Waves and Oscillations
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Lecture - 04
Superposition of Oscillations: Lissajous Figures

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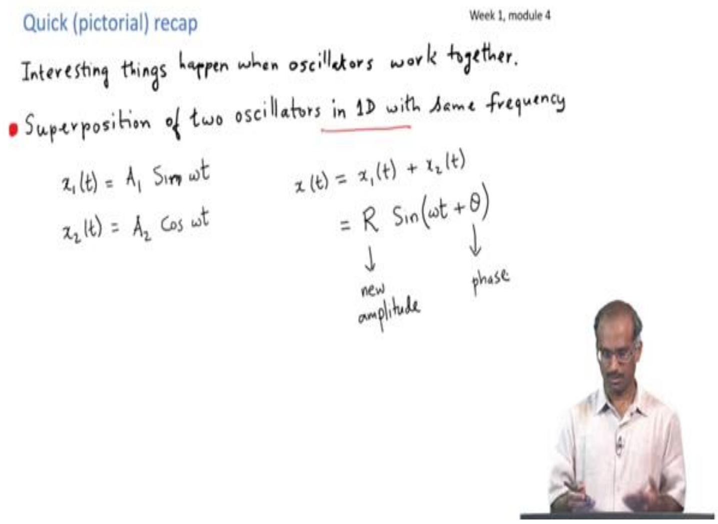

Welcome to the 4th module. As we had planned in this module we will look at what happens when you combine two oscillators which are perpendicular to one another. But before we do that let us quickly recap what we did in the last module basically that was a question about what happens when you combine two oscillators in 1 dimension. So, it is along the same line and we consider two possible cases; one is where both the oscillators had the same frequency. So, for example, I could have had the following possible solutions $x_1(t)$ could have been $a_1 \sin \omega t$ and $x_2(t)$ could have been something like may be $a_2 \cos \omega t$.

So, we wanted to know what happens if you put both these oscillations together and the central physics there when you want to combine not just two, but any number of oscillations is that the net displacement would simply be the sum two individual displacements in this case. So, it would be $x_1(t) + x_2(t)$ and individually each one them is an oscillating system and in this case, you are guaranteed that combined system, the combination two oscillators in the same dimension would also be an oscillating system.

So, in principle we could write the solution as, something like this where R is the new amplitude and θ is the phase for $x(t)$.

So, clearly the problem is simple in some sense all we need to determine is how is this R and theta related to let us say a_1 and a_2 and so on. So, that is one situation that we met.

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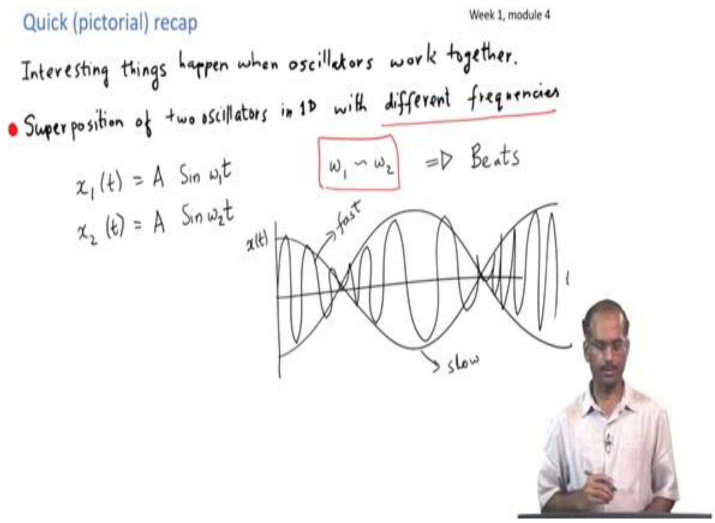

In the second case it is pretty much the same scenario you are superposing two oscillations in 1 dimensions again, but now the two oscillations will have different frequencies. So, it could have been something like $x_1(t)$ may be $A \sin \omega_1 t$ and $x_2(t)$ could have been something like $A \sin \omega_2 t$. In principle you could add additional phases and so on, but it is simple enough and would capture what we need.

So, what we saw was that in the situation where the two frequencies are nearly equal, but not exactly=one another, in that case you get the phenomena of beats. So, there is waxing and veining the oscillations. So here, when I plot $x(t)$. So, x (t) is again $x_1(t) + x_2(t)$. As a function of t you would get possibly a profile, but also a something that has faster oscillatory component. So, this is the one that has the fast oscillations and this one corresponds to slow oscillations.

So, we have a component that would oscillate fast and another component which is oscillating slow. So, what you see really is a super position of both these components together and this phenomenon is what you call beats and it is particularly impressive when the two oscillations have frequencies which are very close to one another.

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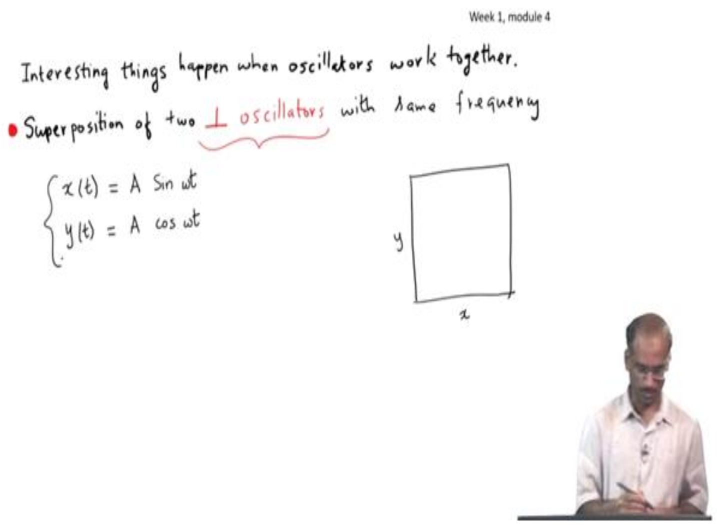

So, today we will look at what happens when we superpose two oscillations, which have same frequency, but in particular they are two oscillators which are perpendicular to one another. So, you could imagine, let us say a pendulum in x direction and the oscillation of another pendulum in y direction and we want to know what the resultant of oscillation or one pendulum which is given an oscillatory input which basically work in both the directions.

So, in such cases, we would like to know what happens to superposition of two oscillations. So, I am going to do this through a series three examples starting from really simple case. So, let me first set the sort of scenario for this. So, I am going to have two oscillators perpendicular to one another. So, let me call them $x(t)$ is one them and may be it could be $\sin \omega t$, this one possible solution. And I could have the second one, which now I will call $y(t)$ especially to indicate that x and y are perpendicular to one another and let us say it is some other a $\cos \omega t$.

So, clearly if I want to visualize this oscillation, I can do it in a 2-dimensional plane where x and y are the two axis that I have here and I can eliminate time and just simply plot the values the displacements, corresponding to x and y oscillator. Hopefully I might get some pattern here, which would correspond to the superposition two oscillations.

So, here is the plan of what we want to do. So, will assume oscillator solutions in x and y direction. So, here is one possible assumption that I have taken, like this and whatever we will do we will try and eliminate time from here. For instance, simplest thing one can do is extract t from here and substituted here, in which case you would have eliminated time and then the resulting equation would simply be a relation between x and y.

So, given a value of x you will get a value for y, in which case you can plot it in two dimensions like this. So, that is exactly what we are going to do and I am going to take you through a series of examples.

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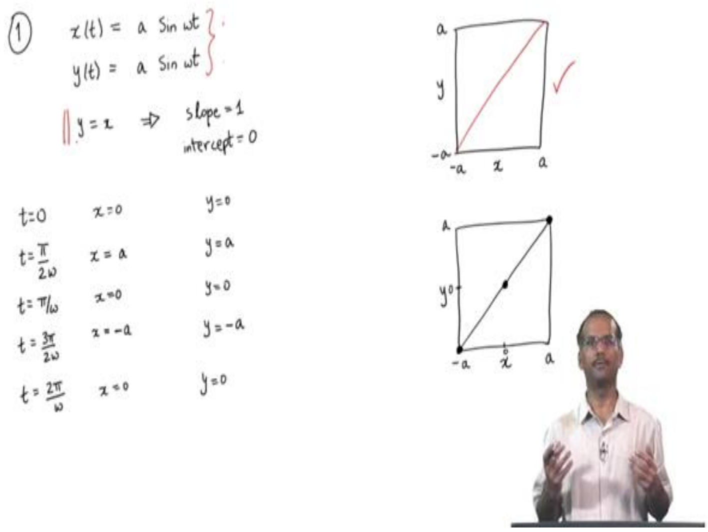

Let me start with the first the examples. So, in this case I am going to assume that $x(t) = a \sin \omega t$ and $y(t) = a \sin \omega t$ as well. So, this is one the really simple cases and let me show you how we can work with this, with these two oscillations and find the effect of superposition.

As I said finally, I want to be able to plot the superposition in x, y space by eliminating time. So, there are several ways of doing this, but in this particular case, it is particularly very simple, simply because x and y are equal both are equal to $a \sin \omega t$. So, y is actually equal to x itself. So, if you compare it with the equation of straight line, this is simply an equation of a straight line that I have here slope 1 and intercept 0. So, here I have this plot the x, y plane which can go from $-a$ to $+a$ because a is the amplitude here and y can go from $-a$ to $+a$. And this result here tells me that, y =x simply a straight line with slope 1. So, I can plot this here.

So, what I have is, I have two oscillations which are perpendicular to one another. So, there is an x oscillator which is oscillating in this direction and y oscillator, let us say in this direction. And I am looking for the resultant and the resultant happens to be a straight line which is, which has slope 1. There is another way of getting the same curve. Let me also show you, because in some cases it is more easier to use that. So, once again let me draw this x, y plane ok.

So, let us go back to this set of equation that I have here. And I am going to put in different values for time and plot the points here. So, let me start by putting say $t = 0$. So, if I put $t = 0$ in these two equations, I will get $x = 0$ and since I know that $y = x$, y is also=0 or you can just directly put $t = 0$ here and that would also give you $y = 0$.

So, $x = 0, y = 0$ is a point and that's here. So, this is the 0 here. So, this point, it is a point through which the superposition will have to pass through, the superposition oscillation. Now, I can choose some other value of time and plot points for instance, let me choose $t = \pi/ 2\omega$. So, if I substitute $t = \pi/2\omega$ in these two equations here and here, I will get $x = a \sin \pi /2$, but we know that $\sin \pi /2 = 1$

So, it is simply $y = a$. And of course, $y = x$. So, y is also equal to a. I am going to have another point here. So, this is one point and 0, 0 is another point. Let me try and get few more points. Let me put $t = \pi/\omega$. When I do that again I get $x = 0$, so, $y = 0$. So, it is basically the same point which here already plotted and let me take one more $3\pi/2\omega$

So, if I do this, in this case $x = a \sin 3\pi/2$, but we know thatsin $3\pi/2 = -1$, in which case this quantity would simply become$-a$. And y is again equal to $-a$. So, $x = -a$, $y = -a$ is this point. And now if you take one more $2\pi\omega$, $t = 2\pi/\omega$ substitute it there, you will again get $x = 0, y = 0$.

So, again it is going to go back to this point. So, it started from here, the point at the center diagonally go in the opposite direction at a, a and finally come to $-a, -a$ and back to 0,0. So, I can join these points together. So, that is exactly what we had got earlier on here, just by saying that $y = x$. So, now, I guess you understand what we are trying to do, basically eliminate time and plot x as a function of y and what you get is the pattern the superposition of two oscillations which are independently oscillating in x and y direction.

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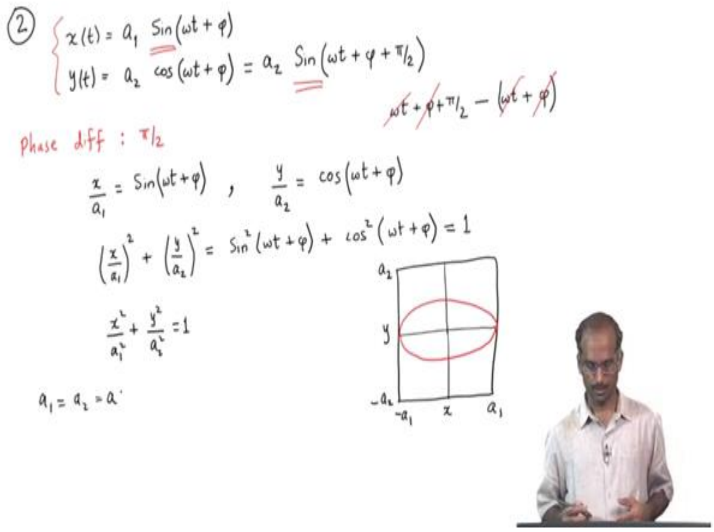

So, now let us do second example. Here I am going to assume solutions again, let us take for example, $x(t) = a_1 \sin(\omega t + \phi)$ and let me take $y(t)$ to be $a_2 \cos(\omega t + \phi)$ ok. So, before we actually go head and plot the dynamics in the x y plane, let us first look at what is the phase difference between these two oscillations ok. So, we already saw the idea of phase in one the earlier modules it is in some sense and initial offset ok. So, now, I would like to know what the difference between those that difference in offsets between these two oscillations.

So, the way to do that is bring the two oscillations to the same form and then subtract their arguments. So, let me just show it here for example, in this case I could do the following. Let me write y (t) slightly differently. So, a_2 and $\cos \omega t + \phi$ can be written as $\sin \omega t + \phi + \pi/2$. Now if you compare this and this both are sine functions and both have some set of arguments. So, the phase difference between these two would simply be the difference between these two arguments.

So, I am going to subtract $\omega t + \phi + \pi/2 - (\omega t + \phi)$. So, clearly $\omega t + \phi$ would cancel out, leaving you with $\pi/2$. So, the phase difference between x and y is $\pi/2$. So, with this background let us go back to what we wanted to do, which is to look at and figure out the superposition of two oscillations. In this case again one can go back to the kind of technique that we used earlier on, which is to say that substitute the values for time and plot the points and then join them to get the pattern of superposition dynamics. But here again this is actually a very simple problem you will now soon realize why it is simple.

So, let me go back to the first equation here and rewrite it differently. So, let me write it as $\mathrm{x}/a_1 = \sin(\omega\,\mathrm{t} + \phi)$ and also let me write the second equation also like that which is y/a_2 which is equal to $\cos(\omega\,\mathrm{t} + \phi)$ So, now, you notice that the right hand sides of both the equations are this sin and cosine functions and we know that $\sin^2\theta + \cos^2\theta = 1$ So, if you exploit that both these both the sides these equations, then we would get $(\mathrm{x}/a_1)^2 + (\mathrm{y}/a_2)^2$ and this is equal to 1.

So, what I have is an equation of ellipse $\mathrm{x}^2/a_1^2 + \mathrm{y}^2/a_2^2 = 1$. And now plotting this is fairly straightforward let me do that. So, the curve would look like this. So, that is an ellipse for you ok. So, if you had two oscillators which are perpendicular to one another and their pattern of individual oscillations is described by this function. The combination these two and when you visualize it in x y plane would be an ellipse. A special case of this is when $a_1 = a_2 = a$, in which case the ellipse would become circle that is a case when the amplitudes of both the oscillations are exactly equal, in which case you would get a circle for the superposition.

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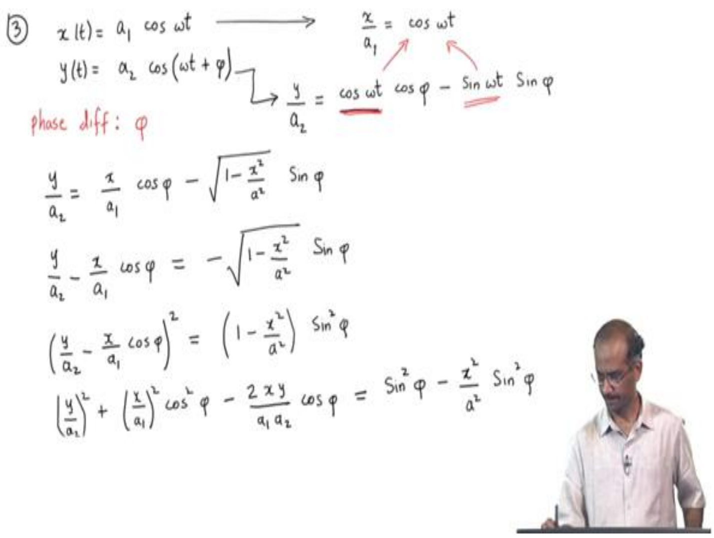

Now, let me go to the 3rd example. So, here once again I am going to have two possible oscillations and it is very easy to see that the phase difference between these two oscillations is simply equal to ϕ. So, same way of doing thing which is $\omega t + \phi - \omega t$ and that would give you the phase difference as ϕ itself ok. So, in most the cases in wave phenomena what is truly important is the phase difference between let us say two waves. So, individual phase in itself is not particularly important what really matters for a whole lot of phenomena that we would be seeing later on is the phase difference among waves.

So, let see if we can get something by way of pictures for these two oscillations. So, I will also show you how to obtain something analytically here. So, starting from this first equation, I will write $x/a_1 = \cos\omega t$ ok. And the second equation let me rewrite it as $y/a_2 = \cos(\omega t + \phi)$. So, let me expand it. So, that is $\cos\omega t \cos\phi - \sin\omega t \, sin\,\phi$ Now the whole aim here is to eliminate time, which means that, I should be able to eliminate this ωt or t from the second equation. And it is easy to do that because I can replace $\cos\omega$ t by x/a_1 from the first equation. So, let me do that y/a_2 will be equal to $cos\,\omega\,t$ which is $(x/a_1)\, cos\,\phi -$ and $sin\,\omega\,t$ is $\sqrt{(1 - cos^2\omega\,t)}$

So, that will be $1 - \cos^2\omega t$ is (x^2/a^2) into $\sin\phi$ So, all I have done is to substitute for $\cos\omega$ t, from this equation and also substitute for $\sin\omega t$ by using $\cos\omega$ t, the relation that $\cos^2\omega t + \sin^2\omega t = 1$. So, having done the substitution this is what I have and to get it in a useful form we will just manipulate it a bit. So, let me take the first term on the right hand side to the left, (y/a_2)– $(x/a_1)\cos\phi = -\sqrt{(1 - x^2/a^2)}\,\sin\phi$

Now, I am going to, so, given that there is a root here I want to get rid the root. So, I will square both sides this equation. In which case I will get $(y/a_2 - (x/a_1)\cos\phi)^2$

if I square it, I will get $(x^2/a^2)\sin^2\phi$. So, now, expand the left hand side the equation that will give me $(y/a_2)^2 + (x/a_1)^2\cos^2\phi$ – $(2xy/a_1a_2)\cos\phi$ and that is equal to $\sin^2\phi$– $(x^2/a^2)\sin^2\phi$

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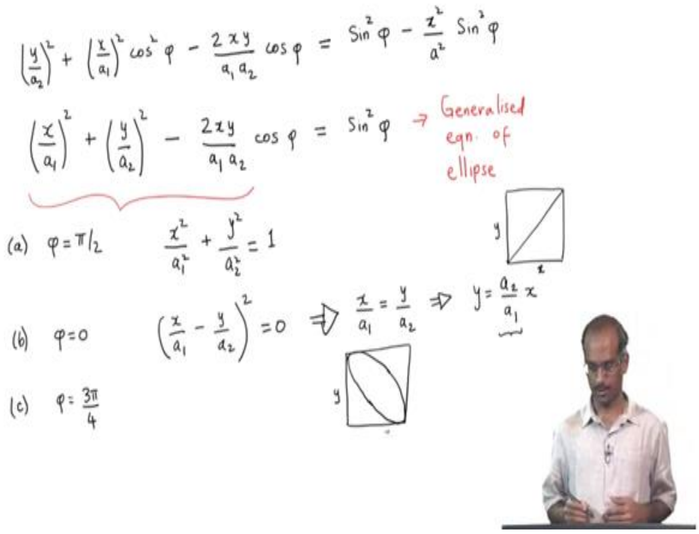

Now, we are very close to getting the result that we want by rearranging this equation, I would get the following equation. So, this is the final equation I have. So, this is in general an equation of ellipse. So, you could say that it is a generalized equation of ellipse. So, generalized in the sense that your ellipse need not always be oriented along the about the x axis, but it could have any orientations. So, this equation provides for that degree of freedom.

So, now that we have this equation let us put in some numbers and see how what kind of patterns we get. So, let me start with the first case. In this case I am going to take ϕ which is the phase difference to be $\pi/2$. If I take phase difference to be $\pi/2$,$\sin \pi/2$ is 1, $\cos \pi/2$ is 0. My equation would become simpler, it would be $x^2/a_1^2 + y^2/a_2^2 = 1$

So, this is simply, an simply the equation of ellipse.

So, what it tells us is that, if we start off with these two sets of individual oscillations and if the phase difference between the oscillation in x and oscillation in y, which is represented by ϕ and $\phi = \pi/2$, we are going to end up with the ellipse, which is exactly the case that we saw earlier on. Let us go to the second case. So, in this case let me take even simpler number $\phi = 0$. The phase difference between the two waves, two oscillations in x and y direction is 0. So, sin 0 is 0. So, the right hand side this equation would be 0 and the left hand side $\cos 0 = 1$. So, I will be left with only these three terms being equal to 0.

But if you look at this closely, this suggests that the real equation is $(x/a_1 - y/a_2)^2$. That is exactly the left hand side here and the right hand side is 0. And this implies that $x/a_1 = y/a_2$, which implies that $y = (a_2/a_1)x$. So, again what we have is equation of a straight line, whose slope is given by $a_2/a1$. So, if I had to plot this function, in this case depending on the value of a_2/a_1. Let us say in the case where both are equal the slope is 1. So, this is what I would get. And I leave this part as an exercise, suppose if I had a third case where $\phi = 3\,\pi/4$.

In this case I would urge you to do the, calculations yourself and finally, verify that the pattern that you will be getting would be an ellipse, but it is ellipse that is oriented like this. So, either you can use the equation, manipulate the equation to realize this or as we did earlier, a longer method would be to substitute various values for time and you would pretty much get the same result. To summarize this part of our discussion, if we are dealing with two oscillations which are perpendicular to one another and we wanted to find out how does the super posed oscillation look like.

So, one way of dealing with that is to geometrically plot that in x y plane. In most cases you will be able to do that by simply manipulating the equations. Ultimately the aim is aim there is to simply get rid the time and directly plot x as a function of y . The other way would be to substitute various values for t or time, plot the points and join them together either ways you should be able to get the same result.

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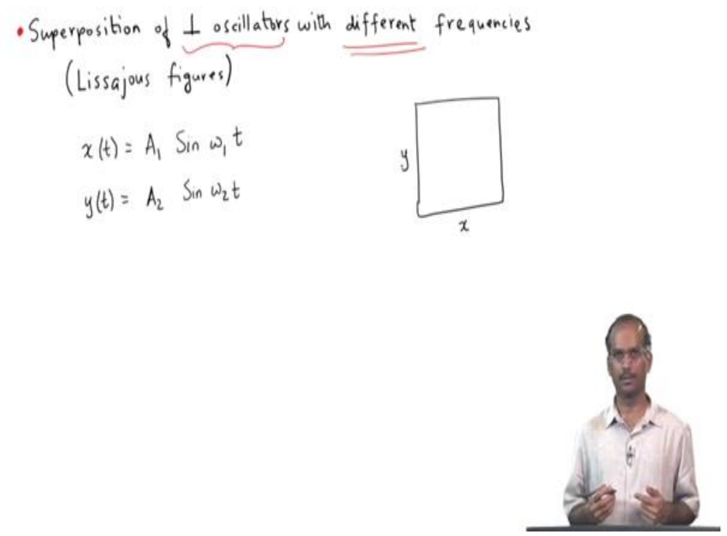

Now, let us go beyond a little more. So, now, let us try and super pose once again oscillators which are perpendicular to one another same as what we just did, but with different frequencies. And these are often called Lissajous figures. So, here again analytically doing this is, little more complicated, in the sense that it gets out of hand. The equations that you get are even more larger and often it cannot easily visualize purely from the equation what kind of pattern that we are getting. Ultimately the aim is simply the same. I am going to write out, let us say start off with two individual oscillations.

Let us say $x(t)$ and $y(t)$ given by a_2 may be $\sin \omega_2$ t. So, now, I have two oscillations with two different frequencies $\omega_1$and ω_2 and they are perpendicular to one another. So, here I want to again find out what is the pattern of oscillation when you look at it in the $x\,y$ plane. So, this again I will do it through a series of examples, but I will actually show demonstrate it to you on a computer.

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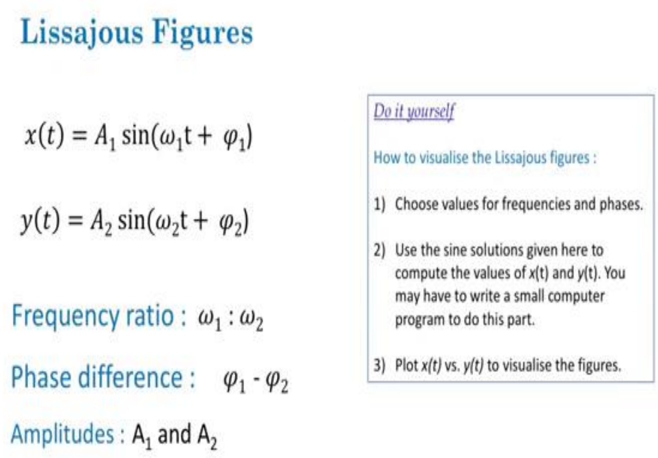

So, now we will see some demonstrations of Lissajous figures. Let me first begin with the case of frequency ratio that is 1 is to 2. So, look closely at the simulation that is shown here.

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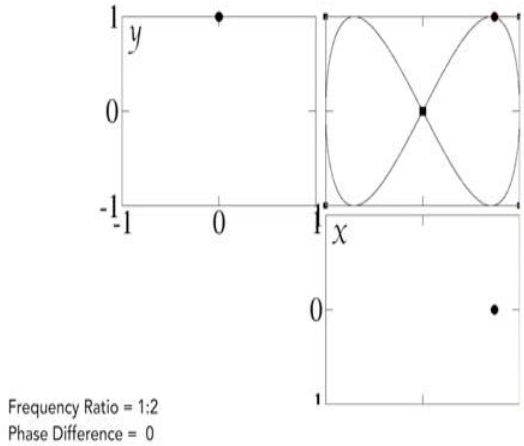

So, at the lower end you see the x oscillator, I will pause it for a second and on the left hand side upper end you see the y oscillator both them are individual oscillators. They are executing oscillations perpendicular to one another. And right hand side top you see the pattern when these two oscillations are combined. So, you already see that we have got something that looks like a two lobes. So, the red dot here traces the superposition of both these oscillations and you should also note that x oscillation is slower by the time x oscillation is completed once, the y oscillation does it twice. So, note these features.

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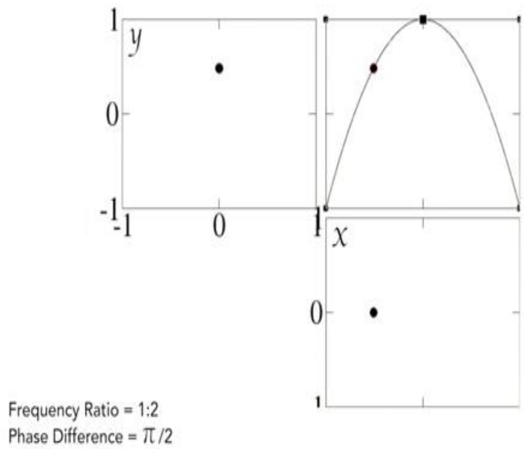

So, in this case the frequency ratios 1 is to 2 as before. Now there is a phase difference of $\pi/2$ between the two oscillators. So, you would have noticed that the starting points of two oscillators were off by a phase of a $\pi/2$ and correspondingly you see that the pattern is now very different, even though the frequency ratios same as it was earlier.

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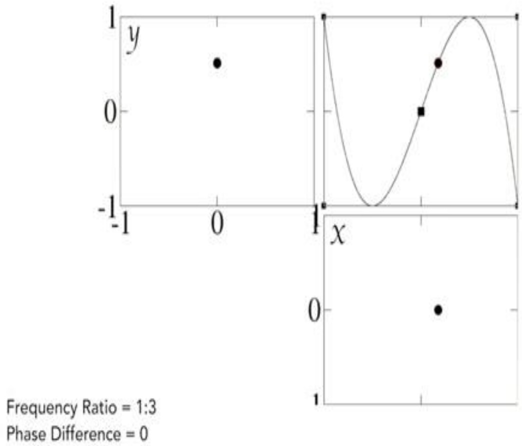

So, in this case the frequency ratios 1 is to 3 and the phase difference between the two oscillators is 0. So, once again should closely note that by the time x oscillator does one oscillation, y oscillator is quite fast it does the oscillation 3 times and you see the kind of pattern that we get.

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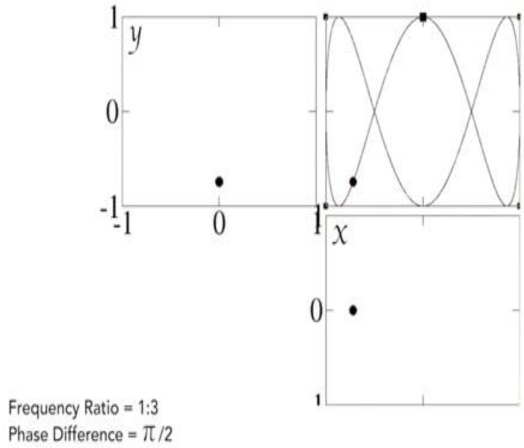

So, this is again the case of 1 is to 3 frequency ratio, but there is a phase difference $\pi/2$ between the two oscillators. So, you see that familiar three lobes in this case, once again note the speed with which the two oscillators are oscillating. So, it retraces the same path again and again.

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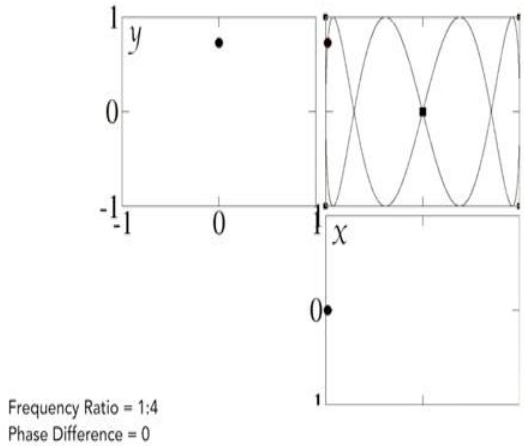

So, this is the case of frequency ratio 1 is to 4. So, you are actually seeing four lobes here and the phase difference between the two oscillators is 0. So, once again you see that it is your superposition is simply retracing the same four lobe pattern again and again.

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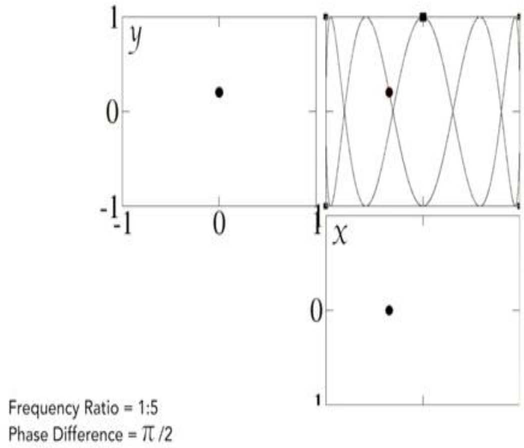

So, as it begins now again the frequency ratios 1 is to 5. So, what should be the pattern be and, in this case, the phase difference is 0. Here you will see a five-lobe pattern that emerges. And finally, it repeats itself again and again. So, whenever you have frequency ratio that is a rational number you will always have pattern that repeats itself.

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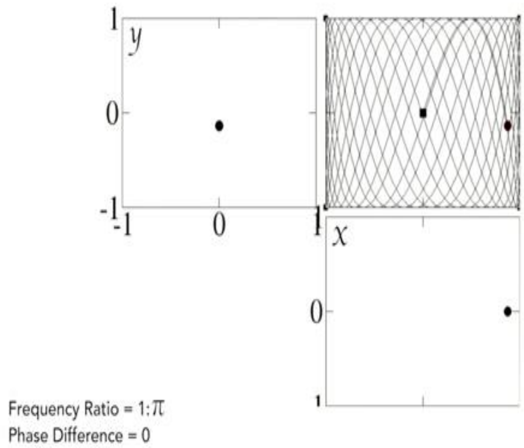

So, let us see one example of a case where the frequency ratio is not rational number. So, the frequency ratio in this case is 1 is to π . So, you will notice that in this case the superposition pattern is never going to repeat itself. So, it slowly retracing a fairly complicated pattern, but you will notice that when it comes back to the starting point it will be little bit away from the starting point.

So, the starting point is the black that is there at the center. So, let us watch it for a moment. It will speed up a part this. You see a very beautiful pattern emerges. And we will wait for until the point comes back close to the starting point. Now it looks like it might go close to the starting point. And as you expect it does not join that and doesn’t retrace the path. So, it’s slightly off and it will keep continuing this and finally, it will fill up the whole space.