Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 43
Fourier Series: Problems

Welcome to the 5th lecture of 9th week. This entire week we were looking at Fourier Series and Fourier transforms.

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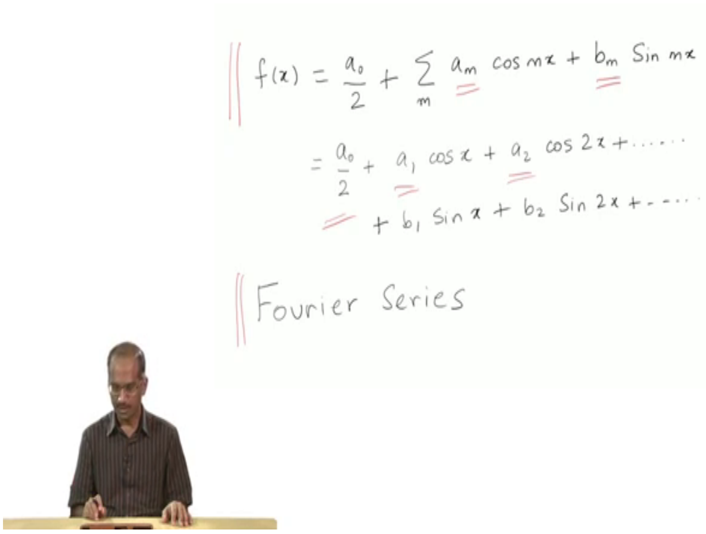

So, let me start by reminding you of what we studied as the Fourier series. This is the Fourier series and $f(x)$ is of course, our arbitrary function. So, as you would remember $\frac{a_0}{2}$ represents the average of the function. And the other terms, the cosine and sine terms capture the oscillation about that average value. For a particular problem, for you if your average is not very important for your purposes you could always set it to 0. But nevertheless, you should keep in mind that there could be instances where the average would be important a_ms and b_ms can be written in terms of an integral.

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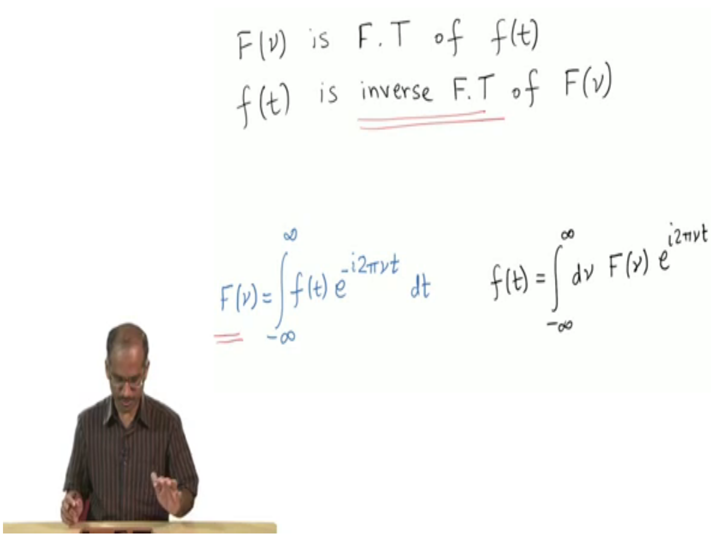

Then we took the continuum limit so to speak. We went from Fourier series to Fourier transform. So, in this case we have a pair of functions. So, given $f(t)$; so here it is more convenient to think of your function as being a function of time, but it does not necessarily have to be a function of time if that were. So, in that case I have this pair of transforms. $F(\nu)$, ν is the frequency in such a case, capital $F(\nu)$ which is this is given by this integral that I have written here and it is simply the Fourier transform of $f(t)$.

In other words, I am looking at the same function in frequency space. So, the function is same, but it looks probably different in frequency space. The important part about this is that the function $f(t)$ need not be periodic. So, remember that when we wrote down this Fourier series, we explicitly assume that function $f(x)$ is periodic, but here we liberate ourselves from that periodicity requirement.

And similarly, you can go the other way around. You can go from capital $F(\nu)$. If a function is given in frequency space you can go to represent the same function in time domain. So, that is called the inverse Fourier transform. With this information, now let us do 3 problems which is what I have planned for this lecture.

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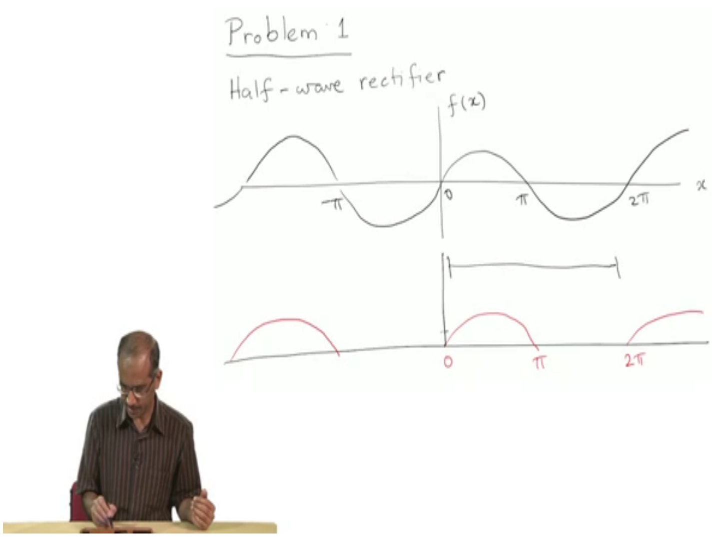

First problem is finding the Fourier series for a half wave rectifier. Now, what does the half wave rectifier do for you? So, it will only keep that positive parts of this waveform, in other words once the rectification has been done our waveform will be simply this. So, it is a periodic function whose basic period is let us say this much going between 0 to 2π, looking at the function you can make some inferences without actually doing any calculation. For instance you can see that the average of this function is not going to be 0. So, clearly the average is going to be somewhere here. It is not exactly a even or a odd function.

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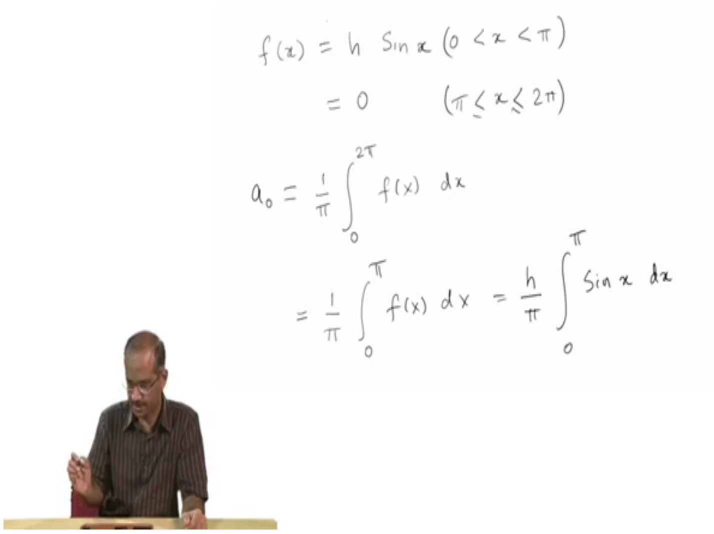

So, here is my function $f(x)$. And even though I have specified the function in the basic unit lying between 0 and 2π, you should remember that it is a periodic function, it spans the entire real axis. Let us first calculate the average value of the function. If you remember this is our basic formula for a_0, the mean value of the function. Conveniently the value of the function between π and 2π is 0, so that would be $f(x)$ is h sin x, 0 to π, sin $x\,dx$. The next quantity I want to find is of course, the a_ns, a_ns are given by $\frac{1}{\pi}$,0 to $2\pi, f(x)$ cos $n\,x\,dx$.

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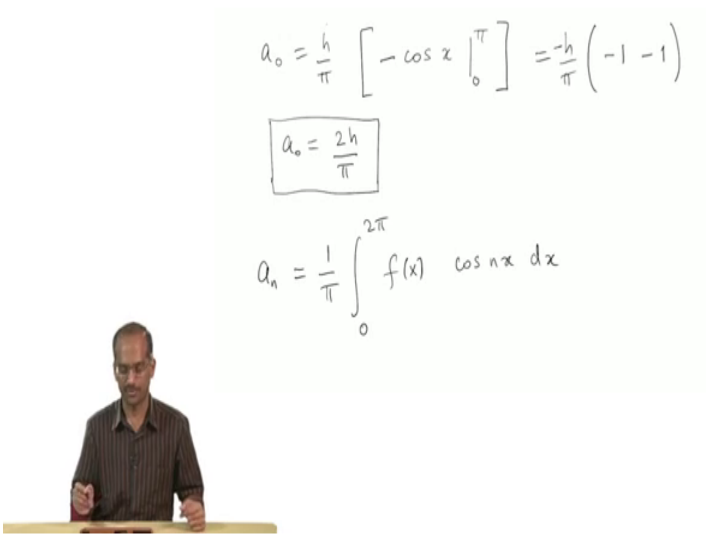

So, remember that these coefficients both a_n and b_n, when you evaluate them through this integral, when the limits of the integral let us say in general go between some range which is l, the factor that comes in front will be 1 divided by $\frac{l}{2}$. So, for instance in this case our x goes between 0 to 2π. So, the factor that comes there should be 1 divided by half of this value. So, half of this is simply π, so which is why I have this $\frac{1}{\pi}$ factor in front of this integral.

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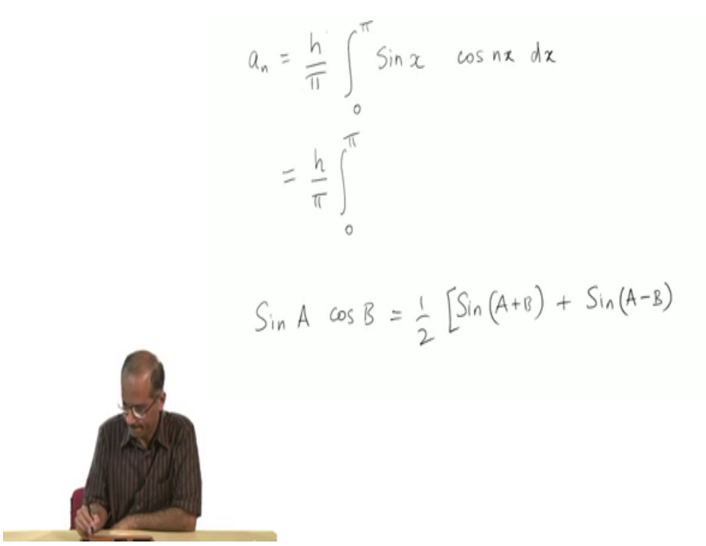

So, I need this relation $\sin A \cos B$ will be $\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$.

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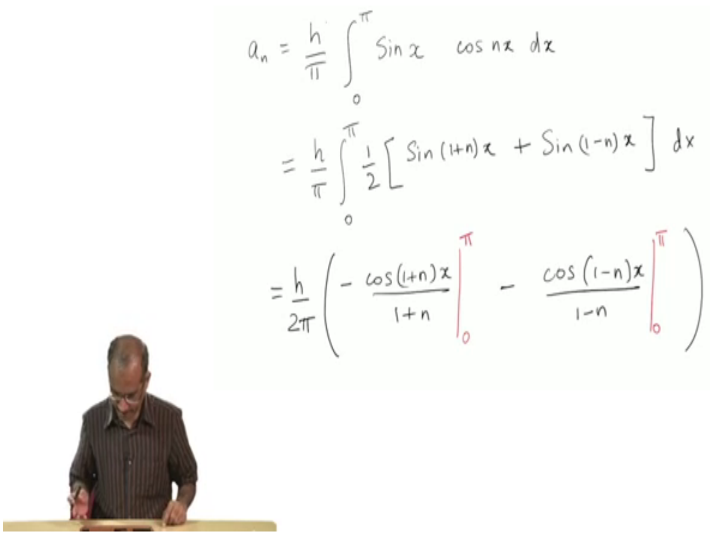

I have done the integral and the limits are there to be put in. So, they were originally two terms, putting these two limits we get 4 terms.

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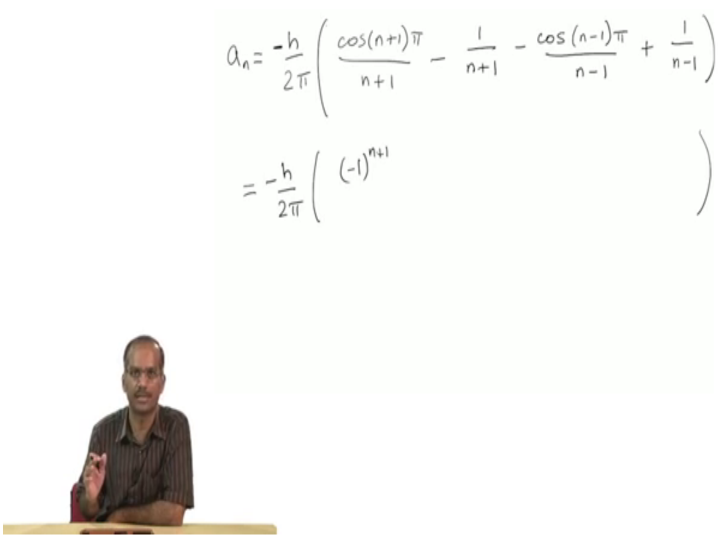

Before I take LCM let me also add this that $\cos(n+1)\pi$ is equal to $(-1)^{n+1}$. So, remember that n is a integer which is why we are able to do this.

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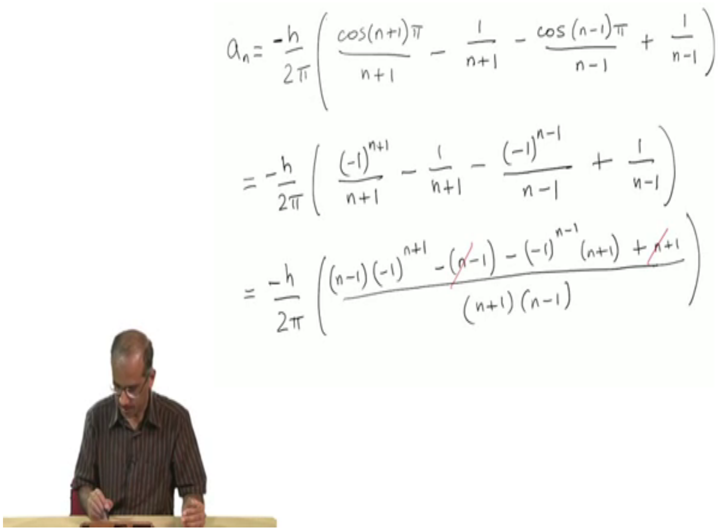

Now, we are ready to take the LCM. So, what I have done is I have taken $-n$; $(-1)^n$ outside, in which case you can put together 4 of those terms like this and this plus 1 and plus 1 together will give you this plus 2 term here.

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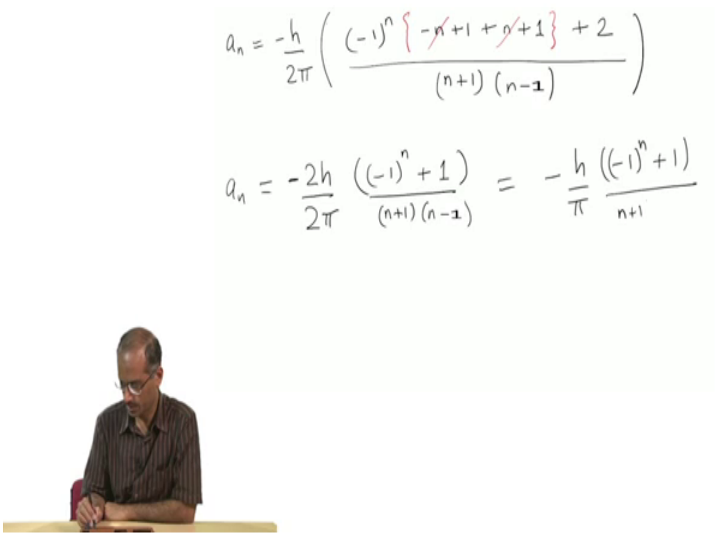

Again, you will notice that *n* and -*n* will cancel out here and then it is simply 2 in the numerator that multiplies $(-1)^n$. So, the entire 2 can be taken out. So, now, I have this final expression for a_n, which will be $-\frac{h}{\pi}\frac{(-1)^n+1}{n+1}$.

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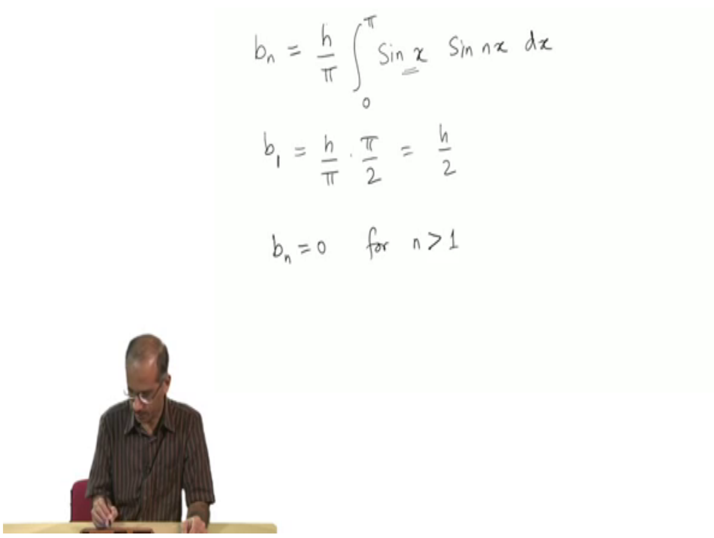

I have written the integral for b_n. Again we need to do the same thing. So, this integral will contribute only when this n is equal to whatever number is here in front of this x. So, in this case it happens to be 1, so the only value for which this integral will be nonzero is if $n = 1$, in which case and all other b_ns are equal to 0 for $n > 1$.

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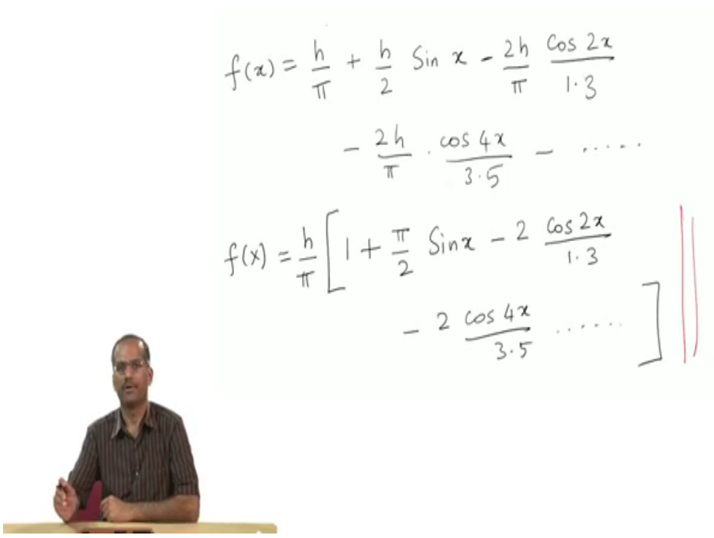

Now, we can put together all the results. $f(x)$ is equal to there is b_1 alone, so let me include that first plus. So, with all these let us reconstruct the function. Now, I can collect together all these terms and write the final expression. So, this is the final result for our function.

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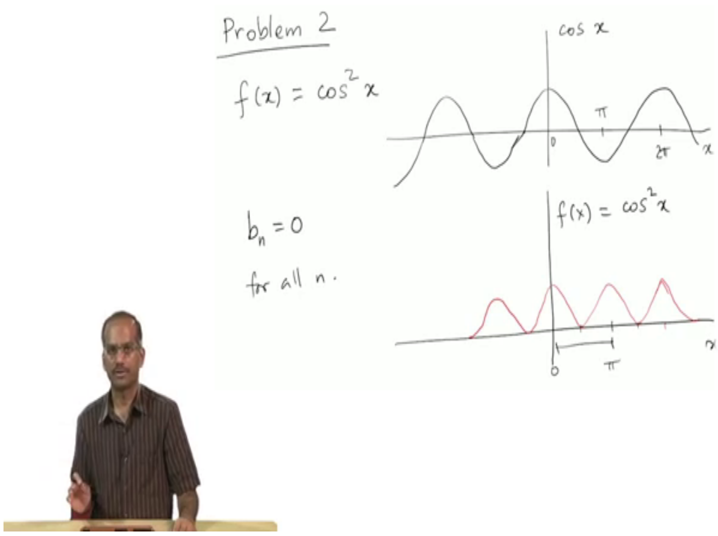

The next problem is a problem where the function is defined only within certain region. Here the function is $f(x)$ is $\cos^2 x$ and we have to find it is Fourier representation. So, let us first plot this function. So, here I have plotted for you the $\cos^2 x$ function. So, one point to notice that it is a even function, that is very clear from the figure and basic periodicity would be from here to here which is going from 0 to π.

So, another thing you can infer is that of course, the average value of the function is not 0 and because it is a even function without doing any calculation you can say that all the b_n coefficients are going to be 0 straight away for all n. So, with this let us calculate only the average value of the function and a_n.

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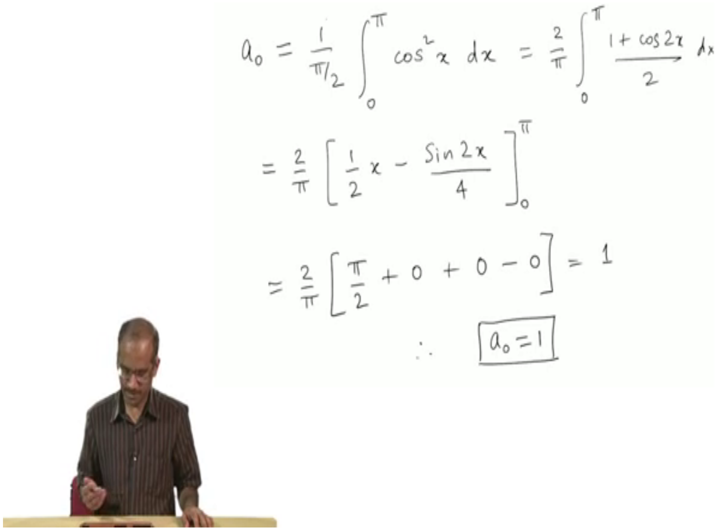

So, here I have written the our standard formula for a_0, the average value of the function, ok. So, I have put in the limits of the integral and it turns out that a_0 is 1.

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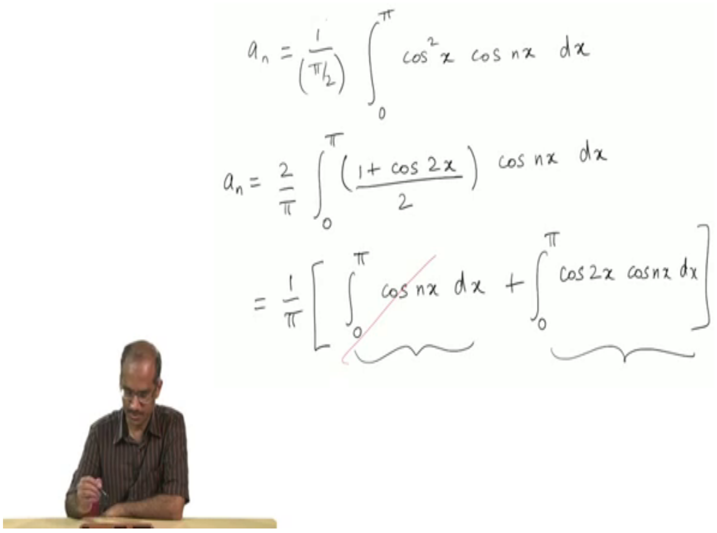

Now, let us calculate all other a coefficients, a_ns. Function is $\cos^2 x$ multiplied to $\cos nx dx$. Again, we replace $\cos^2 x$ by $\frac{1+\cos 2x}{2}$ and the whole thing multiplied to $\cos nx dx$. So, both these integrals are easy to do. Especially, the second integral you should remember that this is like one of the formulas we saw on the very first day when we started looking at Fourier series. In general, what is the integral of $\cos nx$, $\cos mx$.

So, the only term that would survive is when $m = n$. So, looking at this integral it would appear that the only term that would survive from here is when $n = 2$. And you can see that the first term here is an integral over $\cos nx$, that is going to give me $\sin nx$ divided by n and $\sin nx$ irrespective of what the value of n is n being an integer will be 0 both for $x = \pi$ and $x = 0$. So, the first integral is entirely 0.

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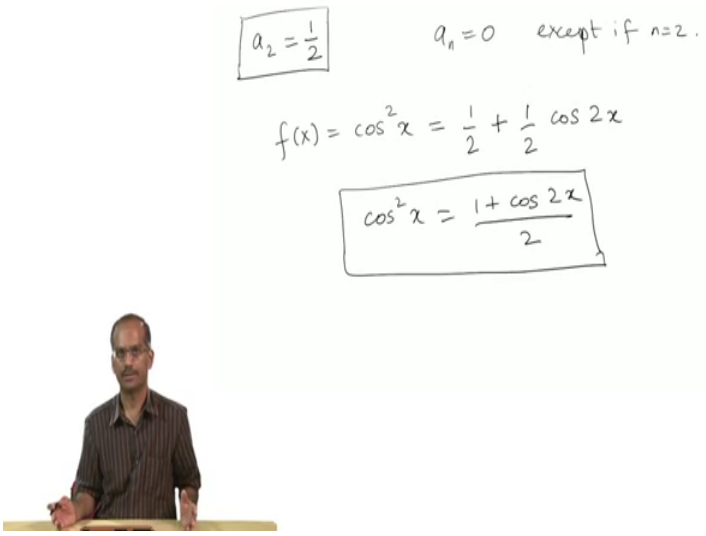

Clearly we are headed towards a situation where all the a_ns are 0 except one for which $n = 2$. So, we will have a situation where a_2 is equal to $\frac{1}{2}$. So, the final result will be $\frac{1}{2} + \frac{1}{2}\cos 2x$ and all other a_ns will be 0 and b_ns are also 0. So, we have the result that $\cos^2 x = \frac{1 + \cos 2x}{2}$, ok. It is not very surprising, this is simply one of the basic trigonometric identities.

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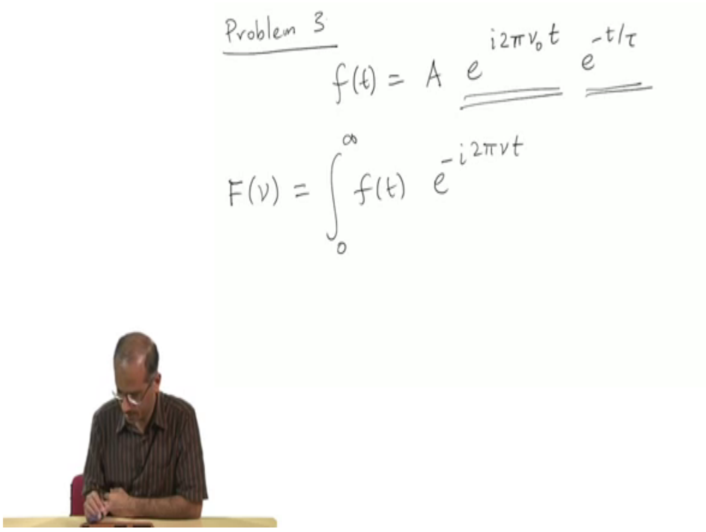

The problem here is I have $f(t)$ which is a product of two exponential functions, one is $e^{i2\pi\nu_0 t}$ and then an exponentially decaying function in time. τ is some time constant. You would notice that this is simply the solution of a damped harmonic oscillator. Now, what I want is to find the Fourier transform of this. So, $F(\nu)$ would be given by since time goes from 0 to ∞ we shall integrate from 0 to ∞. $f(t)e^{-i2\pi\nu t}dt$.

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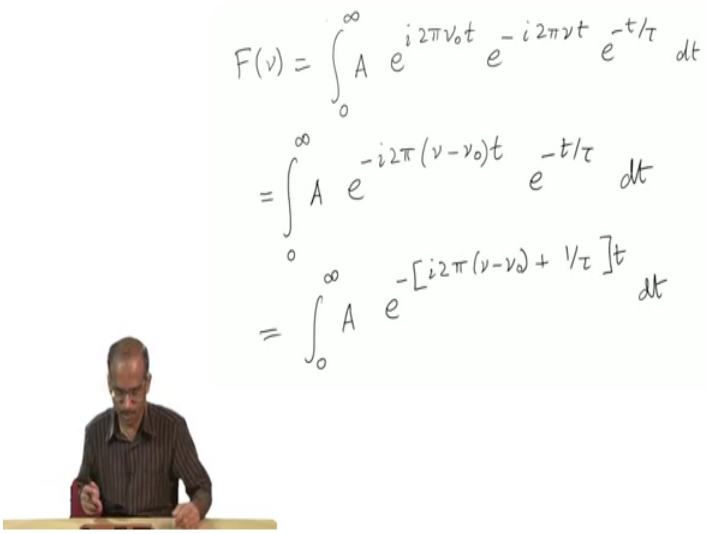

So, I have written it as e power minus some constants into t divide integrated over t.

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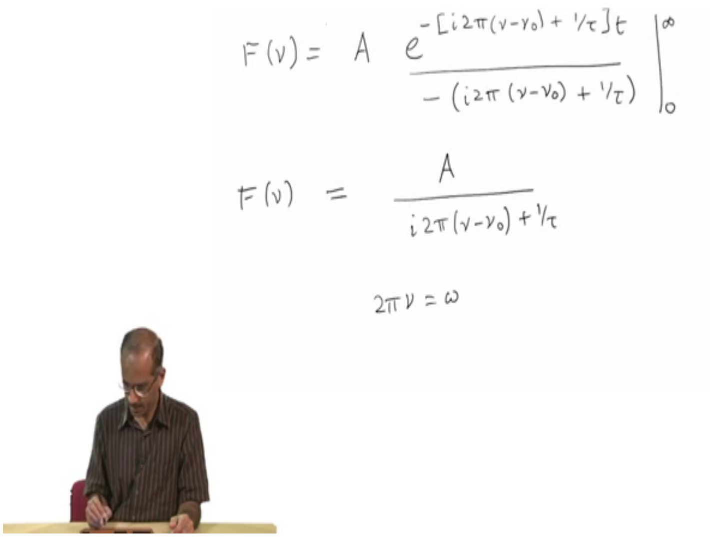

This integral is easy to do again, simply an exponential integral. $F(\nu)$ would be a can be taken outside the integral and this result of integral should be evaluated between 0 and infinity. Now, you can see that if I put the upper limit $t = \infty$, the numerator will simply go to 0 and if I put the lower limit its going to give me 1 and so, the final result will be $\frac{A}{i2\pi(\nu-\nu_0)+\frac{1}{\tau}}$. So, this is my $F(\nu)$. And I will also use the fact that $2\pi\nu$ is ω and $2\pi\nu_0$ is ω_0.

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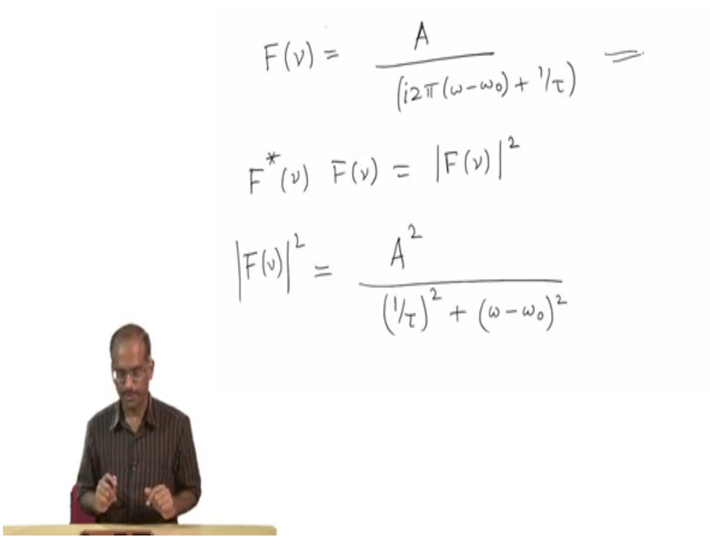

Now, it is more convenient to look at it as what is called a power spectrum. So, we would like to calculate what is $F^*(\nu)F(\nu)$. So, that would give me the modulus square of this and that would be a positive definite function. This i should have been inside. So, let me directly write the result of this $|F(\nu)|^2$ and that is going to give me. So, instead of using ν I have used ν. We know the relation between ν and ω. To make a little more sense of this it is better to see how it looks if I plot it.

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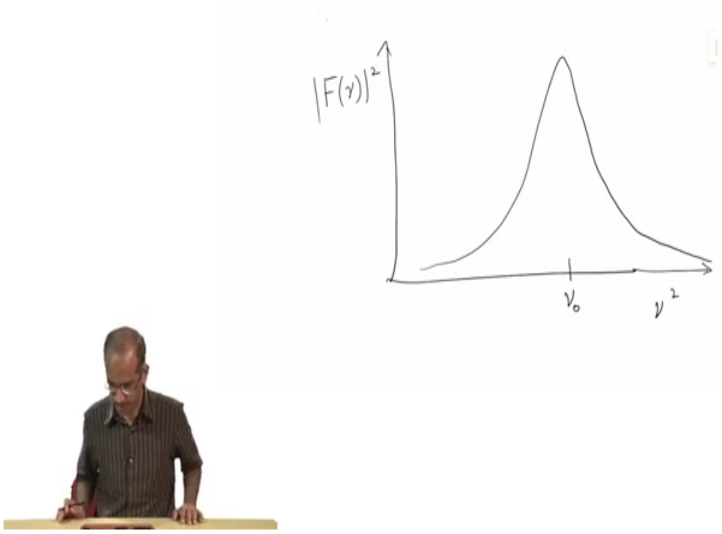

So, this is how the function would look like, the Fourier transform of a function whose amplitude is exponentially decaying. So, this was my $f(t)$ and here I have plotted the power spectrum. The peak occurs when $\nu = \nu_0$. In other words, it is equivalent to saying that ν_0 or the frequency ν_0 contributes most to this function, all other frequencies contribute relatively lesser and there is a decay around ν_0.