Waves and Oscillations
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Lecture – 40
Fourier Series: Part 2

So, welcome to the part two of Fourier Series. In this 9th week we had started with studying Fourier representation of arbitrary functions.

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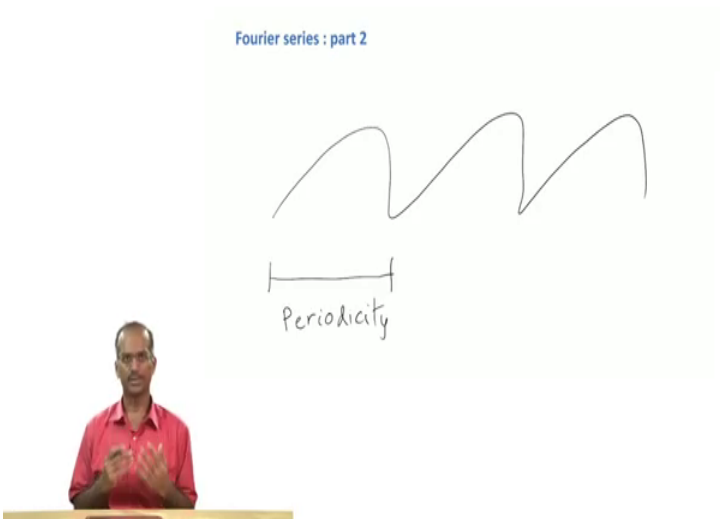

So, the idea here, let me repeat it. The idea here is that given an arbitrary function we allow for it to have finite number of finite discontinuities that can be handled something like for instance this function. So, you could say that it is periodic with this being the periodicity. So, in such cases what we wanted to do is to write these functions in terms of sine and cosines.

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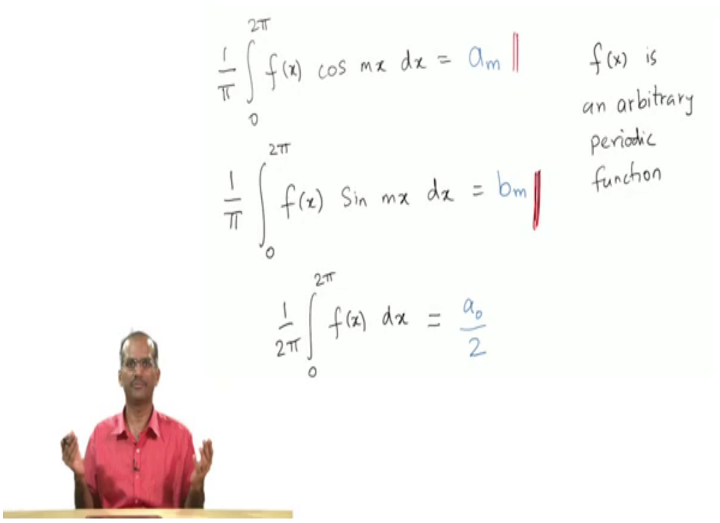

So, given that $f(x)$ is arbitrary and periodic function we calculate these coefficients a_ms, b_ms, $\frac{a_0}{2}$ and so on and finally, assemble them together in a formula of this type.

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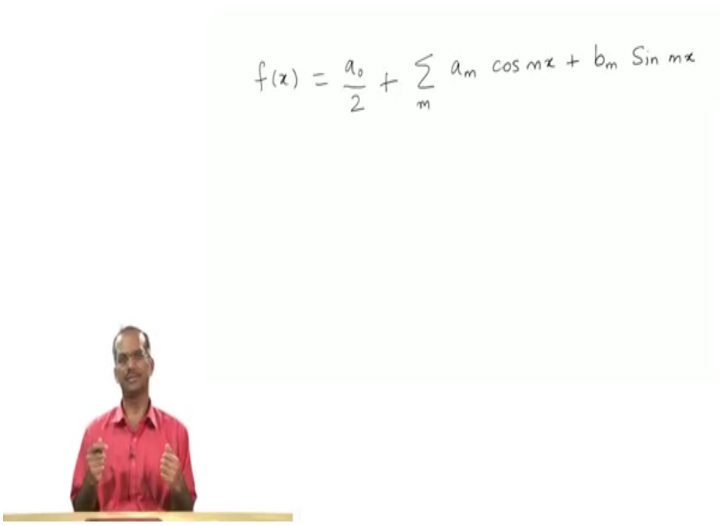

So, $f(x)$ now can be written as $\frac{a_0}{2}$ plus I have an infinite summation so, that would be $a_m \cos mx + b_m \sin mx$. The summation over all integer values of m. The physical interpretation that you can draw from this is that you could say that my periodic function has these following frequencies which with which it is made up of. So, typically if you look at say time series of many weather variables you will see that there is an annual component, there is a seasonal component which would correspond to the kind of annual and seasonal variations that we can immediately recognize.

Today, what we are going to do is to look at how we can liberate ourselves from the necessity of having the function to be perfectly periodic.

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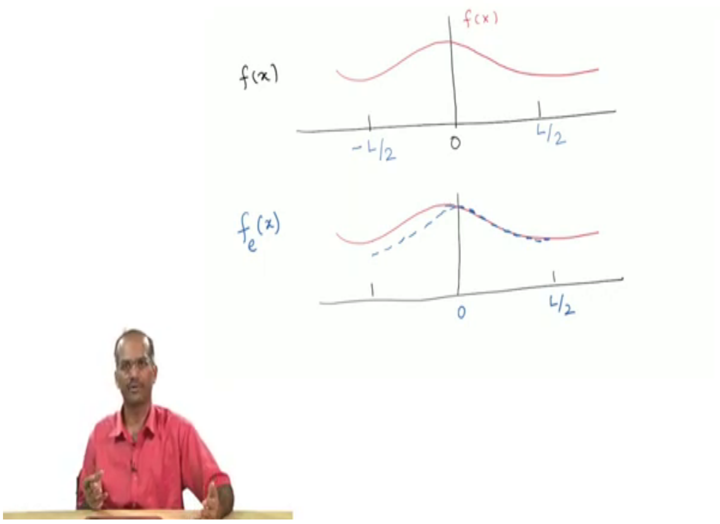

Let us say that I am looking at a function which is not quite periodic, but something like this. So, it is explicitly not periodic, but I am not worried because I want the representation of this function between 0 and $\frac{L}{2}$. In the range between 0 and $\frac{L}{2}$ this function which I will call as $f_e(x)$ to indicate that I am trying to approximate $f(x)$ by an even function which I will call $f_e(x)$. So, in the desired range which is between 0 and $\frac{L}{2}$ it quite nicely matches $f(x)$ and we can also do this with an odd function. Let me show how that works.

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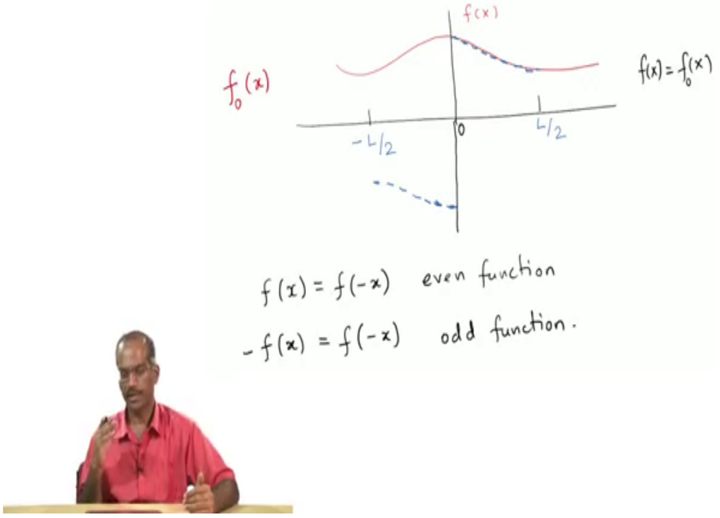

Now, if I match with the odd function which I will call $f_o(x)$ to indicate that I am using an odd function. So, in that case again what you will see is that in the desired range between 0 and $\frac{L}{2}$, the odd function which I have chosen $f_o(x)$ will nicely match my function $f(x)$.

Now, let us write a Fourier series we have all the formulas written down. So, the results are not going to be too different from what we have already calculated.

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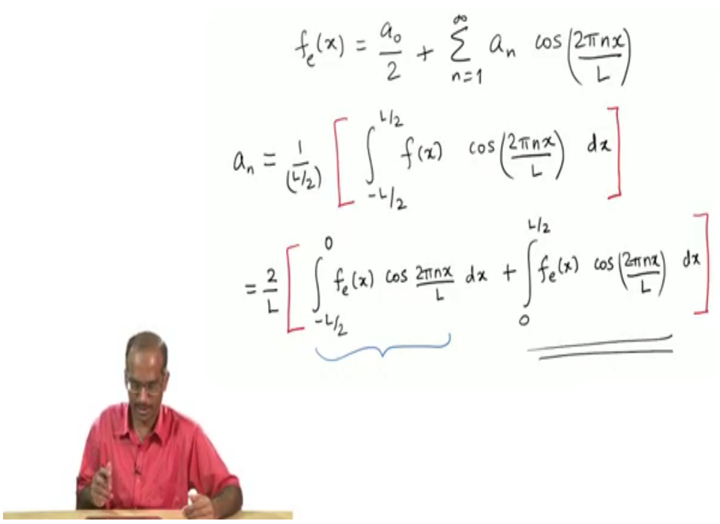

If I am using the even function to approximate $f(x)$ then without doing any further work from what we already studied in the previous module I can directly write down the Fourier expansion or the Fourier series for an even function and we know that for even function the Fourier expansion terms will all be made up of cosine functions.

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Note:

- In the following calculation, a phase factor $2\pi/L$ has been introduced as a multiplier to the argument of the sine and cosine functions of the Fourier series. This subtly different version of the usually used Fourier series is appropriate for application to physical problems where typically x has the dimensions of length, because $2\pi nx/L$ would be dimensionless, as it should be.

But, still I have to determine this a_n for that we just go back to what we studied earlier so, we will use this. You should note one thing about this expression a_m So, the limit of this integral goes from 0 to 2π and here we are dividing by $\frac{1}{\pi}$. So, it is if your interval is 0 to 2π, the basic periodicity of your function $f(x)$ then you would divide by half of that period.

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Note:

- In the following calculation, the evaluation of $\frac{a_0}{2}$ is not done.

So, I have this first expression which is simply based on the expression that we obtained in the last class. Split this integral into two parts one that goes from $-\frac{L}{2}$ to 0 and other goes from 0 to $\frac{L}{2}$. So, now, I have split the integral into two parts.

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Properties of Even and Odd Functions

$$I = \int_{-a}^{a} f(x)\,dx = \int_{x=-a}^{x=0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

If $u = -x$, then $du = -dx$:

$$= -\int_{u=a}^{u=0} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{u=0}^{u=a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

If f(x) is even then:

$$I = 2\int_{0}^{a} f(x)\,dx$$

If f(x) is odd then:

$$I = 0$$

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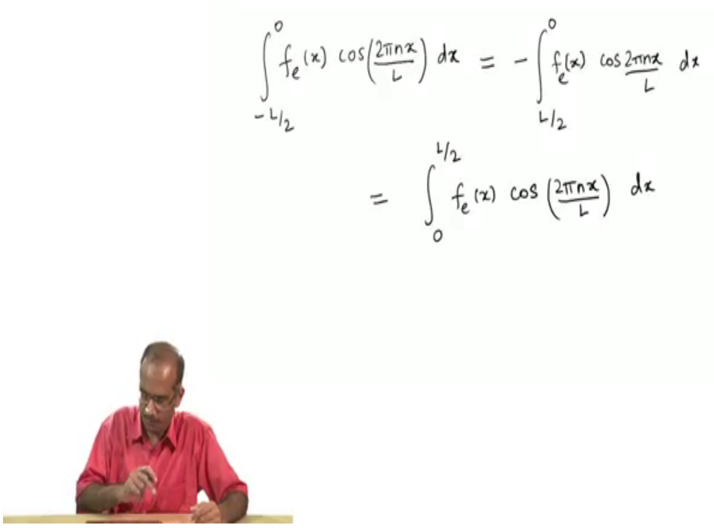

Now, if you compare this with what we have here that is in fact, exactly this second integral. This whole expression is just 2 times the second integral. So, now, I can write the final expression in the following way.

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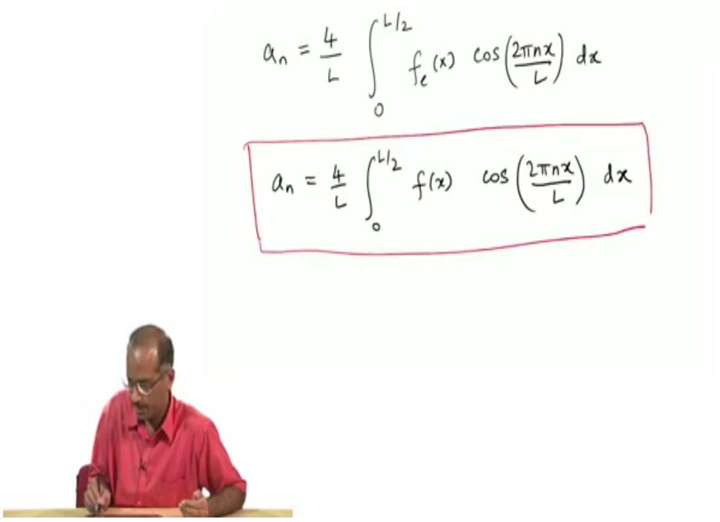

And in the region between 0 to $\frac{L}{2}$, $f_e(x)$ is same as $f(x)$. So, I can replace $f_e(x)$ by $f(x)$. So, here I have an expression for the a_ns. So, now, let us look at the other case when the odd function $f_o(x)$ matches your desired function in the range between 0 and $\frac{L}{2}$.

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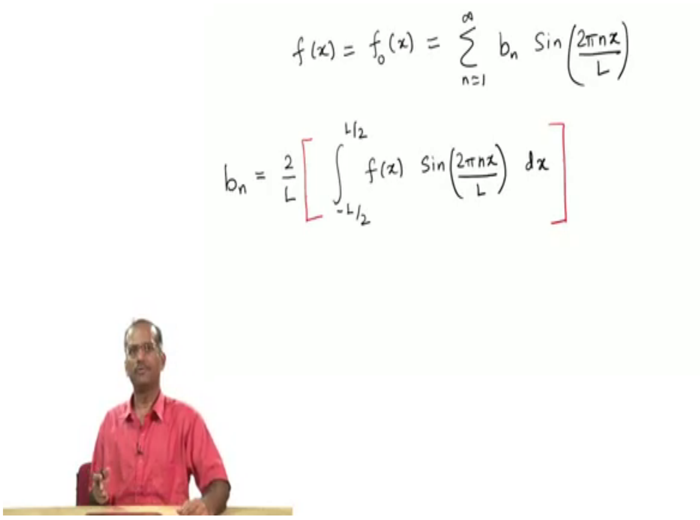

$f(x)$ can be written as expression again taken from what we did earlier on the last module. So, it is simply an implementation of this, you are integrating from 0 to 2π. So, in this case it had the physical meaning of a periodicity the basic period which was 2π and the quantity that you divide here is half of that.

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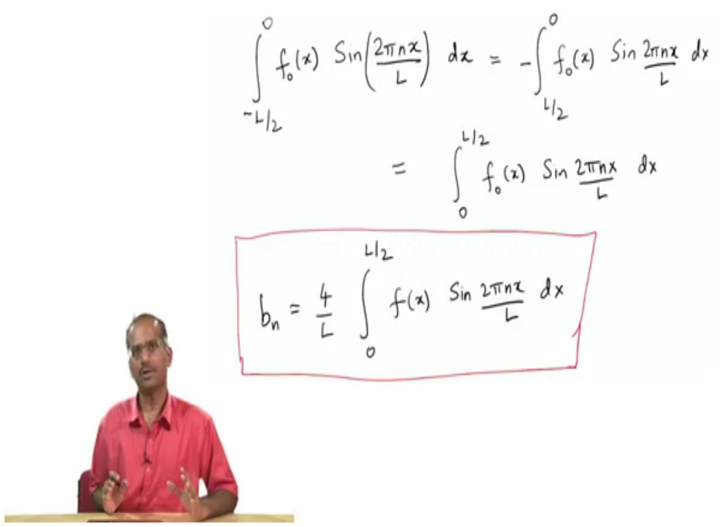

Again, as usual we will split this into two integrals. Again you go back and look at what you have here you will notice that that is exactly this term. So, b_n will simply be $\frac{4}{L}$ integral of 0 to $\frac{L}{2}$ and $f_o(x)$ which is a odd function that I am trying to approximate from my $f(x)$ they are expected to coincide quite nicely within 0 to $\frac{L}{2}$. So, I can replace the odd function by $f(x)$.

So, this finally, gives me the desired expression for b_n and we should keep in mind that the $f(x)$ that we are going to get by this procedure will precisely match the function only in the range between 0 and $\frac{L}{2}$, outside of this range it is not going to match, but then we did not even demand it. So, let us see how it all works by actually doing one problem.

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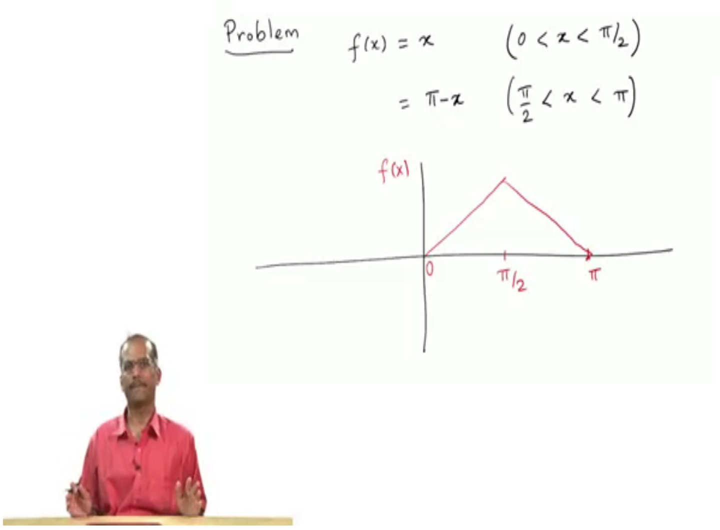

So, this is my function. So, between 0 and $\frac{\pi}{2}$ it is a straight line like this and between $\frac{\pi}{2}$ to pi it is $\pi - x$. So, at $x = \frac{\pi}{2}$, $f(x)$ is $\frac{\pi}{2}$ which is this point itself and at $x = \pi$ it gives you a_0 which means it comes here. So, I should be getting a line like this. This is the function and we do not know how it behaves outside of this.

And, now we want to find out the Fourier series for such a function. Unlike in earlier cases where you were told that this repeats itself we do not know if it repeats itself and we really are not worried about it. So, let us say that we will use the odd function to approximate it. So, we can do either way I mean you can make a choice at this point. You can say that I will use an odd function or an even function, but let me say that I will use an odd function.

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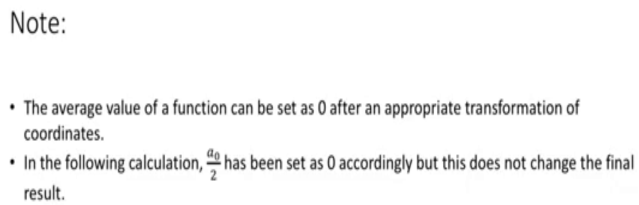

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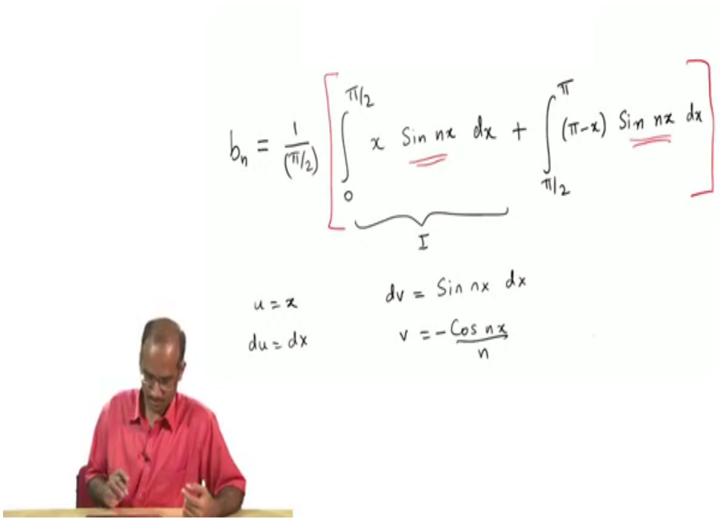

Let us write the expression for b_n. So, remember that our x runs from 0 to π. So, we should take half of that and divide it here one divided by half of that would be $\frac{\pi}{2}$. So, this is my expression for b_n. So, you can see that b is dependent on n and n comes from this sine and x here and of course, we need to do the integral and simplify this : $u = x$, $du = dx$ in that case and dv will be $\sin nx dx$ and v is integral over $\sin nx$ which is $\frac{\cos nx}{n}$.

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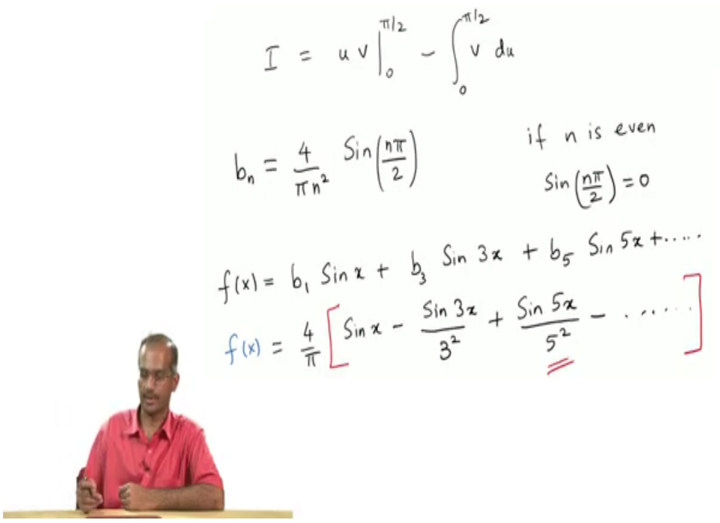

Now, to do the partial integral if I call my desired integral as I, which means that I would call this as I. In that case I is equal to uv evaluated between the limits 0 and $\frac{\pi}{2}$ minus $\int v du$ evaluated between 0 and $\frac{\pi}{2}$ and finally, when you do everything and assemble I will directly write the result for b_n leaving this as an exercise for you to just complete the steps.

Now, I can assemble and write the final result. n is a even number in that case $\sin\frac{n\pi}{2} = 0$ which means that only the odd terms of the series would survive substitute the values for b_1, b_2, b_3 and so on. So, b_1 would be $\frac{4}{\pi}$ and $\sin\frac{\pi}{2}$ is 1. So, that is going to just give me $\sin x$ and in fact, you see this $\frac{1}{n^2}$ term here so, which means that $\frac{4}{\pi}$ is a constant that I can actually take outside. So, the first term is $\sin x$ as usual and second term would be of course, $\frac{4}{\pi}$ which has already been taken outside that is going to generate 9 in the denominator and $\sin\frac{3\pi}{2}$ which will be -1.

So, I need to change this plus to minus and I am going to be left with $\sin 3x$ and since it is $\sin 3x$ it is instructive to write this as 3^2 it makes it nicer and you can even write a compact formula later and now it is very easy to write the next term. So, that would be $\frac{\sin 5x}{5^2}$ minus $\frac{\sin 7x}{7^2}$ and so on. This result that I have for $f(x)$ is a Fourier series for my function which lies between 0 and π. So, if you take into account all the terms of the series there are infinity of them it will exactly reproduce for you this function here.

A general term would be $\frac{\sin nx}{n^2}$. So, which means the numerator which is sum $\sin nx$ is always an oscillatory function and that is always going to lie between $+1$ and minus whereas, the denominator is increasing a square of some integer. So, clearly the weights of successive terms are decreasing. So, effectively it would mean that maybe first few terms would be able to do a good job of reproducing this function for our purposes.

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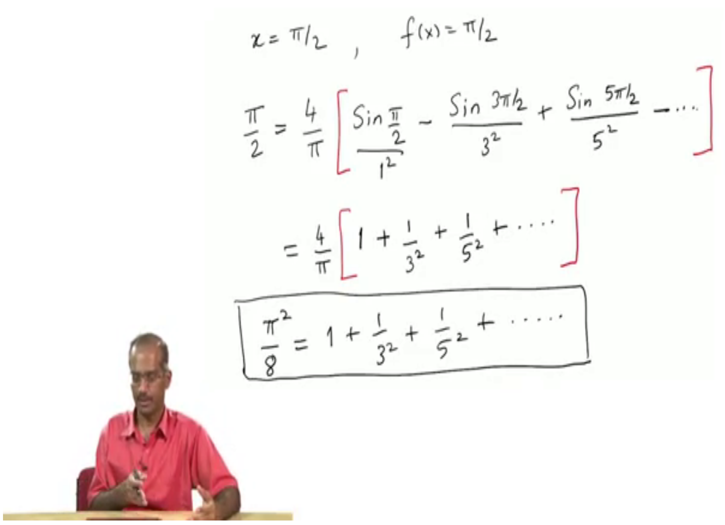

Now, you go back to the definition of a function $f(x)$. It tells me that if x is $\frac{\pi}{2}$, then $f(x)$ is $\frac{\pi}{2}$ as well. So, at $x = \frac{\pi}{2}$, $f(x)$ is $\frac{\pi}{2}$. So, now we can put this in this relation that we have just obtained let us do that. So, it tells me that left hand side $f(x)$ is $\frac{\pi}{2}$ and that is equal to $\frac{4}{\pi}$; first term is $\sin x$ so, that will be $\sin \frac{\pi}{2}$ minus $\sin 3x$ that is $\frac{3\pi}{2}$ divided by 3^2 plus $\frac{\sin \frac{5\pi}{2}}{5^2}$ and so on.

So, this can be now simplified and we can write a nice series. So, this would be $\frac{\pi^2}{8}$ that is equal to $1 + \frac{1}{3^2} + \frac{1}{5^2} +$ and so on. So, what we have is an infinite series for π. So, in principle you can calculate the value of π by just summing this series to any accuracy and precision.

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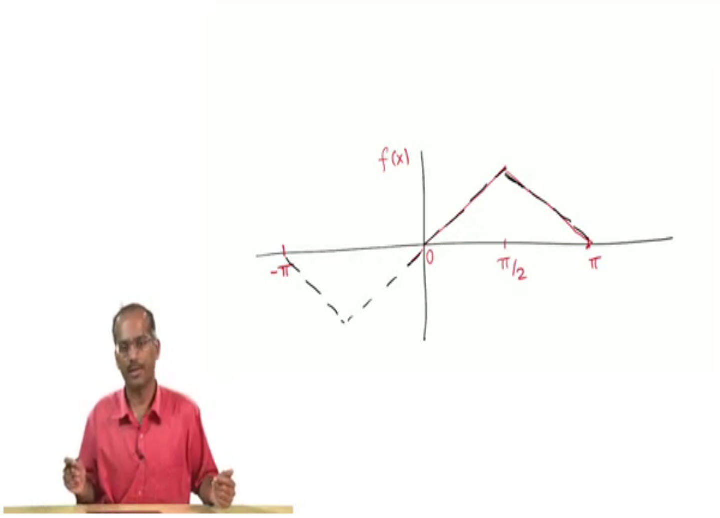

Now, finally, before I close let us look at pictorially what we have done this is the function that we wanted to write a Fourier series for and we do have this result and if you sum everything you should of course, be able to approximate this function precisely like this it would match between 0 and π which is our desired range.

But, now if you ask what would happen between $-\pi$ and 0, you could actually make a guess make an informed guess because we use the odd series to represent the function. So, which means that outside of this range between 0 and π the function would behave like an odd function. The function would continue to be like this in the range when x lies between when x lies between $-\pi$ and 0.