Waves and Oscillations
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Lecture – 41
Fourier Series and Energy of Vibrating String

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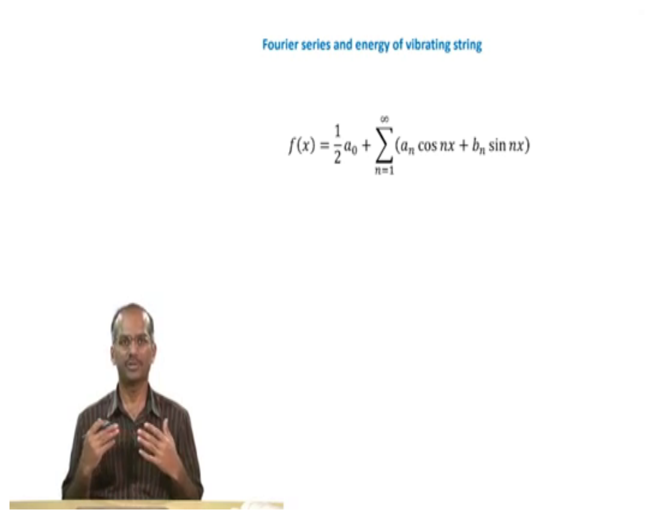

Welcome to the 3rd lecture this week, since the beginning of this week we were looking at the Fourier series. I have written down this formula standard Fourier series formula, here this first term $\frac{a_0}{2}$ tells you the average of your function $f(x)$ and the rest of the infinite terms tell you the oscillations about that level and later on we extended it to cases where the function is not necessarily periodic.

But you can still use this machinery and say that over a small region it can be approximated by a function which is periodic. Now, we shall use all this machinery in this lecture to write down an explicit formula for energy of a vibrating string. So, this is a standard problem we had looked at a few weeks back, the question of a string that is tied between let us say two rigid walls and you wanted to find out what are the various possible frequencies that can be excited on this system.

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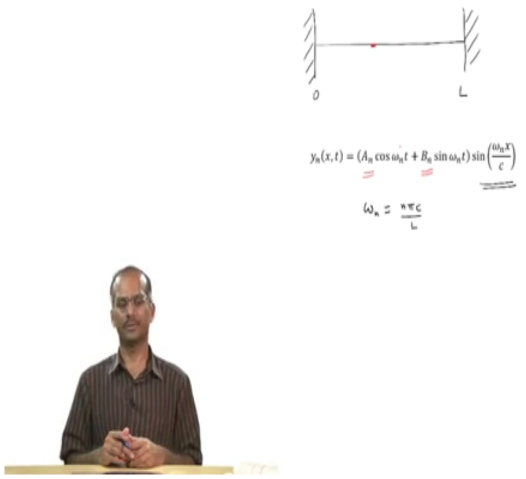

So, here I have the solution for this specific problem with boundary conditions taken into account. The boundary condition says that y which is displacement should be 0 at $x = 0$ clearly this solution satisfies that. And, if you also substitute for ω_n which is the normal mode frequency of the nth mode into this equation you will also see that when $x = L$ again the displacement is 0 and it is so far all the normal modes. So, what we have is the displacement as a function of position along the axis along 0 to L and it is also a function of time.

Now, the question was to calculate the energy of such a vibrating string, this small part which is marked in red, all this small infinitesimal region does is to oscillate up and down. So, it is just executing simple harmonic oscillations and we know the formula for energy of a simple harmonic oscillator. So, you can write down the kinetic energy of this small segment and also the potential energy of this small segment. So, you do that and integrate over the entire length of the string, add all the contributions from every part of the string and then we would have got an expression for energy of the string.

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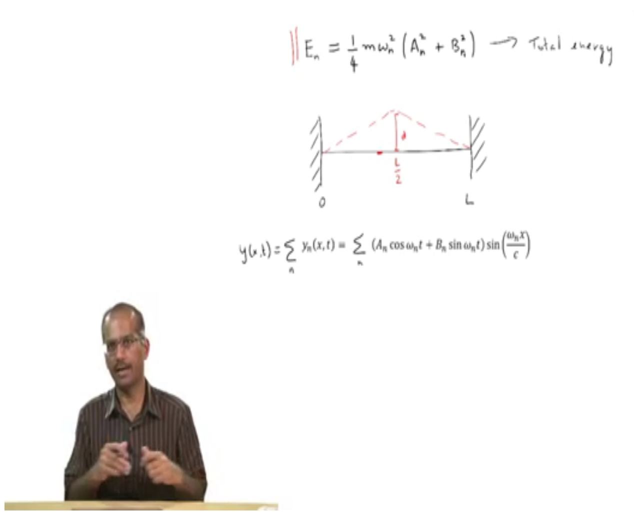

So, let me write that expression now. Now you will notice that this quantity it depends on this A_n and B_n. So, these need to come from the initial conditions the information about what kind of oscillation that is going to result will only come from the initial condition. So, I am going to pull the string right at the center of this string and pull it by an amount d and leave it thats going to excite the string and it will start oscillating, n is an index of which normal mode we are working with. So, when I excite this thing which is the normal mode did I excite.

In this case what happens is that when you arbitrarily excite strings like this, you are not exciting a single normal mode you are most probably exciting a collection of normal modes. Meaning that it is a combination of displacements of several modes, so what I am going to see as vibrating string subsequently will be well approximated by several different normal modes. So, in principle you could say that it is going to get contributions from pretty much all the normal modes. Given that this is my formula for energy, I want to find out what is the A_n and B_n subject to the initial excitation being this triangular form that I have given here.

And, then once I find the energy of the nth normal mode I will add over I will add the energy contributions coming from different normal modes and that is going to give me the final answer. Any arbitrary displacement of a string like this can always be written as a summation over the displacements of all the normal modes. So, my net displacement which will be a function of position and time which I will call as y without any subscript, that will be a summation over this n summation, summation over all the normal modes. So, this y which is a function of x and t tells me the displacement as a function of position and as a function of time for some arbitrary displacement that I have done. Now I want to specialize to the case when $t = 0$. So, I want to know what is the initial displacement.

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$$y(x,0) = \sum_n y_n(x,t) = \sum_n A_n \sin\left(\frac{\omega_n x}{c}\right)$$

$$v(x,0) = \frac{\partial y}{\partial t} = \sum_n \frac{\partial y_n}{\partial t} = \sum_n \left(-\omega_n A_n \sin\omega_n t + \omega_n B_n \cos\omega_n t\right)\Big|_{t} \sin\left(\frac{\omega_n x}{c}\right)$$

$$v(x,0) = \sum_n \omega_n B_n \sin\left(\frac{\omega_n x}{c}\right)$$

So, all we have done is to simply substitute $t = 0$ in this the last equation in this slide. Next is I want to also get the particle velocity at $t = 0$, the displacement at initial time depends on this quantity A and the velocity at initial time depends on this quantity B. For a moment imagine that $y(x,0)$ is some arbitrary function of x, then what you have on right hand side is an infinite series in terms of sin x; sin $\frac{\omega x}{c}$.

So, that should remind you of Fourier series and similarly if you look at this v as a function of x,0, again just for a moment if you imagine that this is some arbitrary function this entire expression looks like a Fourier series. We know how to extract this coefficient A_n and B_n because, in that case A_n and b_n will precisely turn out to be the Fourier coefficients.

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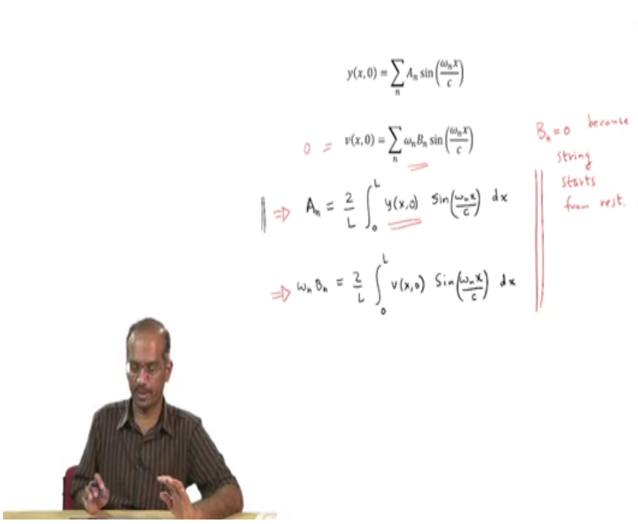

Here I have collected the result that we have till now obtained. So, using the Fourier series that we had learnt now I have these expressions for A_n and of course, you can write only for B_n because, your coefficient here is actually the product of ω_n and B_n. So, I have an expression for $\omega_n B_n$.

So, if we are going to start our string from rest in that case this velocity is going to start from 0. So, velocity at all the positions along the string is going to going to be 0 and if that were the case that the initial velocity is 0 this implies that $B_n = 0$.

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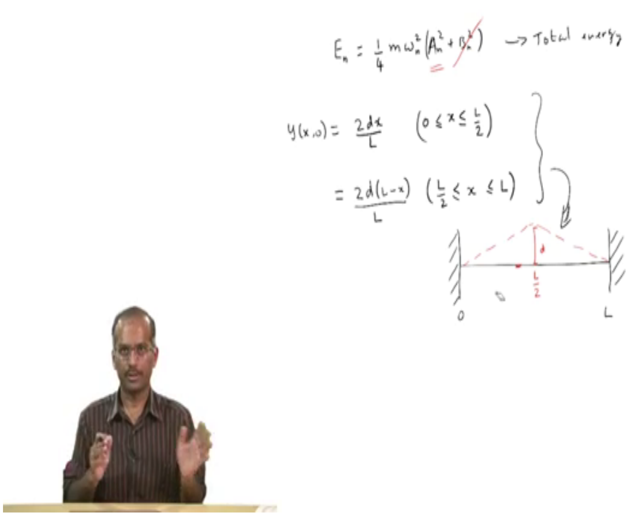

So, in that case our problem simplifies even further because, we can set all these $B_n s$ to be 0, in which case the energy is simply $\frac{1}{4} m \omega_n^2 A_n^2$. So, next let us now find A_n then substitute it back in this formula and to find A_n I need to do this integral here. To do this integral I have to specify this initial displacement and as you can see my initial displacement is that I am actually pulling the string at the midpoint of that string at $x = \frac{L}{2}$ by a distance d. In other words the initial profile of displacement is what is given by this figure that you are seeing right now.

I have written down the initial displacement in analytical form as you can see d is the height by which I am pulling it initially and you will also notice that if $x = 0$ the displacement is 0 and also that if $x = L$ which is the other end of the string again the displacement is 0. To be able to do this integral we needed this $y(x,0)$ now we have this, so we can go ahead and perform the integral and obtain A_n. And as you can see the integral needs to be split into two between 0 to $\frac{L}{2}$ and $\frac{L}{2}$ to L because the functional form of y in these two regions is different.

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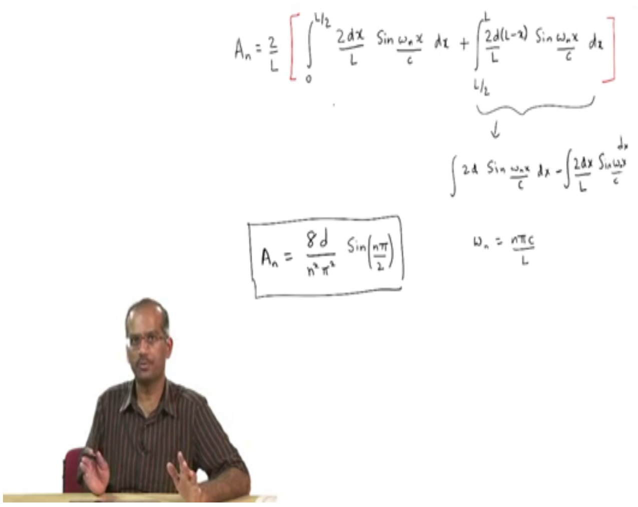

Now, I have written the integral explicitly by substituting for $y(x,0)$ in the correct regions so, one between 0 to $\frac{L}{2}$ and other between $\frac{L}{2}$ to L. You will notice that this integral alone will split into two integrals which could be written as. So, I urge you to do this integral, it is fairly straightforward and simple integral when you complete the exercise.

So, this is the expression for A_n and you should keep in mind that in obtaining this we should substitute for ω_n as $\frac{n\pi c}{L}$. Where, c is of course the speed of the wave, if n is even A_n has to be 0 because in that case the sin $\frac{n\pi}{2}$ will be 0.

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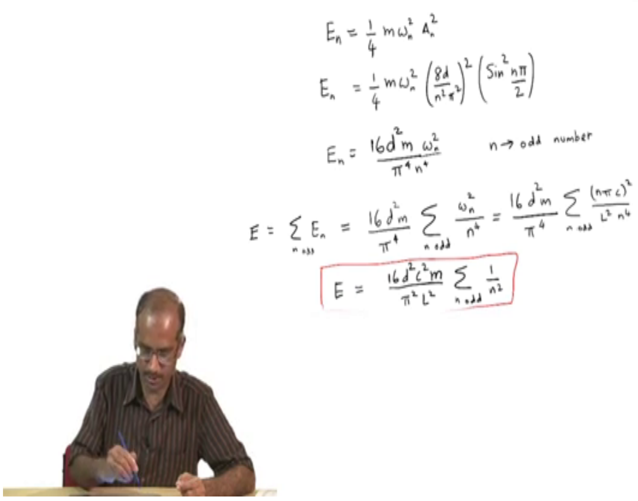

Hence in the expression for energy I can remove this sin square term and say that n has to be only odd numbers, we are nearly in the last step. So, the total energy would simply be sum over all the $E_n s$ and remember that n has to be odd numbers, let me put that in here in the summation.

I have also substituted for ω_n^2. So now, after cancelling everything I am going to be left with the following expression. So, here I have my final expression for energy which is in terms of summation over odd integers. And, if you remember in the last module, we in fact worked out precisely this summation, let me just quote the result for you.

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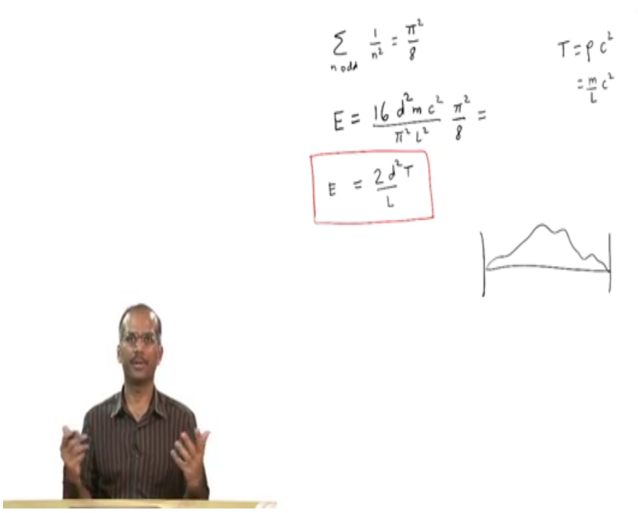

Now, we can substitute this in our expression for energy that we have just obtained and if we do that I get my final expression. The tension in the string is equal to $\rho \times c^2$ and rho is the linear density which is $\frac{m}{L}c^2$. So, this gives me the final expression for the energy. So, you will notice that this relation that we have got depends only on the top level parameters of the problem, one is it depends on the uniform tension in the string.

It depends on the total length of the string between the two walls and it depends on how much or by how much I pull the string away from the equilibrium position. So, if you go back to this figure here, I pulled it by an amount d and it turns out that the energy of the oscillating string is proportional to d^2. Clearly, it makes sense because the energy of vibration being the kinetic energy should be equal to the potential energy that was given to it initially; this is precisely because we have not allowed for any dissipation.

In fact, you could see that total E that I have obtained is independent of time and what we have obtained is the energy of the oscillating system for a specific choice of initial condition. And, here the choice has been such that we were able to use whatever we learnt, suppose for instance I say that I have my string here and this is my arbitrary choice of initial condition. There is no guarantee that a problem like this can be doable analytically, in most of such cases you will have to do the problem numerically.