Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 39
Fourier Series: Part 1

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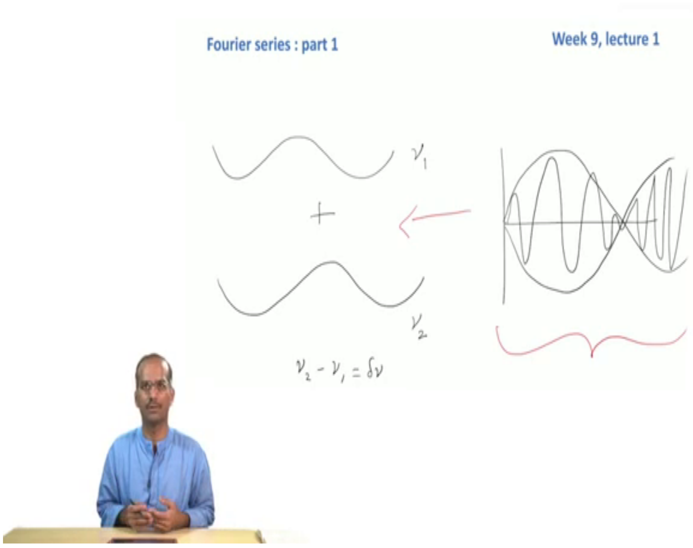

Welcome and as we start week 9 it is a good time to look back at what we had done. So, we started with very simple oscillations of a single object, then we analyzed oscillations of coupled objects. Many beads for example, coupled together and we were interested in the coupled mode of oscillation in other words what is the pattern of oscillation of all the particles together and then we went beyond it brought all the particles close together to form a continuum and it gave us a wave equation.

And, from the wave equation we looked at transverse oscillations described by wave equation, we also looked at the longitudinal waves which are also described by the wave equation. Prominent example of longitudinal waves are the sound waves. So, we derived the formula for speed of sound, we also looked at how the formula would change if we were describing sound propagating in a solid and also what happens when waves in general and in particular sound moves from one media to another media.

Now, what we will try to do is to address a reverse problem for instance we had done this very simple problem where we added two say sin waves with slightly different frequencies. So, let us say that this has a frequency ν; ν_1 and this has a frequency ν_2 and the difference between them is really small. $\nu_2 - \nu_1$ you can assume is really small, think of it as some $\delta\nu$. This when we add them together gave us something like a pattern of beats.

Now, the question is if I were given let us say this object the combined waveform can I get these individual components. So, that is what we are going to do today and this kind of analysis requires this tool which is called Fourier series.

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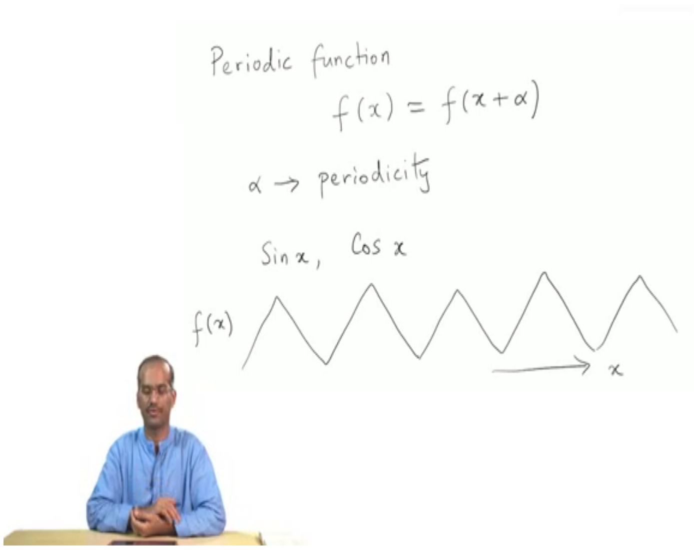

I would say that some function is periodic if let us say $f(x)$ is equal to $f(x+\alpha)$ very simple examples of periodic functions are of course, $\sin x$ $\cos x$ for example, I can construct a periodic function that looks like this. This is also a periodic function. Again, the question is the same; now, can I analyze and find out what components of sins and cosines have gone into making this periodic function?

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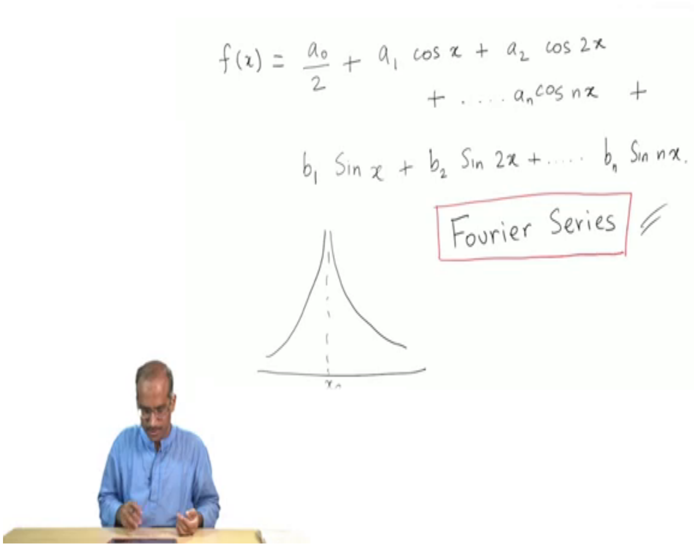

The central idea here is that if I have any periodic function $f(x)$ it can be represented in what is called the bases of sins and cosines functions. So, I can write it as where n can be a extremely large number in principle it can even go off to infinity. So, you will notice that the right hand side which is this summation represents a summation over large number of smoothly varying functions so, which means that we require the left hand side which is the $f(x)$ also be a smoothly varying well behaved function.

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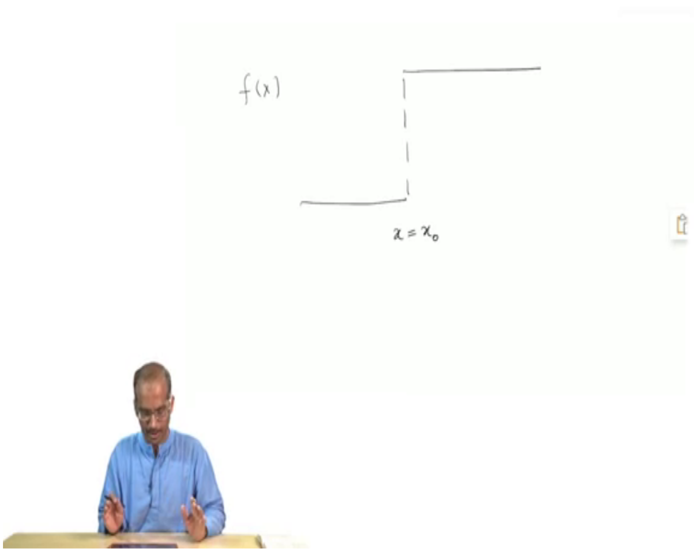

We can still have discontinuities which are finite in nature; for instance, if I had a function of this type. So, let us say that at $x = x_0$, I have this finite discontinuity in the function $f(x)$. This itself is not a problem can be taken care of by this Fourier series.

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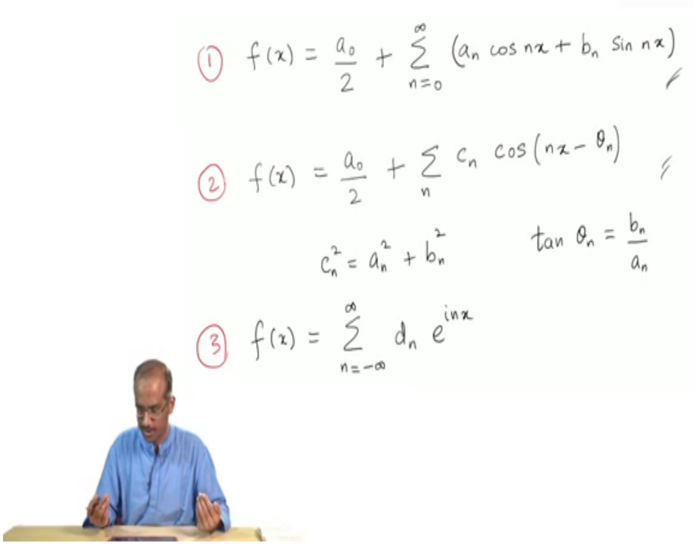

Let me write the Fourier series in a compact notation. $f(x)$ is a_0 by 2 plus I have an infinite summation n going from 0 to ∞. So, I have written down the expression or the Fourier series in a compact notation involving the summation notation. So, the unknowns are actually the a_n's and b_n's. So, those are the quantities which we need to determine if we have to make use of this Fourier series.

So, in general if your function $f(x)$ is sufficiently smooth in the sense that it has only finite number of finite discontinuities in all such cases the summation in the right hand side is known to converge. So, I am going to state this without actually showing you the proof it is written in such a way that we can identify the discrete frequencies involved in making of this function $f(x)$. So, this is another representation of the Fourier series and it does not take much effort to figure out that $c_n^2 = a_n^2 + b_n^2$ and also that $\tan\theta_n = \dfrac{b_n}{a_n}$ and one another possible representation of the Fourier series is in terms of e^{inx}.

So, I have three different ways of representing the Fourier series depending on the problem at hand sometimes one of these is more easier to handle than the other. In either case whichever one that you choose to work with you cannot escape from determining the unknowns.

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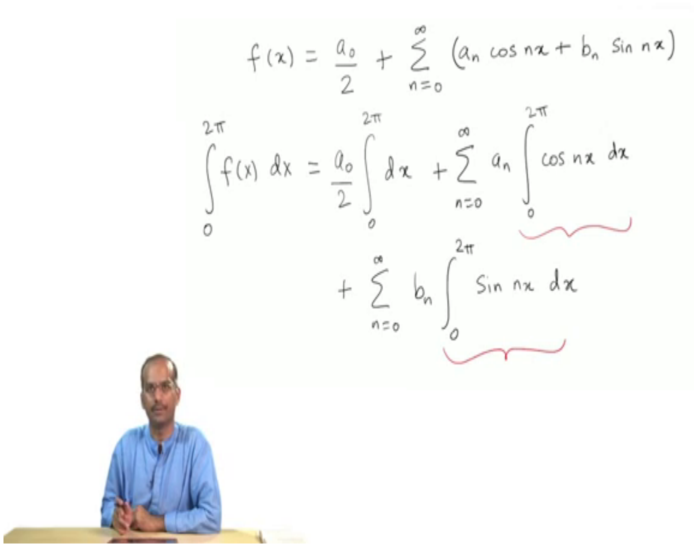

So, let us first determine a_0 and to do that all I need to do is simply integrate the function over the range of periodicity. So, here I will assume that my function $f(x)$ has periodicity 2π $\cos nx$ and the $\sin nx$ integral.

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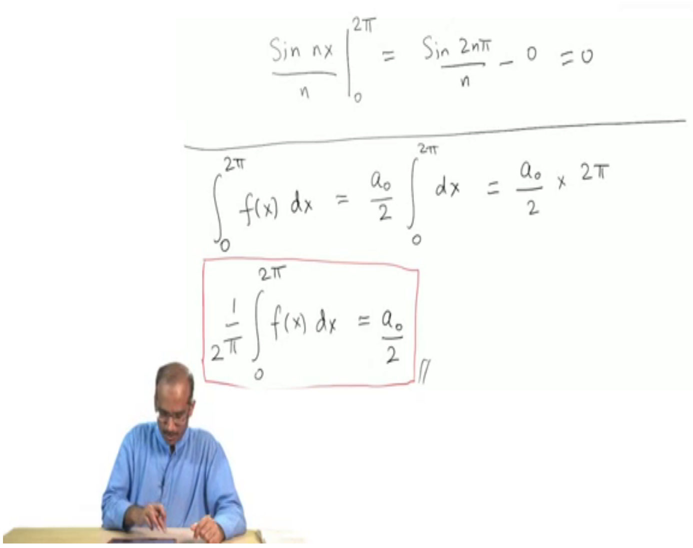

Both will be 0 because for example sin $2n\pi$ for any value of integer is 0 as well. So, this integral is 0. So, clearly the cos nx integrated over 2π is 0. So, that would go away for all values of n and you can do the same exercise for sin nx. So, integrate sin nx between 0 and 2π for any integer value n that would also be 0. So, I am going to have the following result. So, now, I have determined the value of a_0 in terms of the function $f(x)$.

To make this result a little more instructive it is more convenient to divide it by 2 here and divided by 2 here and you will notice that the first term which is $\frac{a_0}{2}$ is very different from all other terms in the series; all other terms involve some cos or sin function oscillatory functions whereas, this one is really a constant. So, this is often called the dc level or it is a constant that tells you where you are and these oscillatory functions oscillate about this the left hand side here is simply the average of the function $f(x)$. Now, let us find out the other two unknowns which is a_n's and b_n's.

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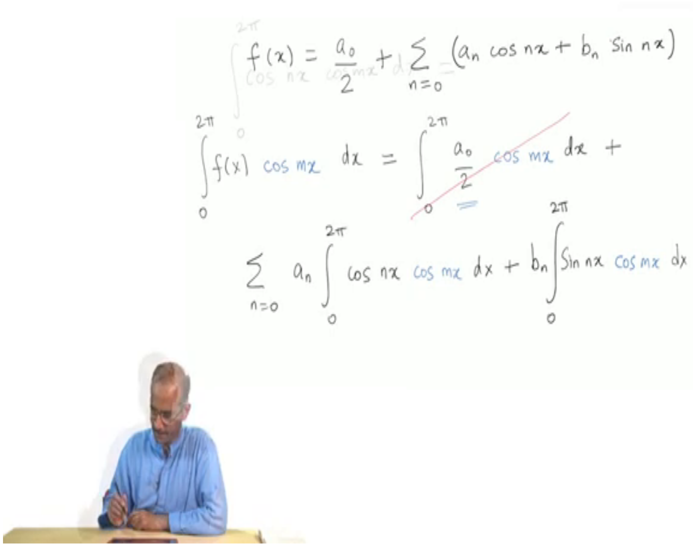

So, what I have done is to simply multiply by $\cos nx$ throughout the equation and integrate over 0 to 2π. So, a_0 by 2 can be taken outside which is this term can be taken outside the integral and integral of $\int_0^{2\pi} \cos nx\, dx$ will give you 0, this is something that we just saw. Hence this term will go to 0.

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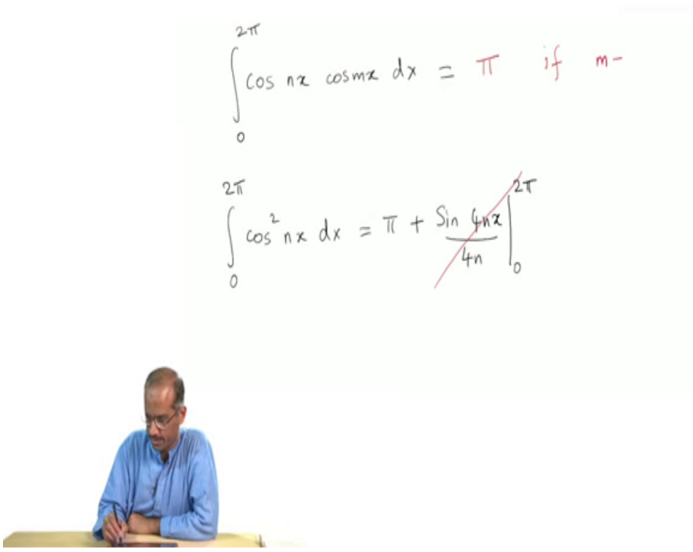

n and m are integers we should keep that in mind. So, let us see what happens if n is equal to m. So, in that case this whole function would simply become $\cos^2 nxdx$ evaluated between 0 and 2π and this integral is very easy to do. So, we are left with only π as the answer. So, let me write the answer here. So, the answer is π if $m = n$.

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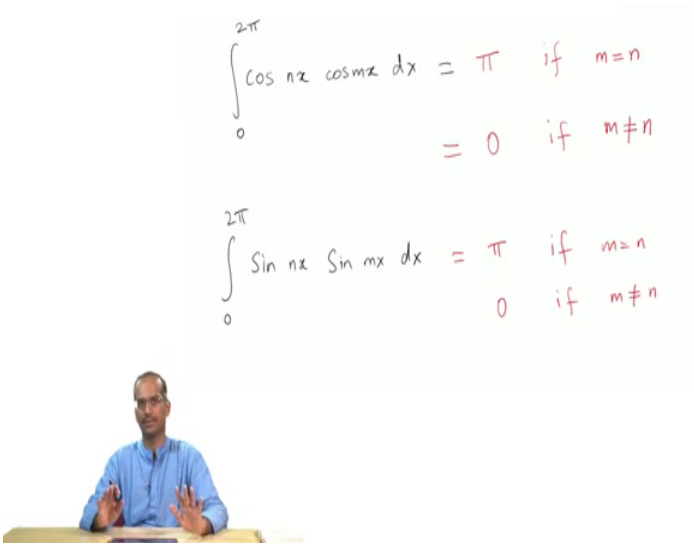

So, if $m \neq n$ you can use the $\cos a \cos b$ formula; so, the result will be in terms of $\cos(a+b)$ and $\cos(a-b)$. So, this integral when m and n are not equal now you can integrate this function very easily it is simply a cos function. So, the result will be a sin function. This entire function irrespective of the value of n and m until they are integers is going to give you 0 if $n \neq m$. So, this will be 0 this will also be 0. So, now, I can again write the result of this integral as being equal to 0 if $m \neq n$.

And, similarly I urge you to work out what happens if you integrate between 0 to 2π the $\sin nx \sin mx$ integral and this is equal to π as well if $m = n$ and it is also equal to 0 if $m \neq n$. Let us say $\cos nx \sin mx$ integrated over dx between 0 and 2π. So, in such a case irrespective of what m and n are until they are integers the result is always 0.

The integral here for example, will survive only if m and n are equal and this one is a combination of sin and cos integral will not survive for any value of n and m any integer values of m and n. So, this will always be equal to 0 whereas, this one will survive if $m = n$ in which case we will have.

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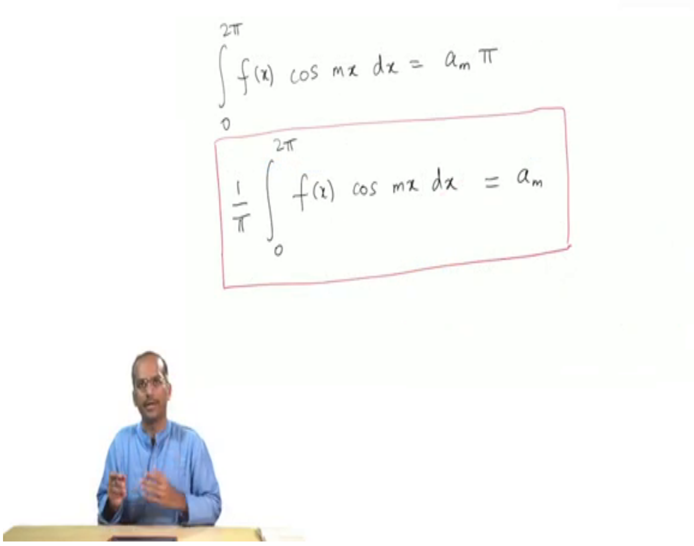

I need to do a very similar exercise to find out b_n's. So, all I need to do is to go back to this the first relation that we wrote down for Fourier series, multiply throughout by $\sin mx$ and integrate over 0 to 2π.

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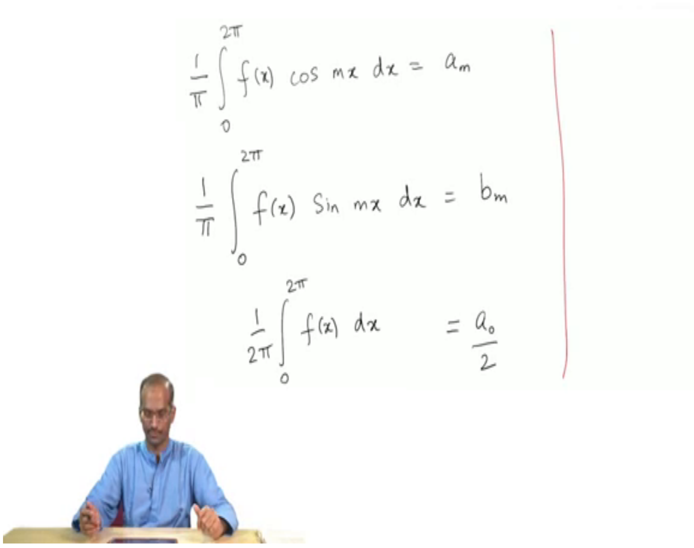

I have collected all the relations here. So, with this in terms of integral over $f(x)$ we know all the coefficients which are a_m's, b_m's and a_0. So, as I said a_0 or a_0 by 2 gives you the average of the function and these other coefficients tell you tells you which frequencies are involved in the construction of the given function.

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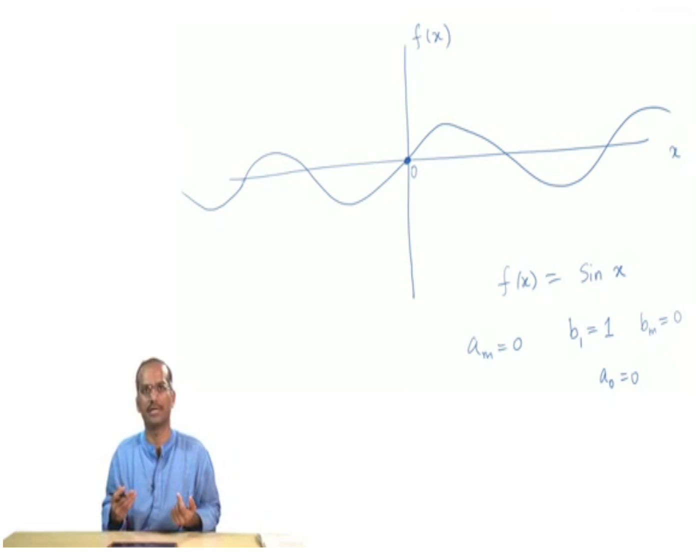

Just to give you a pictorial example for instance I have this function here which I just plotted out and you can see from the looks of the function clearly if I take this to be the origin this point here if that is the origin, then clearly the function is oscillating about x equal to about y equal to 0. So, which means that the average of the function is 0 and the oscillatory component if it is a pure sine curve our analysis will tell us that only one term of this b_m's would survive and everything else will be 0 and none of the cosine integral will survive. So, all the a_m's will be 0 for all values of m and b_m all of them will be 0 except one and $a_0 = 0$.

So, it is a simple exercise it is very intuitive you can try and do it in other words to do it more formally you should assume that your $f(x)$ is sin x. The result of this would be that all a_m's will be 0 for all values of m; b_1 will be 1, but all other values of b_m other than 1 will be equal to 0 and a_0 of course, will be equal to 0 as well. So, a_0 is related to the level the average of the function and just by looking at it you can very easily say that the average of the function is 0.

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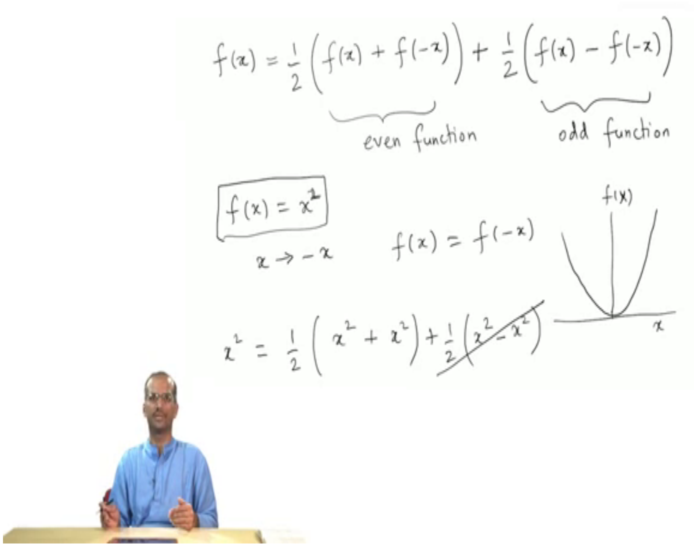

What we know is that if I have a function $f(x)$ a periodic function $f(x)$ I can write it in the following manner. To understand what is happening let us take a simple example this function would be called an even function because under the transformation that x goes to $-x$, $f(x)$ is equal to $f(-x)$. So, you put $x = -x$ the function does not change and in fact, you can even plot the function. So, it would look like this. So, it is a quadratic function in x.

So, on the left hand side of the equation that I have written down I have x^2 and this will be half of $f(x)$ is of course, $x^2 + f(-x)$ for this choice x^2 will be x^2 itself and again half $f(x)$ is x^2 minus $f(-x)$ is x^2. So, clearly the second term will go to 0 and the first term gives you x^2 itself which is identity the point that I am trying to make is that you can write any function $f(x)$ in the form that I have written down here.

So, there is one part the first part which would be which would survive if it is an even function and then there is a second part which would survive if it is an odd function.

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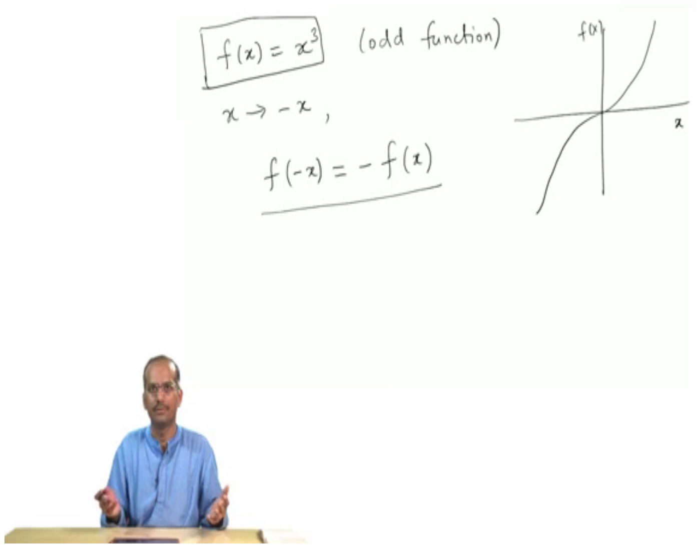

So, this is $f(x) = x^3$. this is an odd function under the transformation that x goes to $-x$, $f(-x)$ will go to $-f(x)$. The x^2 function is symmetric about this line which is $x = 0$ line whereas, this one is sort of anti-symmetric.

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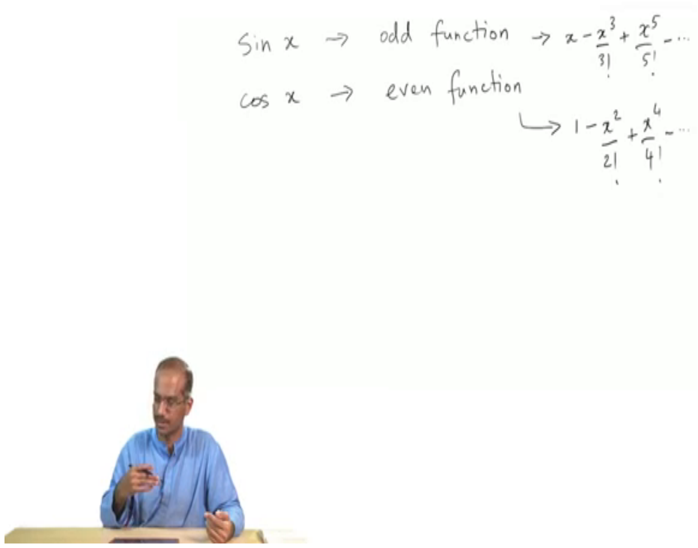

The sin functions are odd functions; sin x is odd function about $x = 0$ and similarly cos function $\cos x$ is an even function. The $\cos x$ function has expansion in terms of x^2, x^4, x^6 and so on whereas, sine function has expansion in terms of x, x^3, x^5 and so on. This general result for $f(x)$ that I have written down it says that any function can be written as an even function plus an odd function.

And, clearly our Fourier series is in that form. So, if my $f(x)$ is an even function then the only contributions would come from the cosine basis which means that all a_n a_m's would be non-zero and all the b_m's would be 0.

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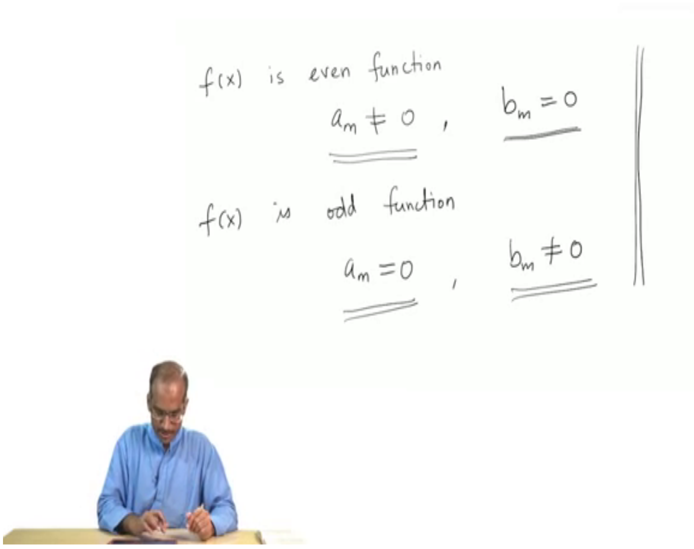

So, this is the gist of what we just learnt about the symmetries of the function. So, if your $f(x)$, the periodic function is a even function then all the contributions will come only from the even part of the expansion which is the cosine part hence all the a_m's will be non-zero and all the b_m's will be 0. So, if you know that your function originally is a even function you do not even have to calculate the values of b_m and similarly, if you know that your function is odd function then all the a_m's are 0 for all values of m and you need to only calculate the values of b_m alone.

But, in general if your $f(x)$ is does not have any particular symmetry of course, there would be contributions from both.

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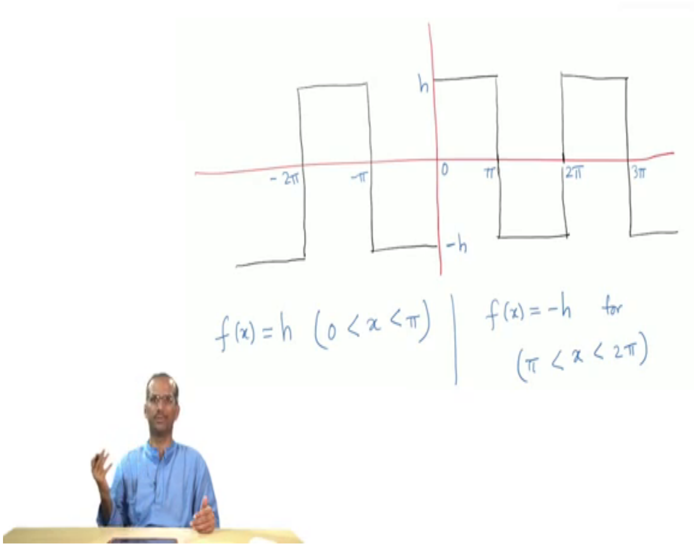

With all this understanding now let us do one non-trivial problem. So, the problem is I have this periodic function which I have drawn for you here. Analytically it is described as follows; clearly the function is periodic as you can see. So, the basic periodicity for instance you can take it either from 0 to 2π or from $-\pi$ to $+\pi$. So, one thing you will notice is that the function oscillates about $y = 0$ which means that the average level of the function is 0.

So, without doing any calculation I can say that the DC level or the steady state level is 0. So, a 0 you should get as 0. That is one thing. Just look at the part of the figure between $-\pi$ to $+\pi$. So, you will see that it is odd function. We just now a while back argued that if we have an odd function and if we are going to do Fourier analysis of an odd function the only surviving terms will be b_m's; b for all values of m and all the a_m's will be zero. In other words, the cosine series should all be zero; it is only the sin series that will survive.

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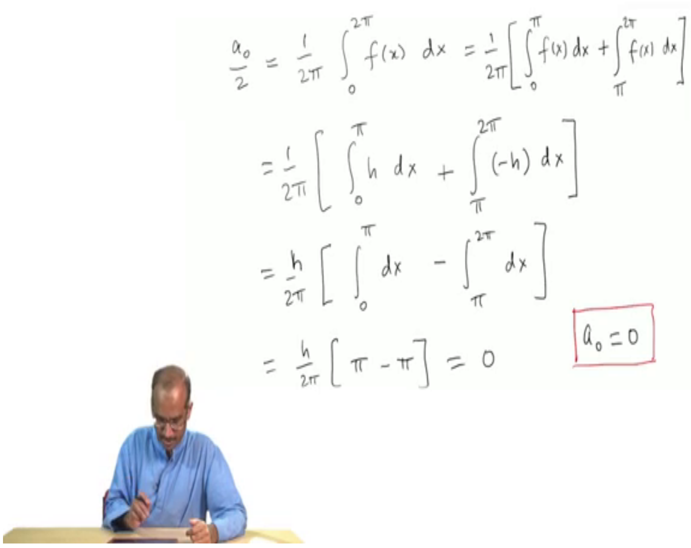

So, let us first calculate a 0, the standard formula which we just derived is already given here and now if I do it between 0 and 2π one complete period I need to split the integral into two parts because my function is defined differently between 0 and π and differently between π and 2π ok. You can see that here. So, I can rewrite this as $\frac{1}{2\pi}\left[\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)dx\right]$. So, this is something we expected based on just looking at the function. Now, let us calculate the a_m's for values of m other than 0.

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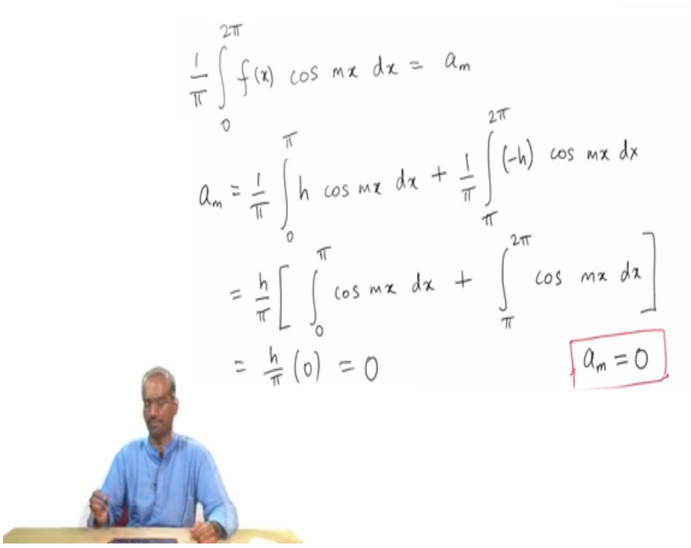

So, to find a_m again we do the same technique of splitting the integral between 0 and π it is easy to do this integral both the integrals will give you a sin function and the sin function is 0 at every limit of this integral at 0 at π and at 2π. So, the answer is going to be $\frac{h}{\pi}$ multiplied to 0. So, the answer is 0. So, the result is that a_m is all equal to 0 for any value of any integer value of m. This is something that we guessed based on the fact that this function is a odd function. Now, finally, let us calculate b_m's.

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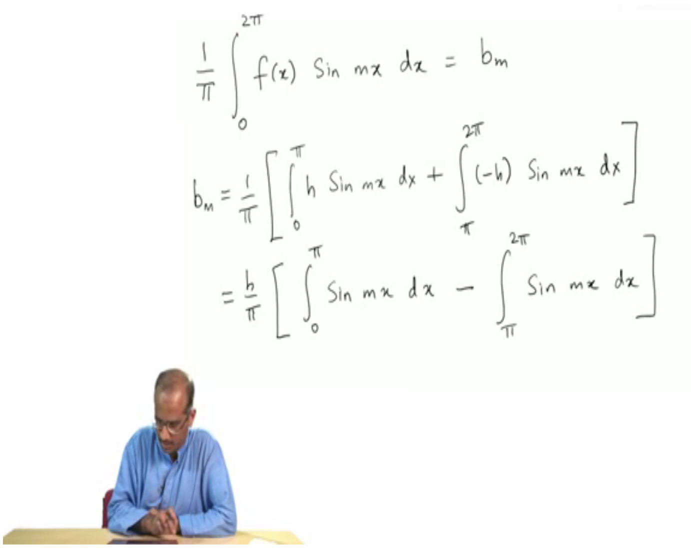

So, let me quickly do the calculations. So, $\frac{1}{\pi}$ and integrating these functions are simple because both are cos functions. So, let us quickly do that.

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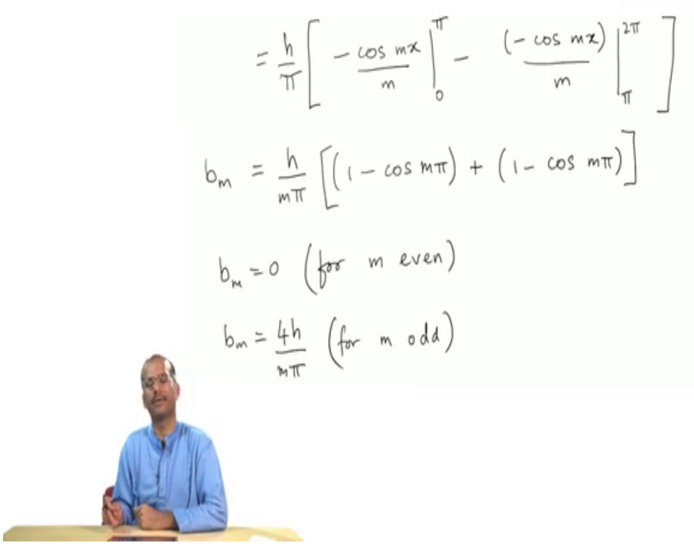

So, that is going to be $\frac{h}{\pi}$. So, I am going to get $-\frac{\cos mx}{m}$ between 0 to π minus $-\frac{\cos mx}{m}$ between π and 2π. If you put in the limits this is going to give me. So, this tells us that b_m is equal to 0 for even values of m b_m will be equal to $\frac{4h}{m\pi}$ for odd values of m.

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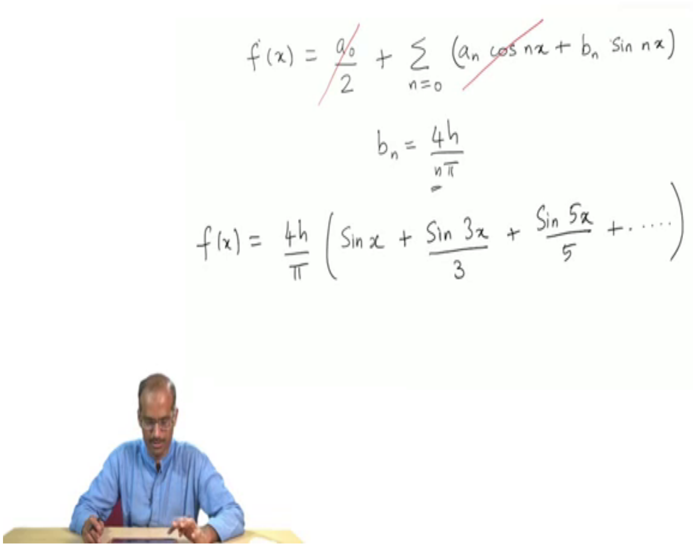

So, this was the Fourier series expansion that we initially wrote down, and now we know that a_0 is 0. So, this term will not exist and we also know that all a_n's are 0 this term also will not exist and from what we just did we also know that b_m's survived only if m is an odd integer. So, the first surviving term will be $n = 1$. So, that will be sin n is 1 so, it is sin x. Next term will be sin $2x$, but we have said that even n's the value of b_m is 0 so, that term would not exist. So, the next surviving term will be sin $3x$ and of course, there will be a divided by 3 here because of this n plus $\frac{\sin 5x}{5}$ and so on.

So, this series that we have obtained represents this function that we have drawn here. So, you can see what we have achieved. So, we have this function which has finite discontinuities at several places at π, at 0, and so on. And, now all that is represented in terms of an infinite series in terms of sine function. So, that is what you get when you Fourier analyze any periodic function.