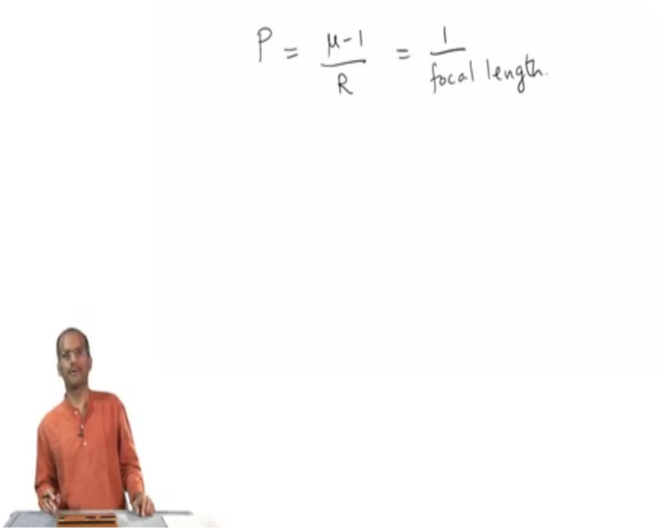


So, here I have essentially equated the times on the left hand side is the time taken to traverse the distance d along the optic axis and on the right inside here is the time taken to traverse some distance in the same time this is going to give me. So, if you remember Z is simply some arbitrary point up till which our ray has travelled in time t value of Z depends on R depends on how far you are from the optic axis and also it depends quadratically on R.

Now, I can go back and draw figure here to indicate this let me show that in blue, so its going to look something like this. What is this tell us? It tells us that all the points which make the wave front they are at a distant Z from this line that I have here and the locus of all the points has this r^2 dependence. So, you could say that in three dimensions the locus of Z is going to define for us as this sphere whose radius is going to be $\frac{R}{\mu - 1}$. This is also means that there is going to be a point of convergence at the centre of that sphere.

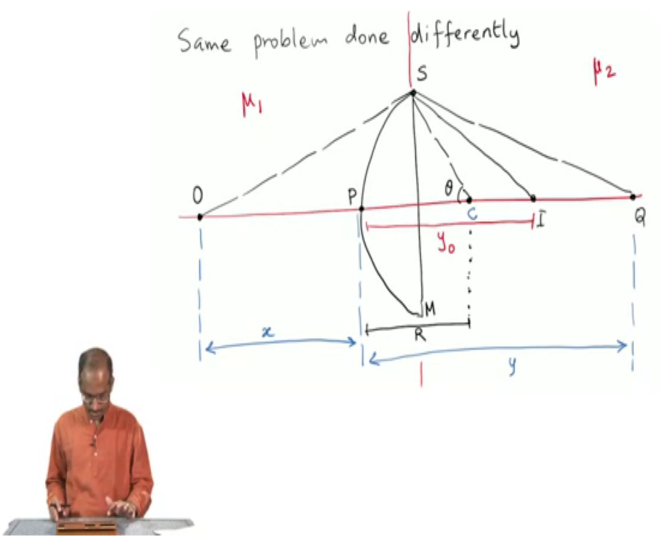
So, in other word if I had an object which is placed at infinity and there is an incoming wave front which you assumed to be a plane wave its hitting this plano convex lens, what this analysis tells us is that all the rays would converge at this point whose radius is given by $\frac{R}{\mu - 1}$.

(Refer Slide Time: 09:20)



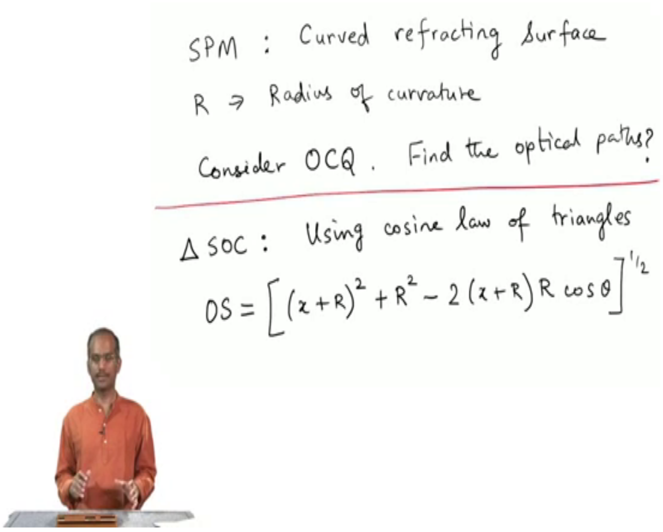
So, power in this case would simply be $\frac{\mu - 1}{R}$ which is equal to 1 by the focal length. So, this is gives as an estimate of power of a lens which is its ability to bend the incoming beam of light.

(Refer Slide Time: 09:40)



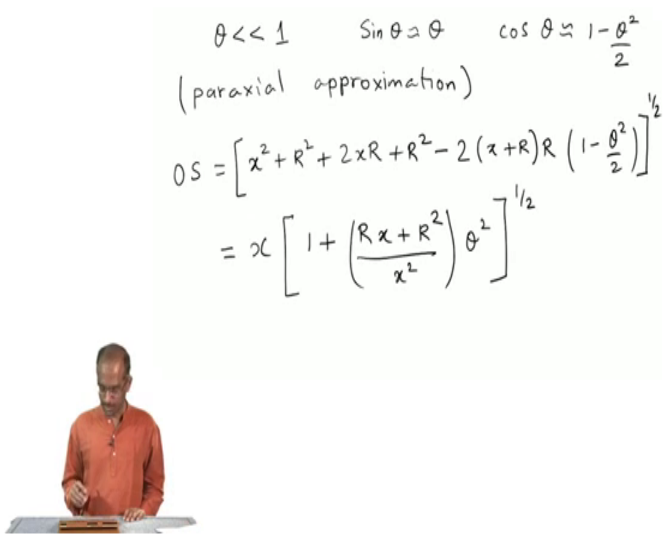
Now, we will redo similar problem, but slightly differently explicitly using Fermat's principle. So, its not a lens what I have is a surface a refracting surface. So, the surface itself is given by this SPM and we have an object let us say at position O here and Q is any point such that OPQ together form a straight line. Of course, given that O is the some object or some position where would this B imaged after refraction on the other side of this curved surface. Capital R is the radius of curvature of this curved surface.

(Refer Slide Time: 10:40)



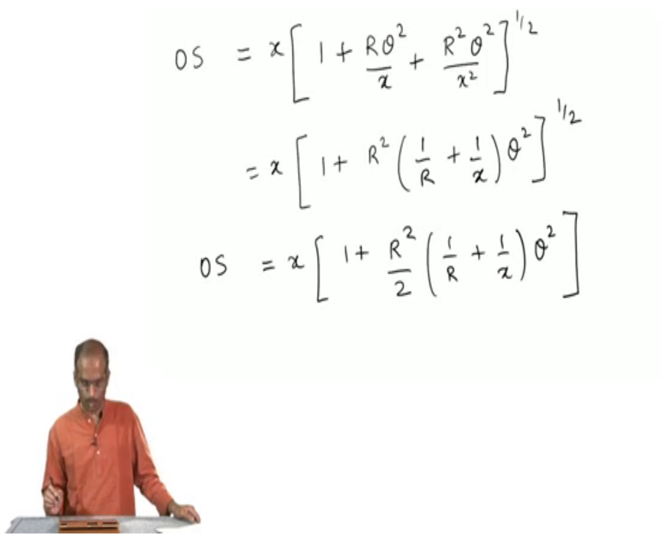
Now, let us start by looking at this triangle SOC using the cosine law of triangles. I can write the following equation for OS you can use cosine law of triangles and write this equation. Before I do anything with these expression I should point out that we will be working in the regime where θ is small.

(Refer Slide Time: 11:04)



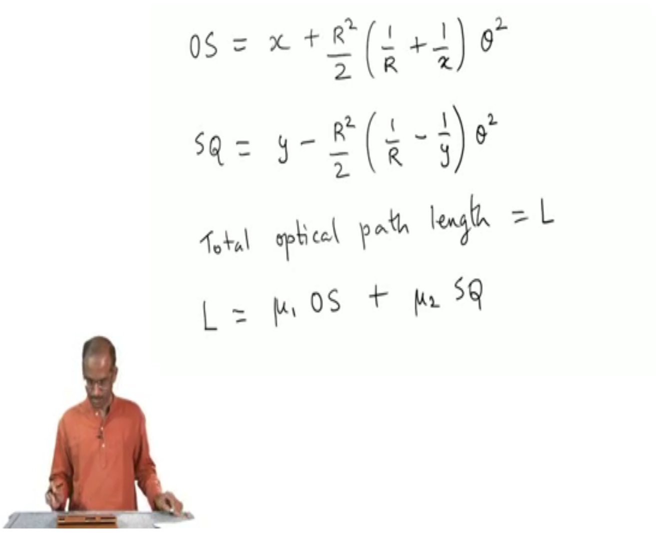
So, this is an approximation in which we will work with and this sort of approximation where the rays or close to the optic axis is called paraxial approximation in optics. So, now I can write OS as follows now this expression can be simplified, I am once again going to use a binomial approximation here, we have an expression for OS now.

(Refer Slide Time: 11:34)



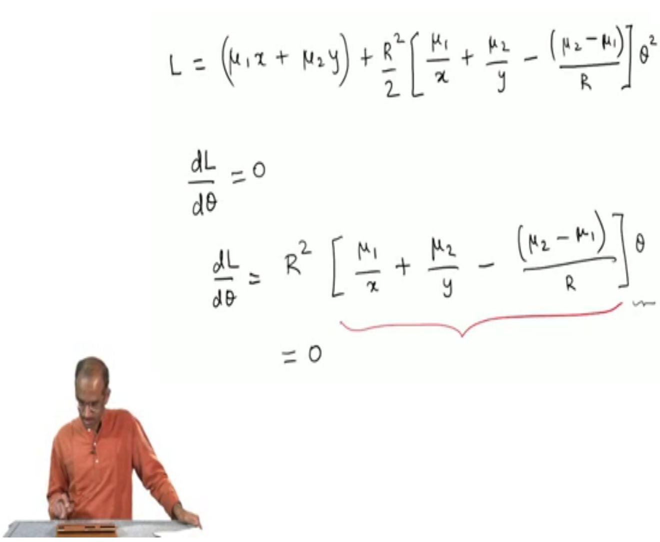
And you can do a very similar calculation to find out SQ. So, I will not do that and directly write the result, but its fairly straight forward just follow the recipe that we just adopted.

(Refer Slide Time: 11:44)



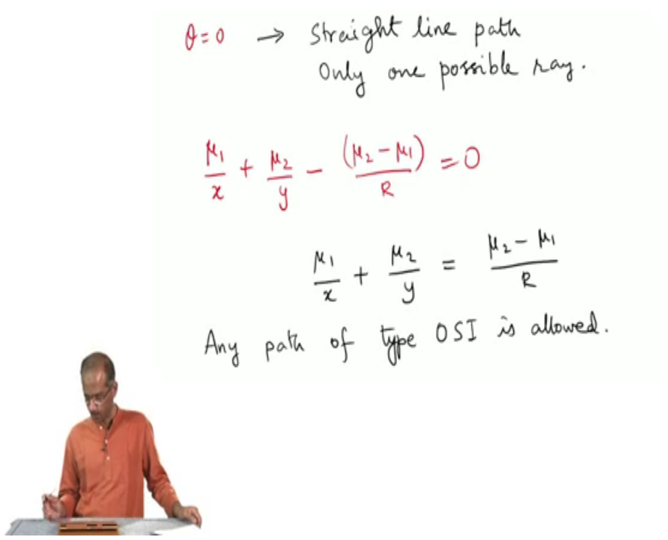
So, I have OS and SQ the total optical path length would be. So, the path length OS as you can see happens in a medium where the refractive index is μ_1 and the path length SQ takes place in a medium where refractive index is μ_2.

(Refer Slide Time: 12:21)



Now, I have collected all the terms together added them and this is my expression for optical path length L. So, I want to find out now $\frac{dL}{d\theta}$ and set it equal to 0, R is the radius of curvature capital R. So, it is not equal to 0. So, what could be 0 or only these two things; one is it is possible that θ could be 0 or this entire expression within the square brackets that could be 0.

(Refer Slide Time: 12:53)

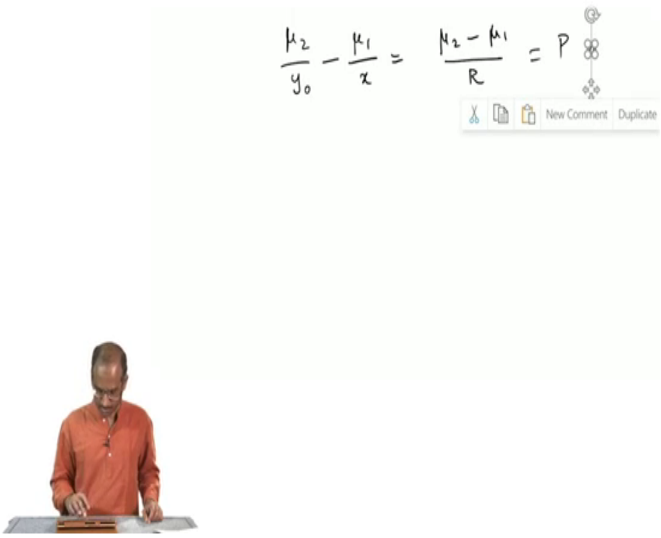


So, theta equal to 0 gives me a straight line path and there is only one possible straight line path. Now, the other way by which $\frac{dL}{d\theta}$ can be 0 is when this quantity within the square brackets is 0 this is what we have obtained from the condition that $\frac{dL}{d\theta}$ should be equal to 0.

If I designate this PI to be some y_0 or the distance of the image from the point P in that case all that this equation tells me is that many possible paths are allowed. So, the point here is that any path of type OSI is allowed provided of course, I identify y with y_0 the distance of the image.

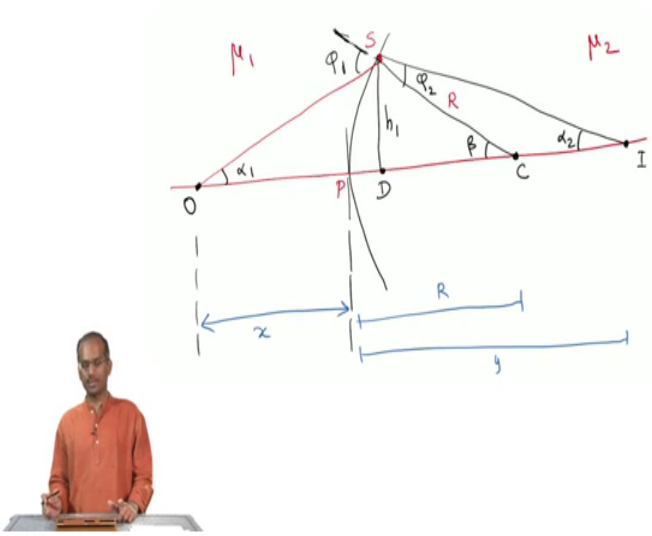
So, typically these kind of equations follow some convention for instance if you go back to this figure that we have here the distances to the left of P generally take a negative sign and the distances on the right of P point P would take a positive sign. So, if I actually apply this convention here this equation could be rewritten as.

(Refer Slide Time: 14:16)



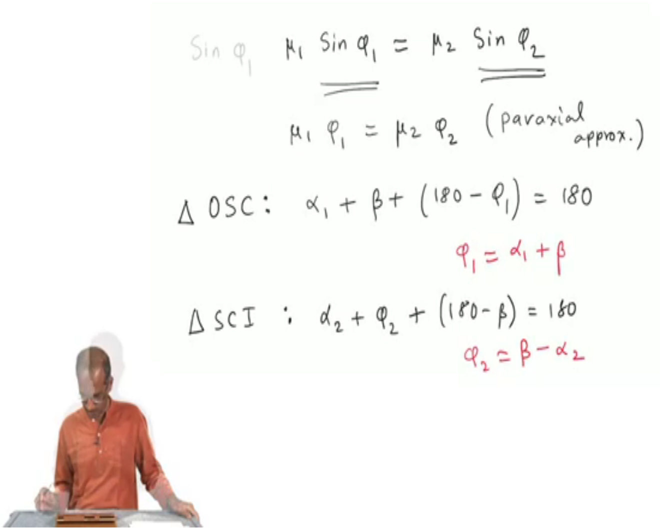
Once again you might recollect that this is simply equal to the power of the lens.

(Refer Slide Time: 14:23)



So, now I have read on the figure slightly differently. So, that we can obtain the same result using Snell's law of reflection. So, there is a beam that goes from point O to S and gets refracted and possibly converges at point I there are various angles it defined α_1, α_2, β and there is also this perpendicular which is of height h_1. So, what is of interest for us are these three distances x which is the distance from point P to O and the distance from P to I which is called y and then there is radius of curvature which is PC in this diagram.

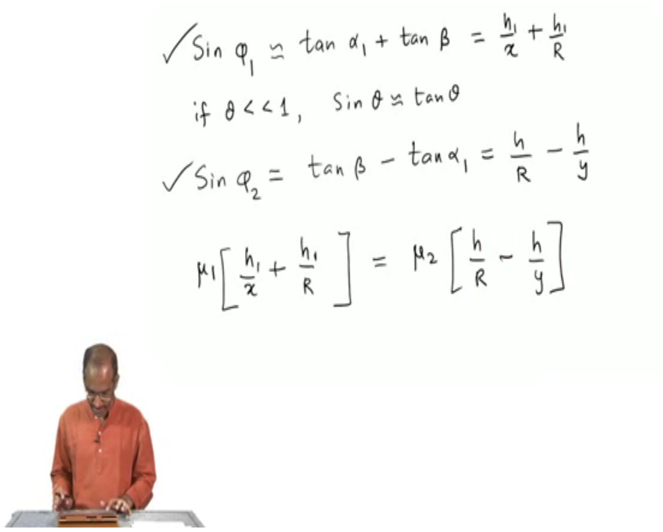
(Refer Slide Time: 15:16)



At the refracting point I should be able to write an equation of this type this is simply a statement of law of refraction, I am going to assume that ϕ_1 and ϕ_2 are small in which case. So, this is what is called the paraxial approximation. So, this line which goes from C to S is the normal at normal to the curved surface at the point S hence this would be our ϕ_1 and similarly this should be my ϕ_2.

So, if I look at this triangle OSC the sum of the angles of this triangle should be 180 degrees this equation would simplified to the following $\phi_1 = \alpha_1 + \beta$. Now, let us also look at the triangle SCI sum of all the three angles is 180 degrees this would give me an expression for ϕ_2 which would be $\beta - \alpha_2$.

(Refer Slide Time: 16:22)

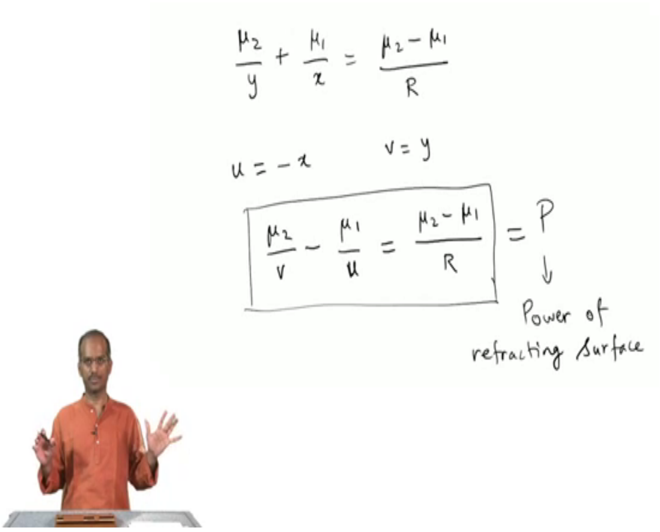


Remind yourself that we are looking at an approximation where all these angles are very small in which case the $\sin\phi_1$ can be approximately written as $\tan\alpha_1 + \tan\beta$. So, I just expand $\sin(a + b)$ and consistently apply that all the angles are small enough and then every time you have a $\sin\theta$ you can replace it by $\tan\theta$.

Again remember that we are working in small angle approximation in which case this distance PD is so small that we can ignore the distance and take this distance O and D to be distance OD to be S self. So, taking PD small would correspond to again thin lens approximation P and D are so close that we can ignore that small difference and take the distance between P and C to be R itself.

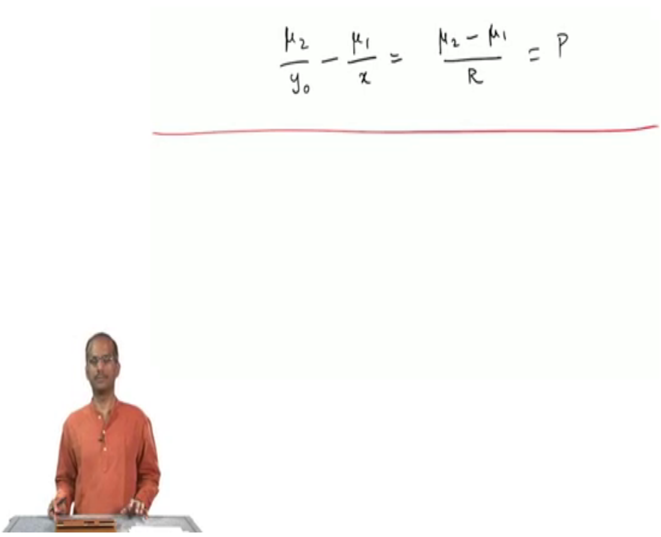
Now, by the same token I can write expressions for $\sin\phi_2$ as well that would be $\tan\beta - \tan\alpha_1$. Now I have expressions for $\sin\phi_1$ and $\sin\phi_2$ I simply need to substitute in this equation. This is the expression that I get when I substituted back in this equation which is the law of refraction. Now, by simply rearrange this equation I can write it in a more suggestive form.

(Refer Slide Time: 18:03)



So, this is the final result I get, but again to put it in a form that is useful for us let us call this distance u to be x, but remember that any distance which is to the left of P is going to get a negative sign. So, u will be $-x$ and v will be y.

(Refer Slide Time: 18:29)



So, this is exactly the result that we just got earlier on. So, we got it by two different methods; one is by actually applying the Fermat's principle and other by using the law of refraction it should not be surprising simply because the law of refraction itself was obtained by using Fermat's principle.

So, it looks like everything is consistent for us and the quantity here on the right hand side $\frac{\mu_2 - \mu_1}{R}$ is again simply equal to the power or P which is the power of lens. You have light beam going from one medium to another medium characterized by refractive indices μ_1 and μ_2. So, we have $\mu_2 - \mu_1$ in the denominator in instead of $\mu - 1$ that we had earlier on.

Now, in the next lecture we will try and put together all these results to obtain equation for thin lens.