Waves and Oscillations

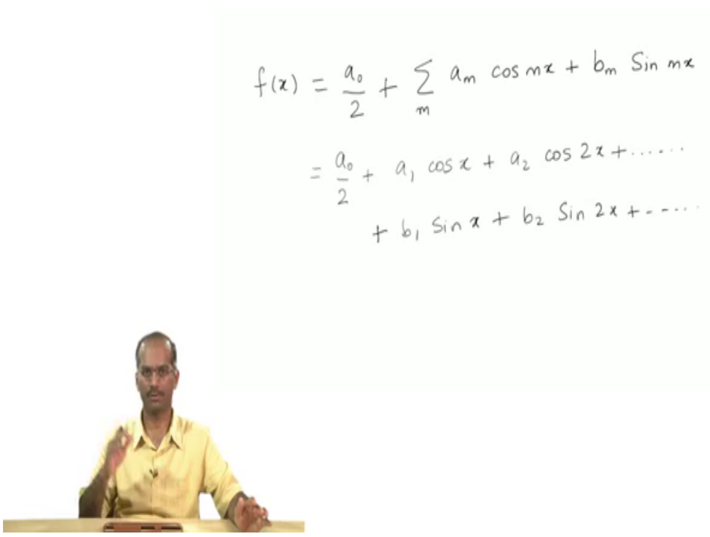
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 42

Frequency Spectrum and Fourier Transforms

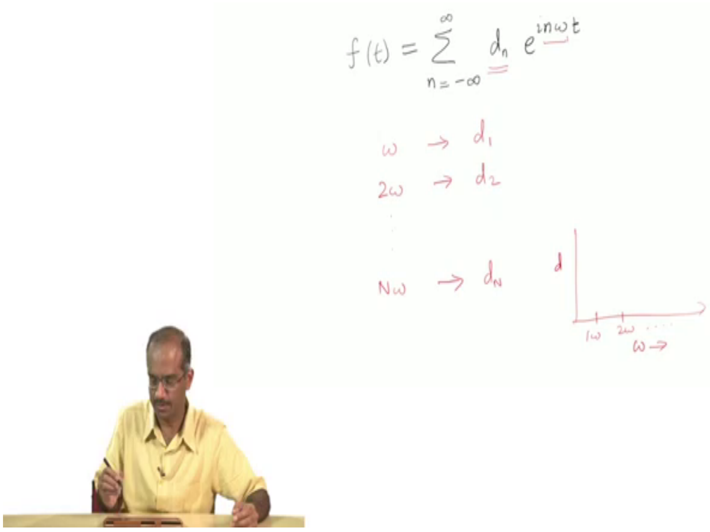
Welcome to the 4th lecture, we are in the 9th week. This entire week we started with the Fourier series and we will continue with that. In today's lecture we are going to look at two different things.

(Refer Slide Time: 00:32)



Let me begin by again flashing the same old slide that for a well behaved function periodic function we defined our Fourier series and it looks like what is shown here. So, you had the average part which is given by $\frac{a_0}{2}$ and then all the oscillatory parts are taken care of by the sine and cosine term.

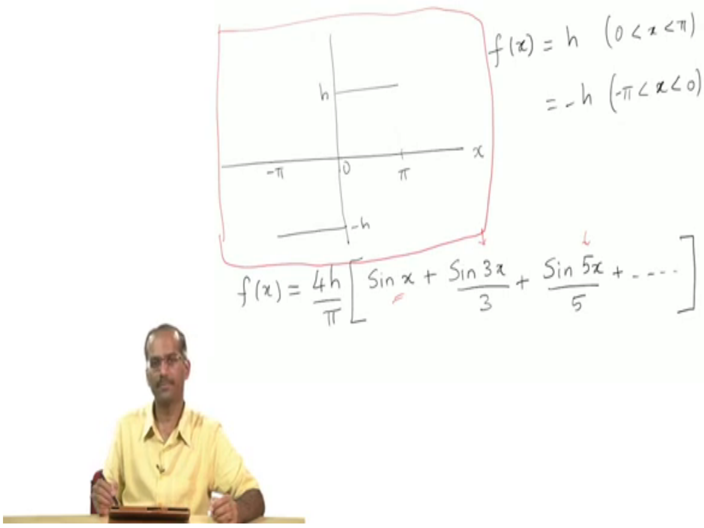
(Refer Slide Time: 00:59)



And let me write the Fourier series in a slightly different form which we did not quite use, but it is also a valid way of writing a Fourier series. On the right hand side I have the Fourier series expansion written in terms of the exponentials $e^{i\theta}$s and crucially you will see that it depends on this $n\omega$ here. I can see this function in the space of these $n\omega$s. So, the space of course, would be in terms of 1ω, 2ω, 3ω and so on and the amplitude corresponding amplitudes would be d_ns; d_1, d_2, d_3 and so on.

In general if I had some $n\omega$ that would correspond to an amplitude d_n. So, I can plot ω_2, ω_3, ω and so on the x axis and plot d_1, d_2, d_3 those values on the y axis. This is equally valid way of stating the same function $f(t)$. It is just that now we are we have transformed it in what would be called the Fourier space or frequency space because as you can see the x axis would now be made up of frequencies. So, it would have let us say 1ω, 2ω and so on and here of course, there will be the values of d that is plotted. Let us go back to the problem that we did in one of the earlier lectures.

(Refer Slide Time: 02:43)

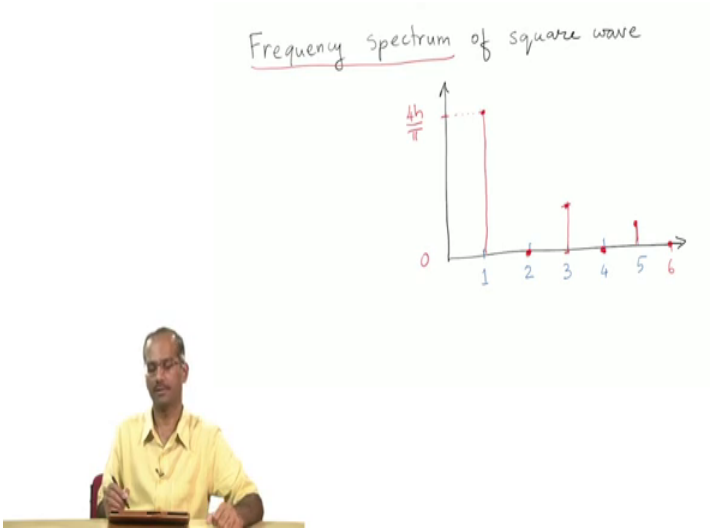


So, that was the problem of square wave. So, between the values of x lying between 0 and π, the square wave had value h and between $-\pi$ and 0 the function was $-h$ and then outside of this range between $-\pi$ and $+\pi$ it repeated itself. So, it is a periodic function. This is its basic periodicity and we calculated the Fourier series for this function, let us recall the result again.

So, $f(x)$ has this common factor $\frac{4h}{\pi}$ and if you take that out then it is a series of sine functions and as I pointed out when we first derived this you will see that the denominator is actually increasing; it is 1, 3, 5, 7 and so on. So, the contribution of the terms with higher frequencies is getting lesser and lesser. Now, let us see it as a frequency spectrum.

So, the x axis is going to be made up of these 1, 2, 3 and so on which are the numbers which are appearing here for instance these ones and I am going to plot the amplitude which is for the case of $\sin x$ the amplitude is $\frac{4h}{\pi}$, for the case of $\sin 3x$ the amplitude is $\frac{4h}{3\pi}$ and so on.

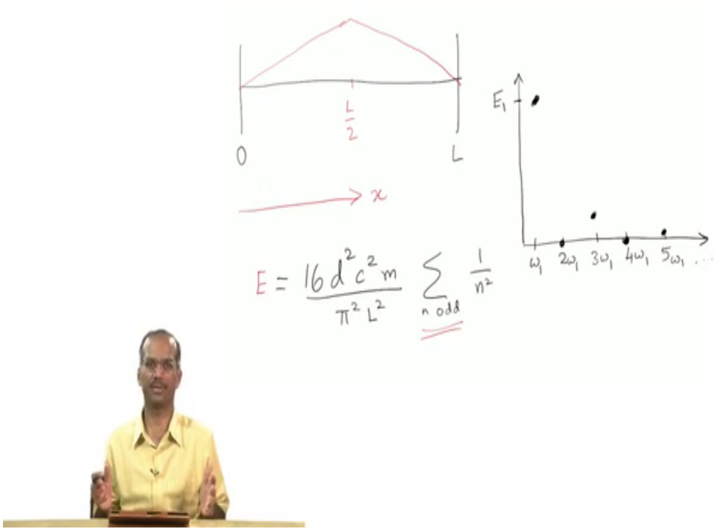
(Refer Slide Time: 04:21)



So, here is my frequency spectrum plot when $m = 1$ you can see that I have drawn a line all the way up to $\frac{4h}{\pi}$ in y axis for the case of 2 it is of course, 0; but for the case of 3 it is $\frac{4h}{3\pi}$. So, it should be one third of this height. So, let us say roughly it is somewhere here and again for $m = 4$ it is 0 and for $m = 5$ it will be one fifth of the height at $m = 1$. So, it is even smaller and so on and of course, at 6 it is going to be 0 again.

So, what we have plotted is a frequency spectrum. With the information given here we can exactly reconstruct $f(x)$ again. Of course, we need the information about all the frequencies, we would just drawn for few of them, but if you have the information corresponding to all the infinite frequencies then you can exactly reconstruct $f(x)$. So, here is my function in the position space and here is the same function in frequency space.

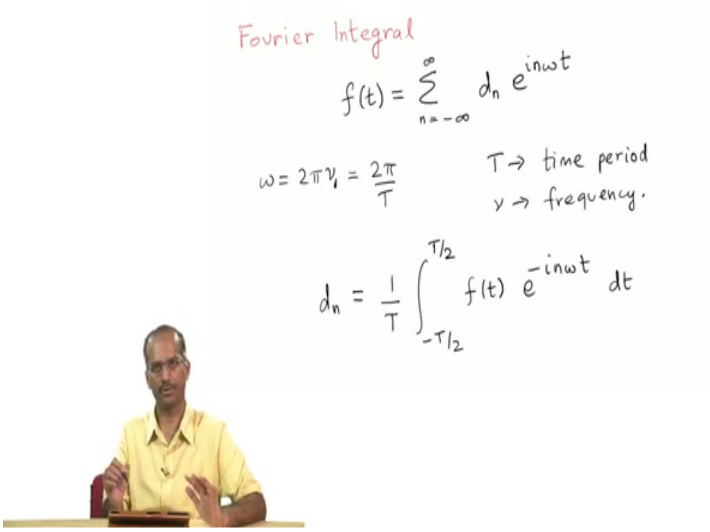
(Refer Slide Time: 06:05)



Let us also look at another example that we did before this was for the case of string that is tied between two ends. The two ends are between 0 and L and we said that we will plug the string at the midpoint and it is going to display standing waves. So, we obtained the energies of the oscillatory system. So, this you can think of as your function in position space. Now, let me plot this in frequency space I am going to label this as ω_1 of course, then there will be $2\omega_1$, $3\omega_1$, $4\omega_1$ and so on.

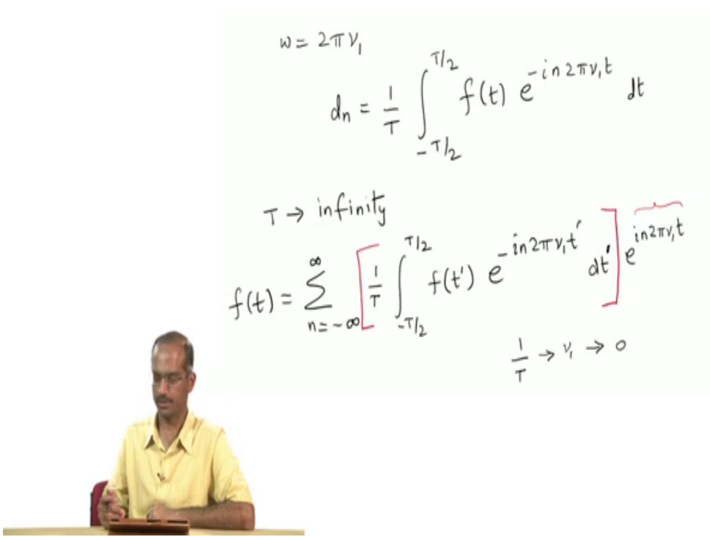
And let us say that the energy in the mode corresponding to frequency ω_1 is E_1 in which case let me say that would correspond to some point here and that is E_1. You can see that most of the energy is concentrated in the fundamental mode and it also pictorially nicely tells us that all the even frequencies do not contribute any energy and the contributions from higher harmonics, higher odd harmonics really get very small very quickly. Let me say that it is a way of seeing the same function in Fourier space.

(Refer Slide Time: 07:42)



If $f(t)$ is written as a Fourier series like this what would d_n be? Simply multiplied by $e^{-im\omega t}$ on both sides and integrate it over time. You will see that on one side only when n and m are equal would that be a non-zero contribution and hence we will get this result.

(Refer Slide Time: 08:10)

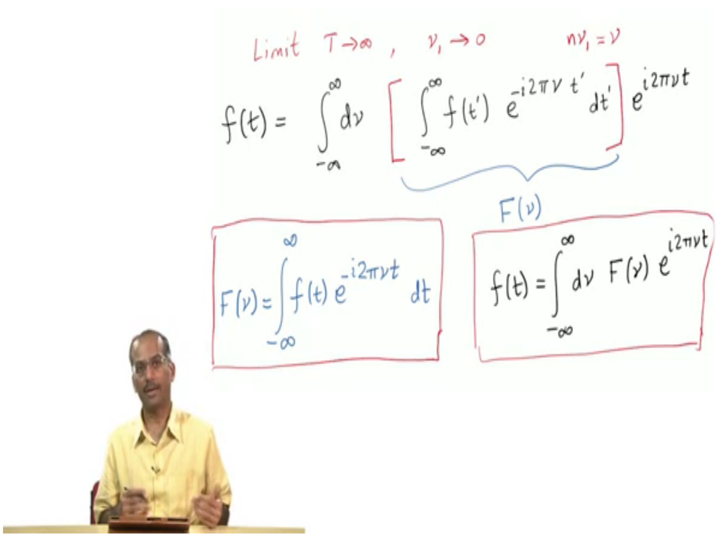


I am going to introduce slightly different variable let me call ω as $2\pi\nu_1$ instead of ν itself. So, let me also make that change here. So, all I have done is to simply substitute ω by ν_1. So, you can think of this as your fundamental frequency and all other frequencies are some integer multiple which is why you have that n here. So, it is integer multiple of this basic frequency ν_1. Now, I am going to take the limit of $t \rightarrow \infty$. This will allow me to handle all kinds of functions even when they are not periodic.

Now, what I am going to do is to substitute this d n in this expression for $f(t)$ that I have on the top. So, d_n is the one that comes within these red square brackets since $f(t)$ involves the $e^{in\omega t}$ I need to differentiate that t from the t that would occur in the definition of d_n and hence I have put t' here; should not matter because it is simply a local t' is simply a local variable or what would often be called a dummy variable.

Now, let me take the limit that $T \rightarrow \infty$. So, when T becomes really large $\frac{1}{T}$ becomes small $\frac{1}{T}$ which is $\nu_1 \rightarrow 0$ and also notice that when $T \rightarrow \infty$ I can take the limits of these integrals within the red square brackets to be going from $-\infty$ to $+\infty$. So, they are going from $-\frac{T}{2}$ to $+\frac{T}{2}$. Now, since T goes to infinity that will change and $\frac{1}{T}$ is going to become $d\nu$ we have this factor $n\nu_1$ here in this in the exponent here n times ν_1 you can take n sufficiently large enough such that $n\nu_1$ is a variable which I would like to call ν.

(Refer Slide Time: 10:34)

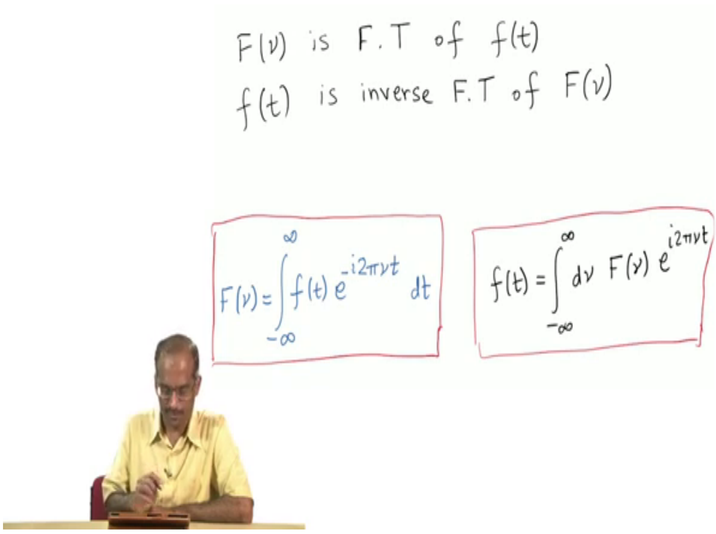


Since I am replacing ν_1 by ν, now it becomes a continuous variable because n goes from -∞ to $+\infty$. So, in that case I can replace this infinite summation here by an integral over

ν. The quantity inside the red brackets would be our $F(\nu)$ let us separate it and write it out. In this case small $f(t)$ would can be written as follows that would be $-\infty$ to $+\infty$ and of course, this is $F(\nu)$ times the exponential.

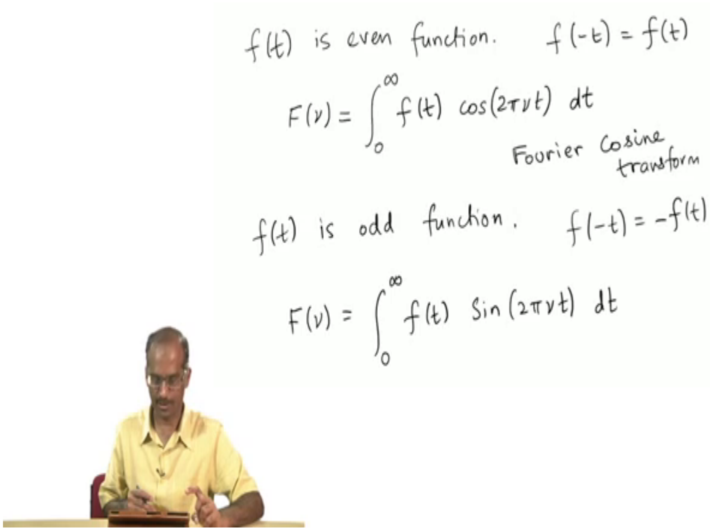
The relation that you have on the left hand side tells you that if you integrate $f(t)$ over $e^{-i2\pi\nu t}$ integrate over time that gives you $F(\nu)$ capital $F(\nu)$ which is the Fourier transform of $f(t)$.

(Refer Slide Time: 11:56)



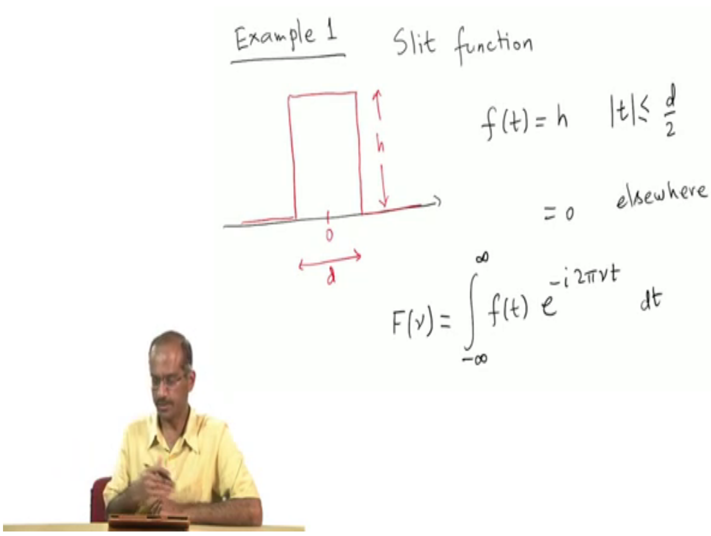
So, the whole exercise can be restated as follows that. So, this you can think of as a Fourier series that is valid for continuous variables and a nice feature is that now you do not even require your function, let us say $f(t)$ to be a periodic function. So, you done this trick of saying that the function has infinite periodicity. These would be the general definitions for Fourier transform and in the literature you will find many different ways of writing the Fourier transform. There would be simplifications if you know some symmetry properties of the original function $f(t)$.

(Refer Slide Time: 12:43)



Suppose, you know that function $f(t)$ is an even function in other words $f(-t)$ is equal to $f(t)$. If this property is satisfied in that case the sine part of this exponential will drop out. So, this is called the Fourier cosine transform. Similarly, if $f(t)$ is odd function by which I mean that $f(-t) = f(t)$ in that case you will get Fourier sine transform of course, I should correct this factor here this should have been $f(t)$. Now that we have all these definitions in place let us see what kind of results it gives for some typical cases.

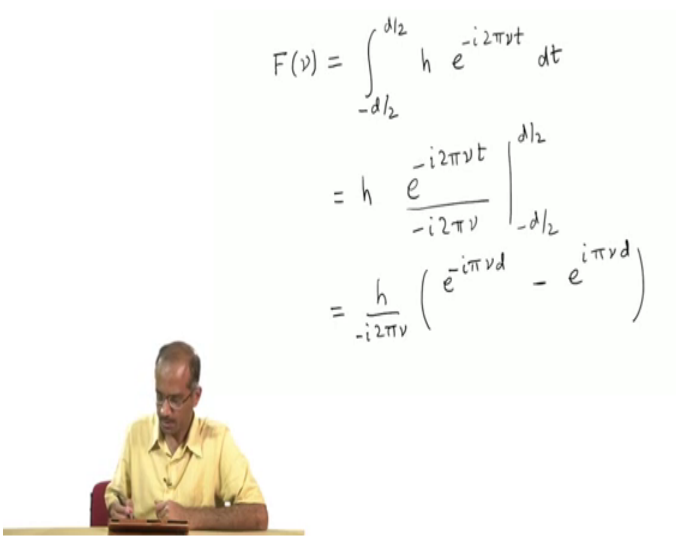
(Refer Slide Time: 13:35)



My first example is what would be called the slit function. So, you can imagine that this rectangular form or the slit function is defined in time domain. I want to do a Fourier transform and look at the same function in the Fourier space or in frequency space. So, the function can be defined as $f(t) = h$, if t is lesser than or equal to $\frac{d}{2}$ or maybe more correctly if modulus of time is lesser than or equal to $\frac{d}{2}$.

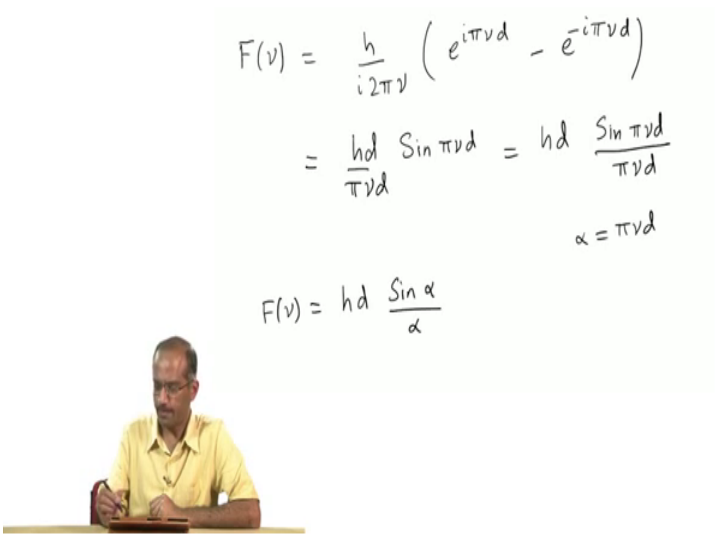
Now, to obtain the Fourier spectrum we simply need to use the formula that we just obtained for Fourier transform and before we do that I should also point out that this function will be 0 elsewhere. And $f(t)$ here is h, but it is equal to h only in the narrow region where $|t|$ is less than or equal to $\frac{d}{2}$ and everywhere else the function is 0. So, the integral itself would go to 0.

(Refer Slide Time: 14:45)



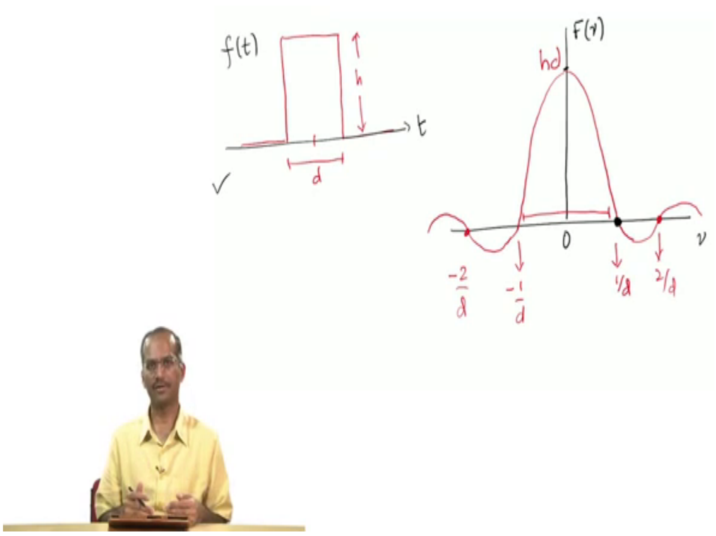
And, this integral is again easy to do; you can take h out of the integral. So, I have substituted the limits now and it is easy to see what manipulation needs to be done.

(Refer Slide Time: 15:00)



If you would like you can just change the notation and call this $\pi\nu d$ to be α in which case $F(\nu)$ would become $hd\frac{\sin\alpha}{\alpha}$.

(Refer Slide Time: 15:22)

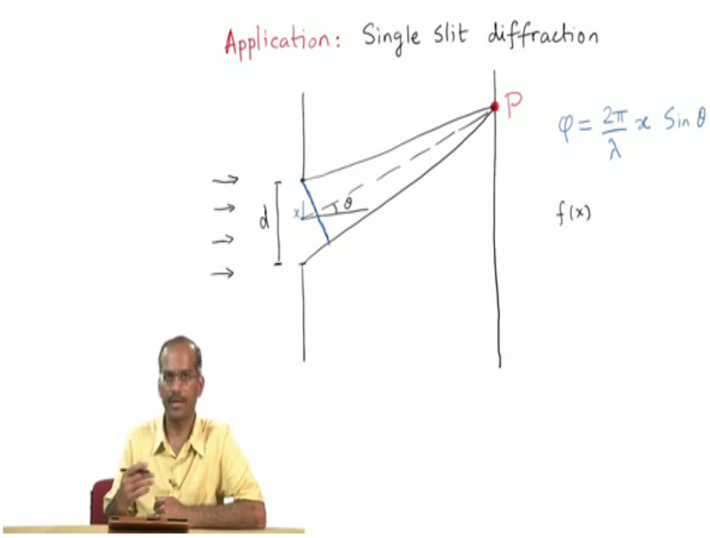


Now, we can plot the Fourier spectrum. So, the x-axis is ν and y axis is of course, capital $F(\nu)$ and as $\nu \to 0$, $\alpha \to 0$ which means that $\frac{\sin\alpha}{\alpha}$ is 1 at $\nu = 0$ the value is hd here and it is going to decrease on either side of 0. And, since we have already seen this kind of a relation let me directly plot the function. Now, if you focus on this point. So, this is the point at which $F(\nu) = 0$.

So, you can ask the question when does $F(\nu)$ become 0. So, that would happen if ν equal to $\frac{1}{d}$. So, this value is $\frac{1}{d}$ and similarly, this value should be $-\frac{1}{d}$ this you can take as one measure of maybe the width of your Fourier transform function. So, that is twice $\frac{1}{d}$ or $2 \times \frac{1}{d}$ for comparison I have shown the original function $f(t)$ here which is a function of time. You will notice that d is the width of our rectangular slit function.

Now, when you look at the Fourier transformed function you see that the width is actually a function of $\frac{1}{d}$, in this case it is $\frac{2}{d}$. So, broader of function in time domain narrower it is going to be in frequency domain. It is not something very specific to this choice of function; slit function in fact, it is a more general truth. Let me now mention one physical application of this Fourier transform of a slit function.

(Refer Slide Time: 17:30)



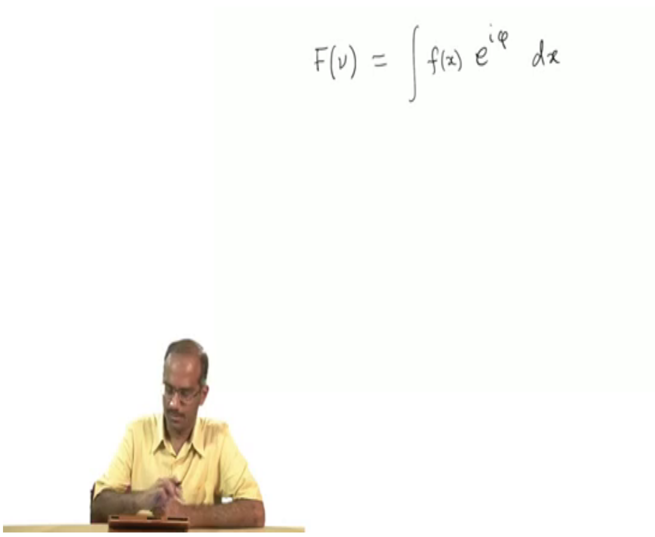
The application is the single slit diffraction we will probably see it in little more detail in subsequent weeks, but let me just mention how Fourier transform is used here. So, the problem is the following we have a single slit that is shown in their diagram the width of the slit is d and you have parallel beam of light coming from left hand side and we will assume that the wavelength of light is about the same as d which is the slit width.

Now, the amplitude of this light as it passes through this slit would be like this $f(t)$ which is shown here which is why it is called the slit function; at this slit diffraction takes place. What I would like to know is the following: what is the intensity at some point p which is some distance away from this screen which has the slit d?

So, if I choose x to be some distance away from the midpoint the phase difference would be a formula like this $\frac{2\pi}{\lambda} x \sin\theta$. To find the intensity at certain point p like this I need to simply add the contributions of light coming from every point along this slit and each point acquires a slightly different phase from the previous one.

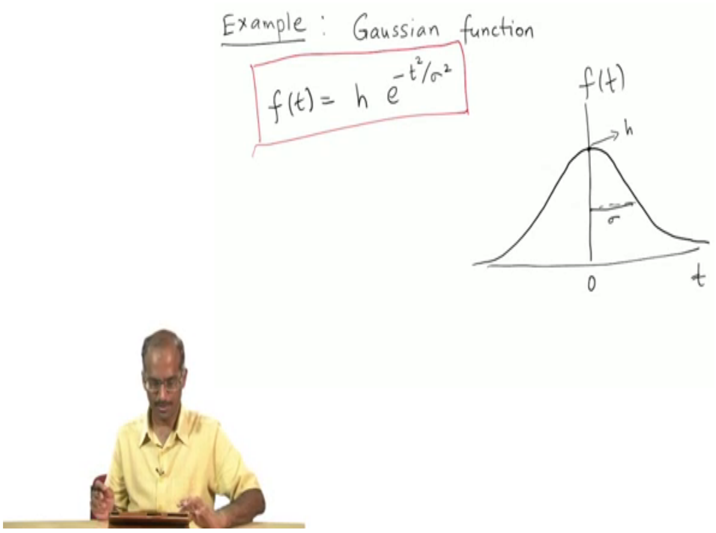
If you count x from the midpoint onwards as a function of x it tells you what is the phase. So, phase changes linearly with the x that is what is expressed by this formula provided of course, θ is the angle of diffraction. Let us say at the midpoint my function is $f(x)$ a little bit away from the midpoint my function will be $f(x)$ multiplied to $e^{i\phi}$. I need to integrate over these phase differences which means I need to integrate over x itself.

(Refer Slide Time: 19:52)



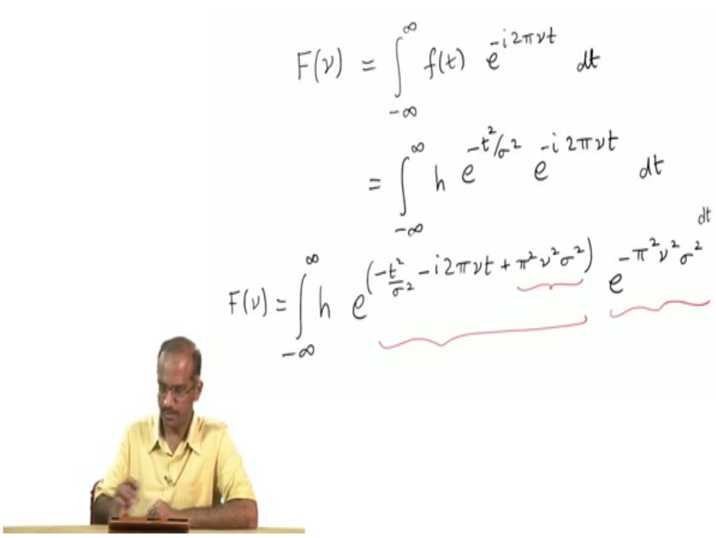
So, I can write the intensity in Fourier space you will realize that the result of this whole process is that finally, it gives you a function $F(\nu)$ which looks like what we have calculated for the slit function. So, in other words the intensity at the central point will be maximum and it is going to decrease on either side. So, that is going to be this succession of dark and bright bands.

(Refer Slide Time: 20:21)



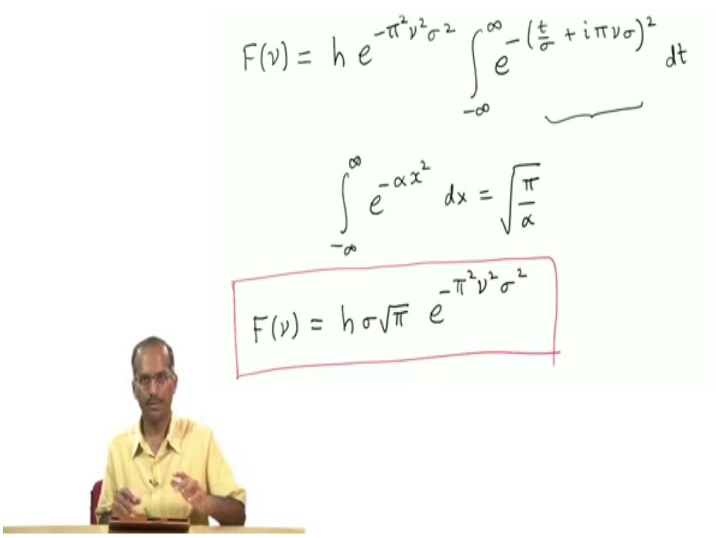
Another example that I would like to give you is that of a Gaussian function. So, I written the Gaussian function here let me just plot it for you $f(t)$ as a function of t look something like this and this is $t = 0$. So, this value is h and at half that value this width would be σ. So, it is a function of t, there is a parameter which specifies the width and another one which specifies the height; height is given by h, σ is the width. I need to find the Fourier transform of this function.

(Refer Slide Time: 21:05)



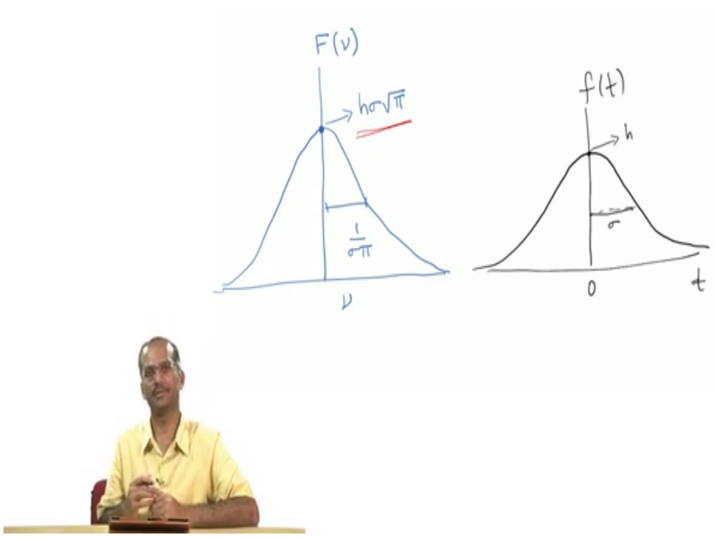
As usual our function is defined of course, in the domain between $-\infty$ and $+\infty$. So, t can go from $-\infty$ to $+\infty$. So, $F(\nu)$ the Fourier transform of $f(t)$ would be. So, the way that one handles these kind of integrals is by a process called completing the square. What I have done is to multiply and divide by $e^{\pi^2\nu^2\sigma^2}$, three terms in the exponent becomes this quantity here.

(Refer Slide Time: 21:38)



Now, we can use the standard result that $e^{-\alpha x^2} dx$ when integrated between $-\infty$ to $+\infty$ gives you $\sqrt{\frac{\pi}{\alpha}}$. So, if I use this result $F(\nu)$ can be written as this is the final expression for the Fourier transform of a Gaussian function. $F(\nu)$ which is a function of frequency is also another Gaussian function. So, what is the difference? If the width of your original function is large the width of the Fourier transform function even though it is still a Gaussian, it is going to be much smaller. Let us see that visually by plotting it.

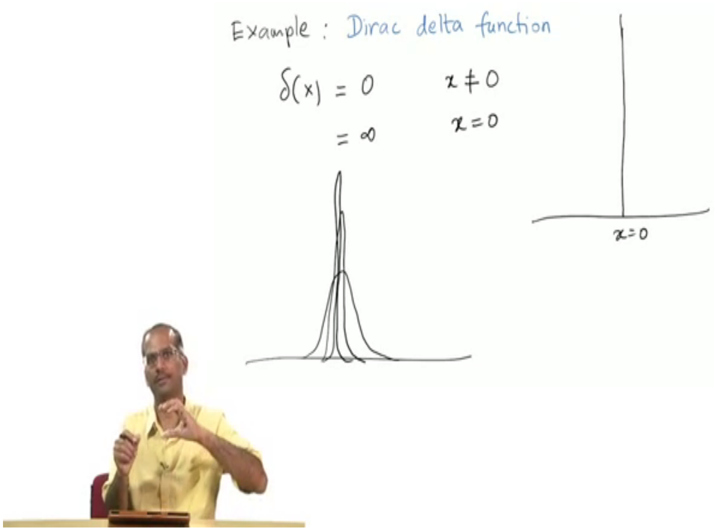
(Refer Slide Time: 22:36)



Here I have plotted both the functions. The original function $f(t)$ is there on the right hand side. So, you notice that height of the function is h and width of the function is given by σ. On the other hand, on the left hand side I have plotted the Fourier transform function we just obtained this result for the Fourier transform of a Gaussian function and it turns out that Fourier transform of a Gaussian is another Gaussian.

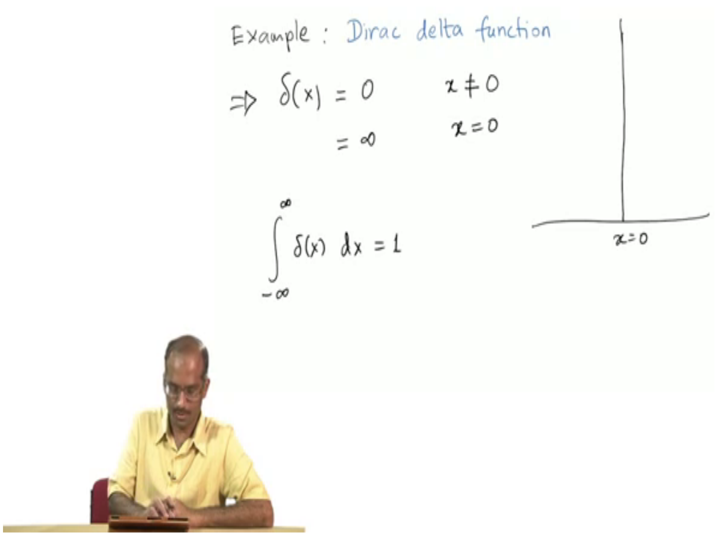
In terms of their shapes they are nearly the same, they look alike. They are not quantitatively equal. The width of $f(t)$ is σ for the Fourier transform function the width is $\frac{1}{\sigma}$ times some constant. If the original height is h and if you assume that σ is much larger than 1 it is going to be h times σ. So, what you are going to really get is a function that is narrow in width, but very sharply peaked.

(Refer Slide Time: 23:32)



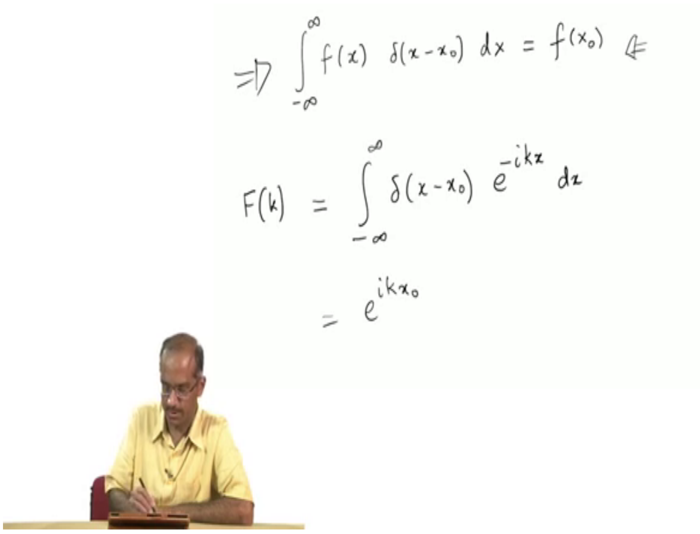
Let me conclude this section with the last example which is the Dirac delta function it is equal to 0 if $x \neq 0$ and is equal to infinity at $x = 0$. To think about it you can imagine that you have Gaussian like function and the width is getting narrower and narrower. As we just saw as the width gets narrower and narrower the height increases. So, you are going to see something like this. So, you can think of your Dirac delta function as a limiting case of this process of making the width smaller and the height getting larger and larger.

(Refer Slide Time: 24:17)



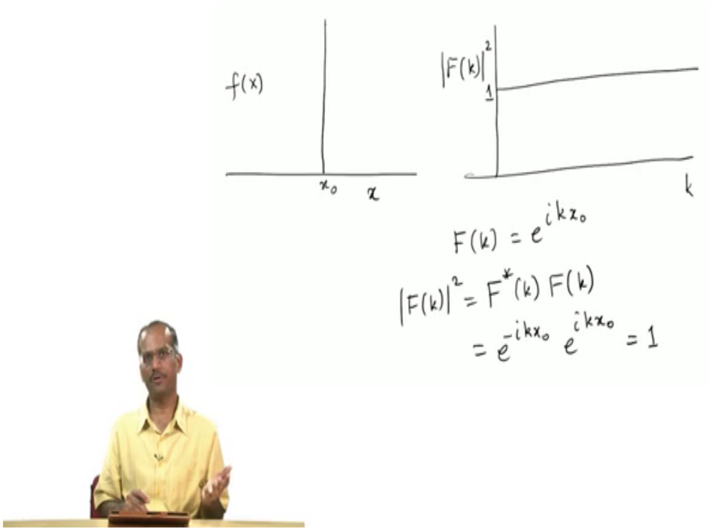
It is defined by the property that δx integrated over dx is equal to 1 over all space. If I take the Fourier transform of this function which is really very very narrow, I should get a function which is highly extended very broad in its width. So, this is what I am expecting.

(Refer Slide Time: 24:45)



So, $\delta(x - x_0)$ is generalization of this δx such that at $x = x_0$ the function is infinite and everywhere else it is 0. The integral of this would be equal to $f(x_0)$. Earlier we were working with time domain function. So, it made sense to work with $e^{i2\pi\nu t}$, but here our function is defined in some sense the position space. So, I am going to work with e^{ikx}. So, there is no ν here ok. But, nevertheless this is still Fourier transform and now, if I use this property which is given here the result of this is e^{ikx_0}.

(Refer Slide Time: 25:32)



So, my $f(x)$ is a delta function sharply peaked at $x = x_0$. Now, the Fourier transformed domain if I plot mod square of the mod square of $F(k)$ it will be a constant because $F(k)$ is e^{ikx_0} and $|F(k)|^2$ will be $F^*(k)F(k)$ and that is going to be a star of k is e^{-ikx_0} into e^{ikx_0}. So, it would cancel and give you 1.

Now, you look at what we have got. In position space it was a sharply peaked function and we took the Fourier transform and look at the same function in the k space it extends all the way from $-\infty$ to $+\infty$. So, it is extreme example of what we have been seeing, something that is sharply peaked in one space is extended over the entire space in the Fourier domain.