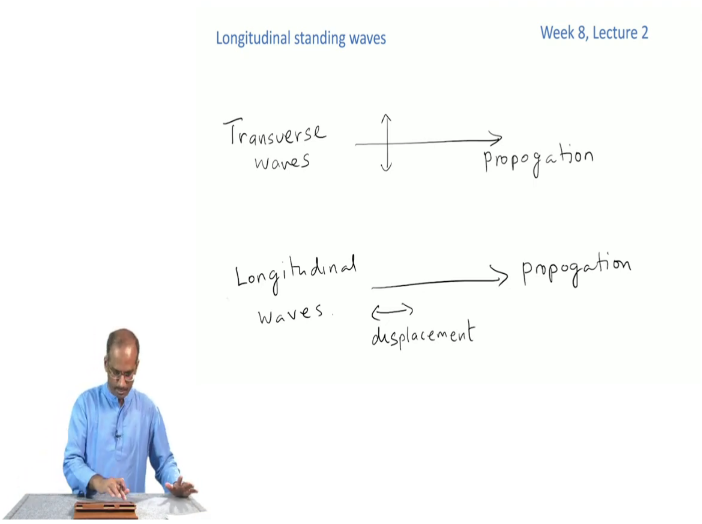
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 36
Longitudnal Standing Waves

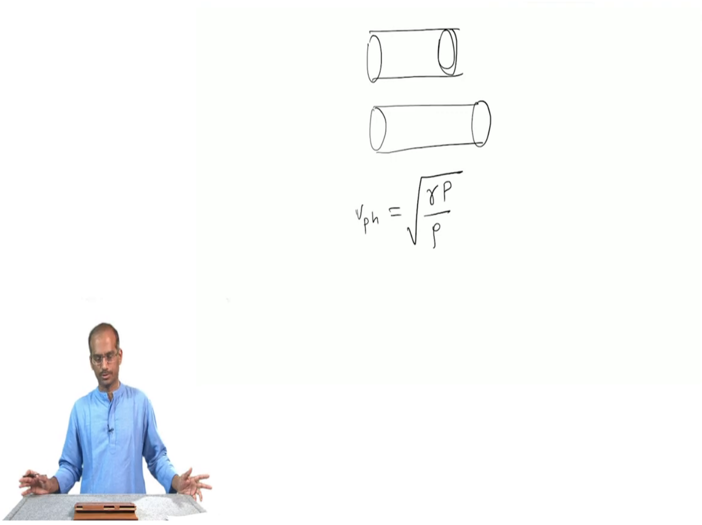
Welcome to this second lecture, we will be studying Longitudinal Waves. So, originally we were looking at transverse waves for most of the time which means that the direction of displacement and the direction of propagation were both perpendicular to one another.

(Refer Slide Time: 00:37)



Then we started looking at the longitudinal waves. So, in this case the direction of propagation and the direction of displacement both are parallel to one and another. So, we saw one prominent case of longitudinal waves which was that of a sound wave.

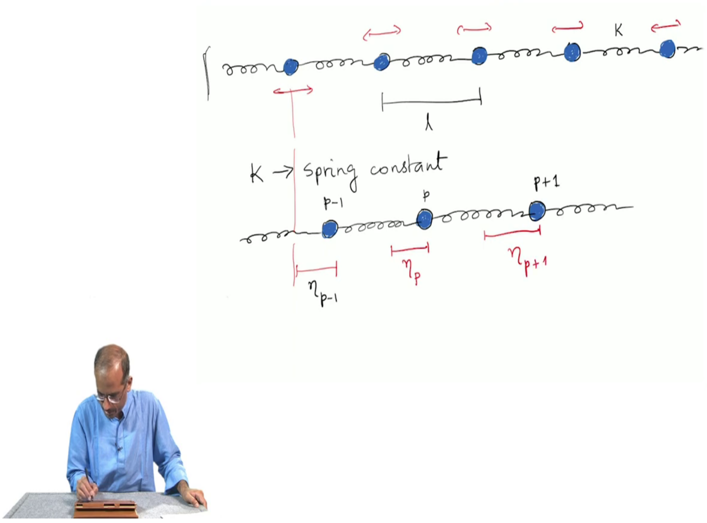
(Refer Slide Time: 00:58)



And of course, the basic mechanism of sound waves is simply the propagation of pressure differences or pressure disturbances. And you say that in the absence of any other external force things and so on this parcel of air is in equilibrium with all the surroundings which is rest of the air. You can obtain the phase velocity of such a wave and phase velocity is simply the velocity of sum or speed of sound in this case and it is given by this expression $\sqrt{\frac{\gamma P}{\rho}}$.

So, γ is the ratio of specific heat capacities, P is the pressure and ρ is the density of gas. We should be able to see pretty much all the phenomenon we encountered in the case of transverse wave for instant standing waves, we looked at the case of standing wave in good detail. So, we should be able to see standing waves for longitudinal waves as well.

(Refer Slide Time: 02:16)



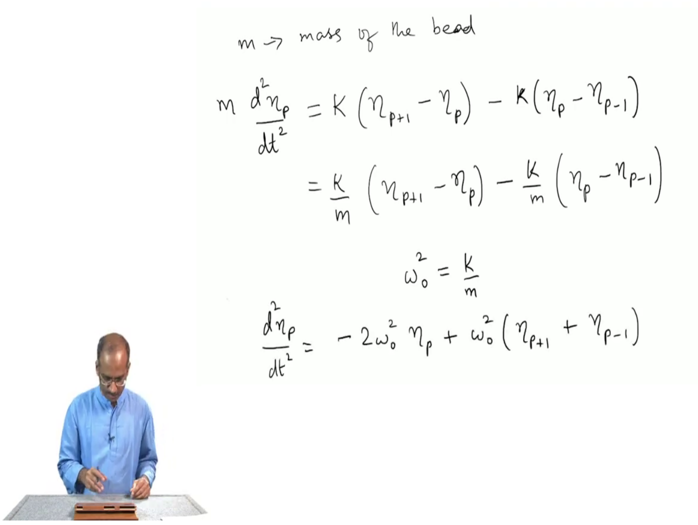
But before we get into looking at the longitudinal standing waves lets once again go a step back and consider a simple model. So, I am having a large number of beads like this which are connected by springs and I am going to take this length between any two of these beads to be l and these springs I will assume that all of them have spring constant K. And what I would want to do in this case it is not to look at the transverse oscillations of this collective system.

So, its again a coupled oscillator kind of problems we made several times earlier on, but now I want to look at how the waves are propagated in the case when each of these beads that we have oscillate in the horizontal direction. Which means, they are not going to oscillate in this direction, but each of these beads that you see here would oscillate about the mean position in the horizontal direction like this.

So, towards this end now let me draw one possible configuration oscillating configuration of this system. Let me say that this is the Pth or $P-1$th bead and this is P th bead and $P+1$ th bead; so, my displacement with respect to the equilibrium configuration. So, the equilibrium configuration is this one which is shown at the top and I have indicated by this vertical red line the position of the $P-1$th bead in the equilibrium configuration. And with respect to that the new configuration is has been displaced by an amount η_{P-1}.

And similarly I can indicate this displacement of the Pth bead as η_P and similarly for the case of $P+1$th bead. So, that is displaced by an amount η_{P+1}. So, the equation of motion would turn out to be something like this.

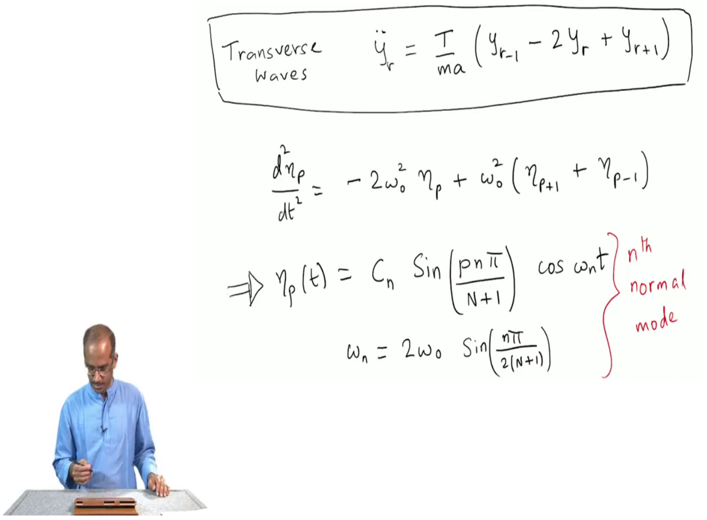
(Refer Slide Time: 04:51)



So, I am going to in specific focus on the Pth bead, write the equation of motion for the Pth bead. In this case I am going to get the following relation together we have our equation of motion. So, I will let you do that that this equation of motion indeed gives you the correct limiting case when you hold the η_{P-1} are actually the $P-1$th the bead and $P+1$th bead fixed in position.

So, the only oscillating object is the Pth bead at the center. So, with all these let me rewrite this slightly differently as $\frac{K}{m}(\eta_{P+1}-\eta_P)-\frac{K}{m}(\eta_P-\eta_{P-1})$. And as usual I am going to identify ω_0^2 as $\frac{K}{m}$ in which case this equation can be rewritten in a slightly different way. So, this is a sort of standard form for our purposes.

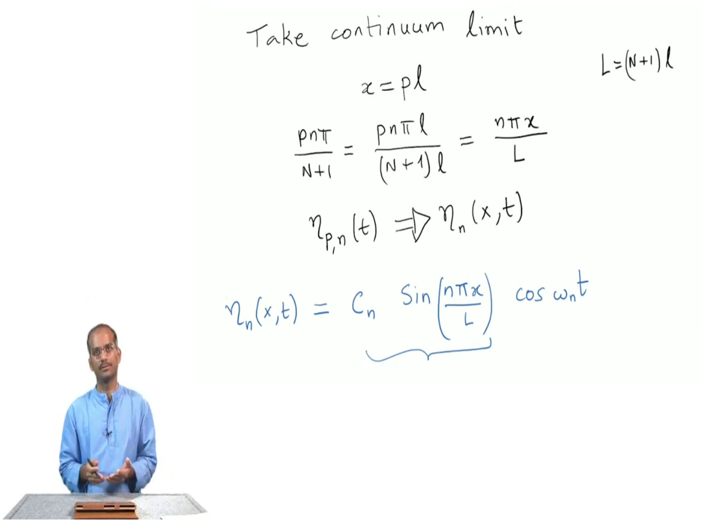
(Refer Slide Time: 06:23)



So, this is to remind you of what we had got for the case of transverse wave few weeks back and I would like you to compare this equation with the equation that we have just now obtained here which is this equation. Given that the equation of motion is identical in both the cases, we just need to use the same solution that we had written down earlier on. So, let me write down the solution for the Pth bead. So, η_P will be a function of time that is a solution that we are looking for.

So, in writing this solution you assume that there are N of these beads when we go to the continuum limit it really would not depend on this N, but for now we will just keep this N and remined ourselves that this N is the number of beads. So, this two set of equation that I have written down here; one is η_P displacement of the Pth bead in nth the normal mode. So, that is what this one gives me and ω_n is of course, the frequency or the normal mode frequency for the nth mode.

(Refer Slide Time: 07:52)

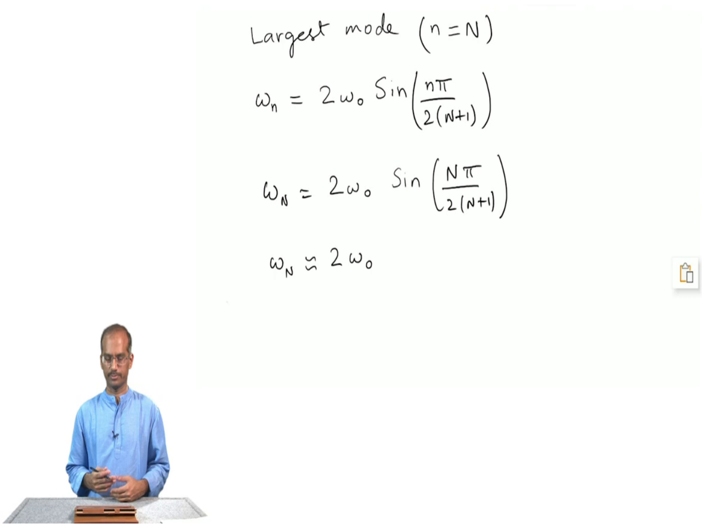


So, we can go to the continuum limit, I can introduce this variable x which is Pl, you can multiply and divide by l. So, it will be $\frac{Pn\pi l}{(N+1)l}$ So, this would simply give me as you can see Pl as x. So, it would give me $n\pi x$ divided by; $N+1$ is that yeah, N is the total number of beads multiplied by the length between two successive beads will give me the total length. So, a L is equal to $(N+1)l$.

Now, I can plug this in my equation and before I do that we also do this small change of notation $\eta_{P,n}$ indicates nth normal mode, Pth bead, η is the displacement. So, this we shall be designated as η_nth normal mode x, t. So, now I have replaced P by x reflecting the fact that we have gone from the discrete set of points to continue. So, now, its straightforward for me to write the result.

So, if you notice the kind of solution that I have written down especially this part here $\frac{\sin n\pi x}{L}$, it will tell you immediately what kind of boundary condition has already been imposed in the problem. So, at $x = 0$ the displacement is 0 and at $x = L$ the displacement is 0 again, even though we did not put in any boundary condition its actually implicit in the solution.

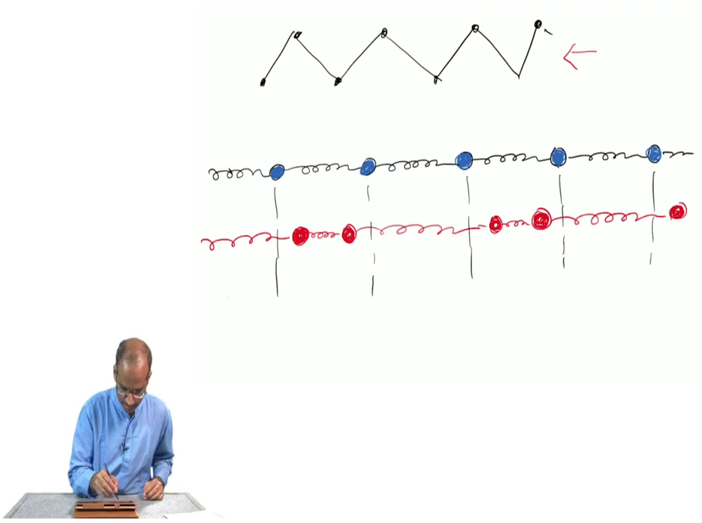
(Refer Slide Time: 10:00)



So, with this understanding of the solution that we have got let us look at the largest mode which means that n which indexes the mode I will take it to be N. And now if I plug in that n is equal to N, ω_N will be equal to $2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$ and in the limit that N is very large $\frac{n}{N+1}$ is approximately 1. So, inside this term it will be $\sin\frac{\pi}{2}$ which is 1, hence ω_N is approximately equal to $2\omega_0$.

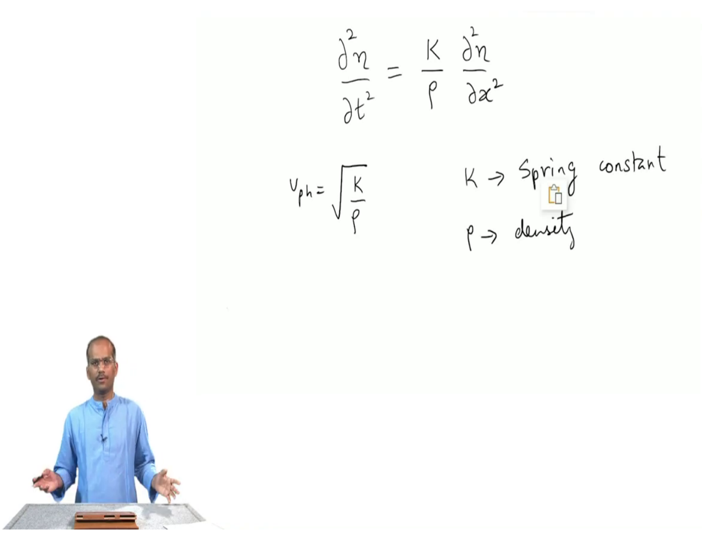
So, this is the largest frequency that will be supported by the coupled oscillator system that we have considering. Again this result is reminiscent of what we had seen for the case of transverse waves as well even in that case the largest frequency; the largest normal mode frequency was indeed $2\omega_0$, twice the smallest frequency of the problem. So, in this case also the smallest frequency is ω_0 and the largest frequencies 2 times that. So, this is how the pattern would be for the case of largest frequency.

(Refer Slide Time: 11:25)



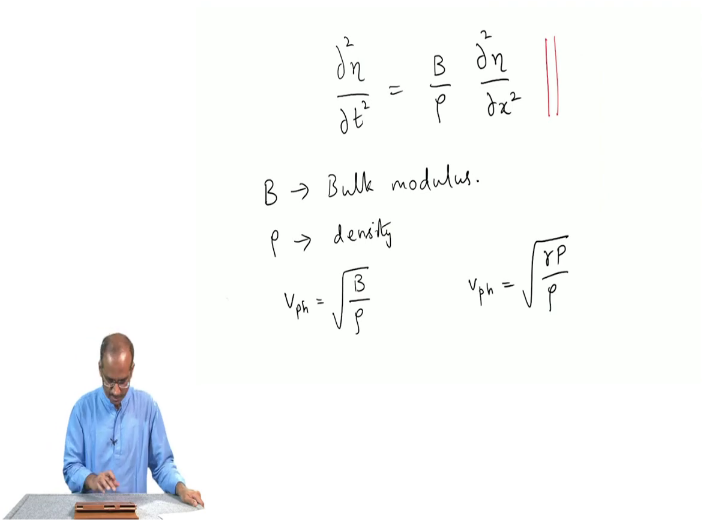
So, we will seen this and also worked out the conditions under which this comes. Now, what is the equivalent of this in the case of longitudinal waves? So, in the case of longitudinal waves what you could expect is that the adjacent springs would be alternatively compressed elongated; compressed elongated. So, you could see that alternatively we are seeing that they are compressed and elongated; now we are ready to write the governing equation.

(Refer Slide Time: 12:13)



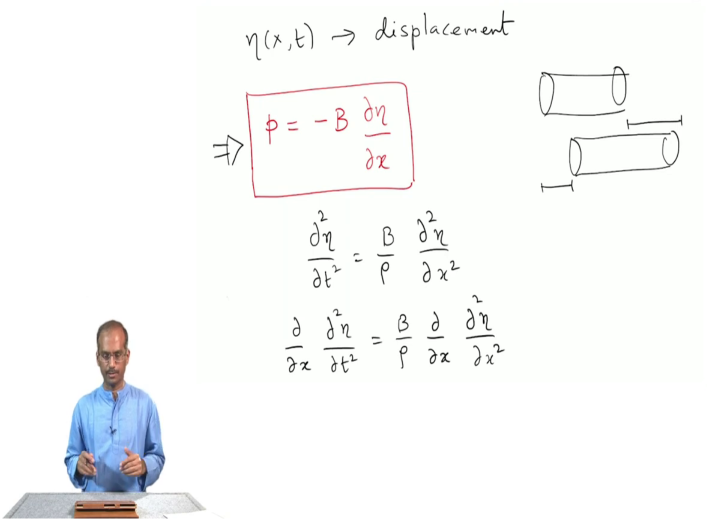
So, it is going to look like $\frac{\partial^2\eta}{\partial t^2}$ and that would be equal to $\frac{K}{\rho}\frac{\partial^2\eta}{\partial t^2}$ and here the phase velocity will be $\sqrt{\frac{K}{\rho}}$.

(Refer Slide Time: 12:42)



Now, it is straightforward to extend this to all the cases of longitudinal waves. So, this is my wave equation and B is the bulk modulus. In the previous module I was using K as the bulk modulus, but since it can clash with k being used as a notation for wave number, I have change the notation for bulk modulus to be B. And the phase velocity would simply be $\sqrt{\frac{B}{\rho}}$ and in the specific case of sound waves in air bulk modulus can be calculated starting from gas laws in which case you could write the phase velocity as $\sqrt{\frac{\gamma P}{\rho}}$.

(Refer Slide Time: 13:36)



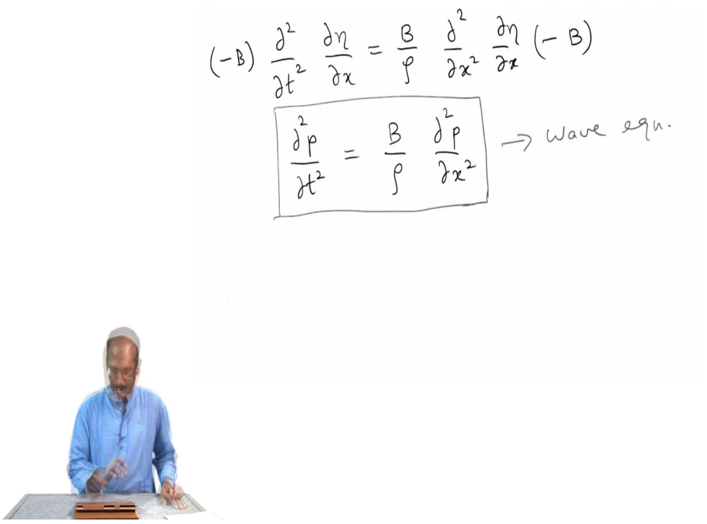
So, as I said this equation has been written with η as displacement. So, η is a displacement and it is a function of position at time, we had a small parcel of air. So, there was one end of it which moved by some amount and the second end which moved by a slightly different amount and we said that when we meant displacement we actually mean the average of these two displacements.

So, that is one way of looking at it and we have written down our equation of motion in terms of in some sense think of it as the average displacement of both the ends of the parcel of gas. We took $\frac{P_1 - P_2}{x_1 - x_2}$ and in the limit where the length of this parcel $x_1 - x_2$ is sufficiently small enough, we said that $P_1 - P_2$ can be written as ∂P. So, it becomes $\frac{\partial P}{\partial x}$.

So, the difference in pressure really is the pressure difference with respect to atmospheric pressure in a practical case of sound waves in air. There are two possibilities with this wave equation; one is I can write the wave equation in terms of displacement of this parcel which is what I have done here. But, now I can also rewrite this same wave equation in terms of P; P being the difference in pressure with respect to atmospheric pressure, P which is that pressure difference that is $-B$, P is bulk modulus into $\frac{\partial \eta}{\partial x}$.

So, now we are going to start using this equation and together with this that we have will combine both to write out a wave equation in terms of P, let me start with this equation in terms of displacement. Now, what I want to do is to take the derivative with respect to x on both sides. So, I will have $\frac{\partial}{\partial x}\frac{\partial^2 \eta}{\partial t^2}$ is equal to $\frac{B}{\rho}$, B and ρ are constants, so I will be able to write it something like this.

(Refer Slide Time: 16:16)

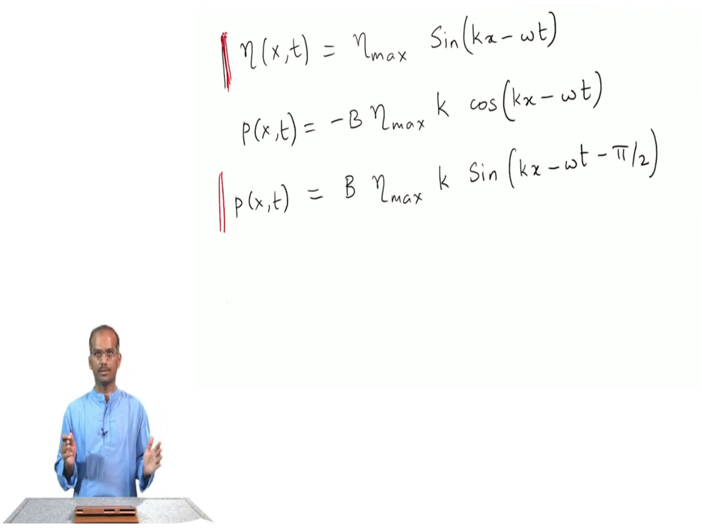


On the left hand side we will exchange the derivatives. So, I am going to rewrite it as $\frac{\partial^2}{\partial t^2}\frac{\partial \eta}{\partial x}$ is equal to $\frac{B}{\rho}$ and again do a exchange of derivatives, so that. And this kind of exchange of derivatives is allowed here mathematically and allowed operation.

So, now let us appeal to the relation that I have here this $P = -B\frac{\partial \eta}{\partial x}$. So, if I want to write my wave equation in terms of P, let us multiply both sides here by $-B$. So, now, you will see that $-B \times$ this $\frac{\partial \eta}{\partial x}$ is simply B. So, this left hand side would become simply $\frac{\partial^2 P}{\partial t^2}$ and that is equal to $\frac{B}{\rho}$ and here again it is the same thing $-B\frac{\partial \eta}{\partial x}$ is P.

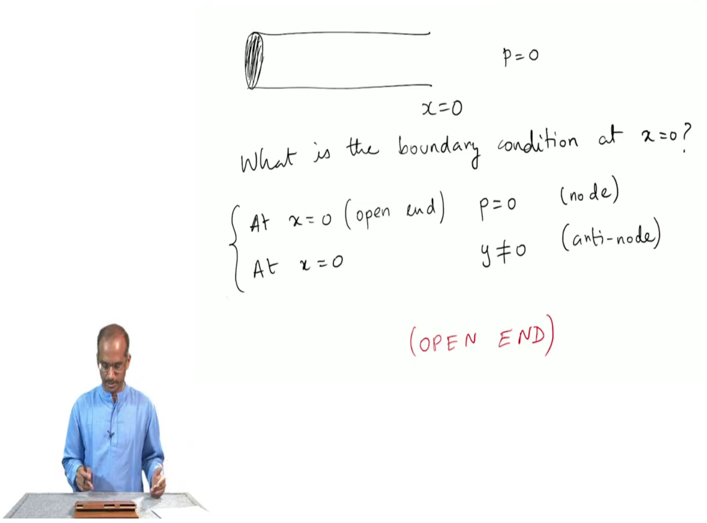
So, I am going to have $\frac{\partial^2 P}{\partial x^2}$ and it is very important to note that this P is related to displacement through this equation and this negative sign is very important. Now, we will see why this negative sign is very important and what is the implication of this relation that P is related to the gradient of displacement.

(Refer Slide Time: 17:54)



So, this is an admissible solution for this wave equation, P would be equal to $-B\frac{\partial \eta}{\partial x}$. So, I can calculate $\frac{\partial \eta}{\partial x}$ from this equation. So, that is my expression for P in fact, I should more correctly write this as P which is a function of position and time. So, in this case I would like you to look at both these functions. So, displacement and pressure they will maintain a phase difference of $\frac{\pi}{2}$ which is why this negative sign is very important.

(Refer Slide Time: 18:35)



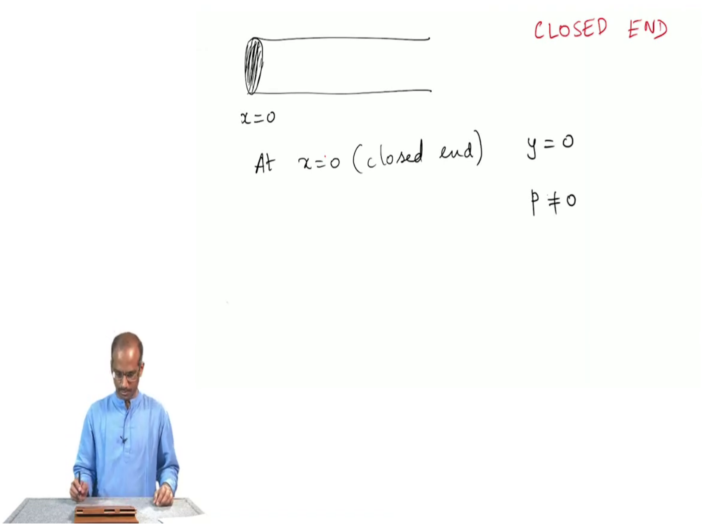
Let us look at what happens when you have longitudinal wave setup in a tube like this. So, I would taken a tube with one end closed. So, the first thing that we need to worry about is the boundary conditions. So, to help us understand the boundary conditions let me denote this open end by $x = 0$. So, $x = 0$ right now corresponds to the open end. The question is what is the boundary condition at $x = 0$?

So, now we have two ways of looking at it either you look at it from the point of view of displacement or look at it from the point of view of pressure. Since, it is an open end, at that end the air is basically in communication with rest of the environment where presumably the pressure is the atmospheric pressure.

So, which means that outside of this tube $P = 0$ and inside of the tube if you do not want physically unrealistic things happening pressure is and should be continuous in which case at $x = 0$ which in this case implies that at the open end of the tube, the boundary condition is $P = 0$. If $P = 0$, it implies that it is a node as far as P is concerned, but because there is a phase relationship between P and y. This would also imply that if $x = 0$ in this case the open end corresponds to a node, it has to correspond to anti node as far as displacement is concerned.

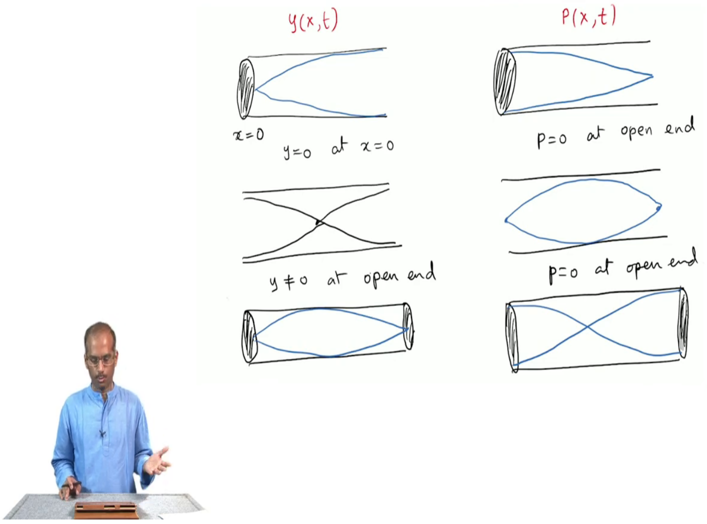
So, which means that at $x = 0$, y should not be equal to 0. So, it will be an anti node. So, if it is a node this would be an anti node. So, let us now look at what happens at the closed end of the tube ok. What is the boundary condition at the closed end?

(Refer Slide Time: 21:13)



At the closed end of the tube if there is going to be any kind of displacement of the particles there either the particles will have to move into the body of the cross section that covers this tube or they will have to move away in the other direction. If they move away in the other direction of course, they would be creating vacuum there and neither can this mass of air enter the inside the material there at the closed end at $x = 0$. So, in this case it is the closed end $y = 0$ and again since P and y maintain a phase difference. So, at closed end $P \neq 0$.

(Refer Slide Time: 22:07)

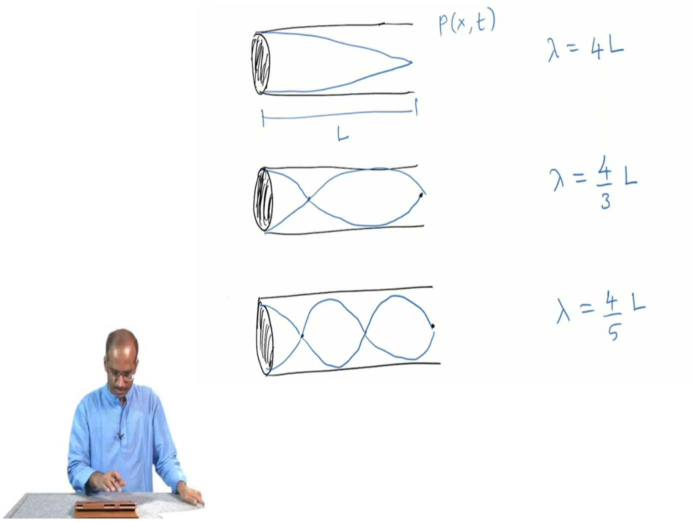


Now, we can actually look at some of the patterns of oscillation in tubes of various kinds. So, let us see the same thing from the perspective of displacement as well as from the perspective of pressure. So, let me show here the perspective of displacement and here the behavior as seen for pressure. At an open end the boundary condition is that pressure is 0 and at the closed end the boundary condition is displacement is 0. The lowest frequency of a standing wave this is how it would look like. So, this respects the boundary conditions that we just obtained. So, this is the pattern of oscillation for the lowest mode. Now, suppose I had a open tube that it is open on both ends, what would happen.

So, in this case as far as the displacement is concerned; the displacement is not 0 at both the ends. So, which means that it can support a node at the center, but at any of its ends it is going to be like this. Whereas, for the pressure at the open end pressure is 0 which means that this is going to generate a node at the open end and it will look like this.

Now let us look at the case when both the ends are closed. So, the lesson is that the pressure is 0 at open end, but not 0 at close end which is what you see on the right hand side here, and the displacement is 0 at the closed end and is not 0 or anti node at open end.

(Refer Slide Time: 24:10)



So, similarly you can see the pattern that is emerging and try and draw for one or two more cases to be sure that you understand what is happening here. So, the central idea that I want to convey here is that for the general case of longitudinal waves we get a wave equation which is similar to the case of transverse waves. And we need to look at separately two possible cases; one is displacement and other is the pressure difference. And we looked at what kind of boundary conditions would be suitable for both these cases for open end and for closed ends.

So, we saw that for the case of open end; at the open end the pressure has to go to 0 and for the case of closed end displacement has to go to 0 and pressure will not go to 0. So, it will be a node for displacement, but an anti node for pressure. So, with all these information assembled, we looked at the lowest mode for three possible cases where you have a pipe with one end closed, both ends closed; both ends open.

So, these are the patterns of the lowest frequency modes or the lowest normal modes for three different cases. And finally, I took one case of pipe with one end closed and drew the first three normal mode patterns. So, we will deal with some more of problems related to longitudinal waves in the next lecture.