Waves and Oscillations

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Lecture - 34
Bandwidth Theorem and Problems

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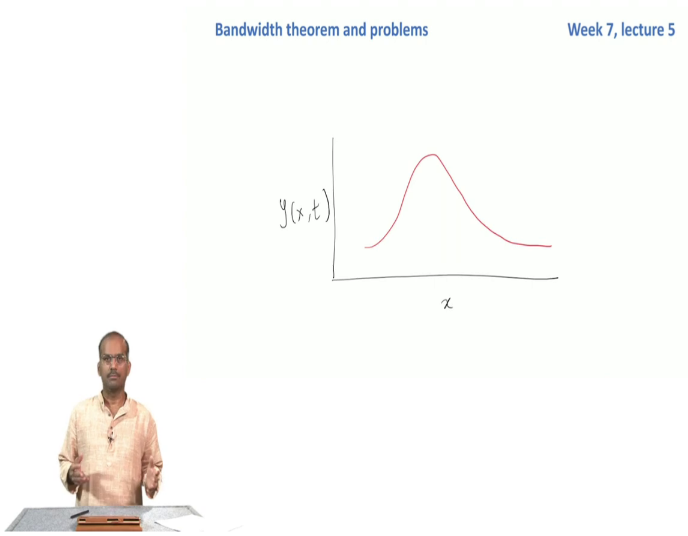

Welcome to the last lecture of this week. So, one of the striking things is that it is possible for a group of waves to either travel faster than their individual components or slower than their individual components. So, typically what happens is that the group velocity is smaller than the velocity of the individual components.

So, let us say that I have a group of wave constructed from let us say several frequency components put together and maybe the profile or the shape of the wave is how I have drawn it here that is right in front of you.

So, it shows y as a function of x at a fixed time t. So, what we know is that if there is no dispersion of course it is going to maintain the profile and go ahead. But can we define a time frame or a time scale over which this profile will be maintained even in a dispersive medium. Let us start by writing down a series of waves whose frequencies are slightly different from one another.

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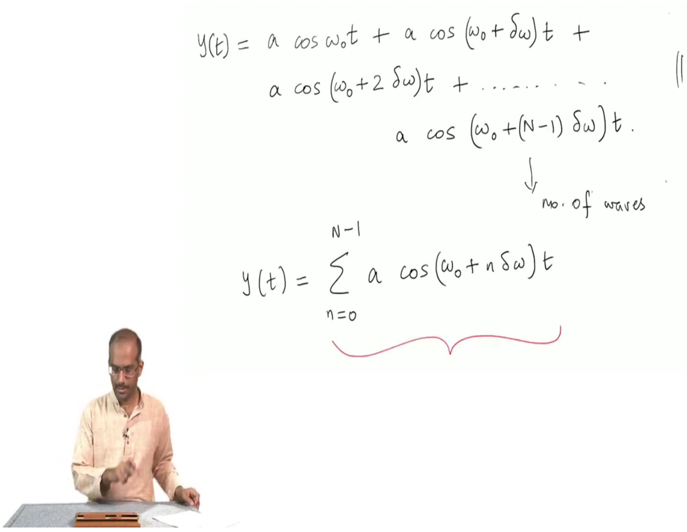

So, it is a group of waves and here I have suppress the space part because we are not really going to be interested in that immediately, N is the number of waves that are super posed and $\delta\omega$ is of course the small change in frequency small change from ω_0. So, if you expand this you will exactly get this expression and you should keep in mind that n is just the running variable, fortunately it can be some I am going to give the result now and proceed and I will prove that as a problem a little later.

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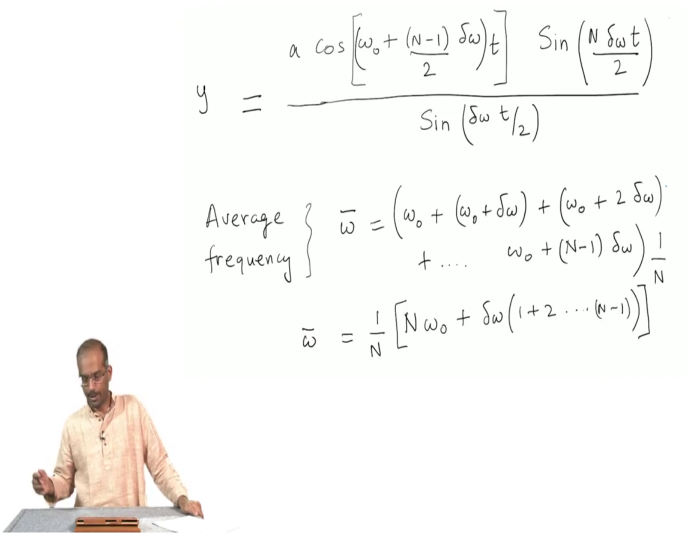

So, as I said I am going to do the summation separately as a problem, but if you do it this is the result that you will get. Since we want to get to the central physics of the problem it is better to stick to simplistic assumptions that would not hide the physics of the problem. So, here that is the reason we stick to all the individual components having same value of amplitude a.

Now, let us define an average frequency, let us call it $\bar{\omega}$ and they should be simply sum of all the frequencies divided by N. So, you have ω_0 then there is $\omega_0 + \delta\omega$ and so on until $\omega_0 + (N-1)\delta\omega$ and there are N of these. So, you divide by N. So, let us do that. So, this can be easily simplified, so I am going to have $N\omega_0$. So, you will notice that there is one ω_0 in each of the terms that are here and there are N terms.

So, there is going to be $N\omega_0$ plus there is of course, if you take $\delta\omega$ outside you will have one plus two all the way up to $N-1$ that will be $\bar{\omega}$ and this submission is easy to do one plus two plus three up to $N-1$ is a very simple submission to do.

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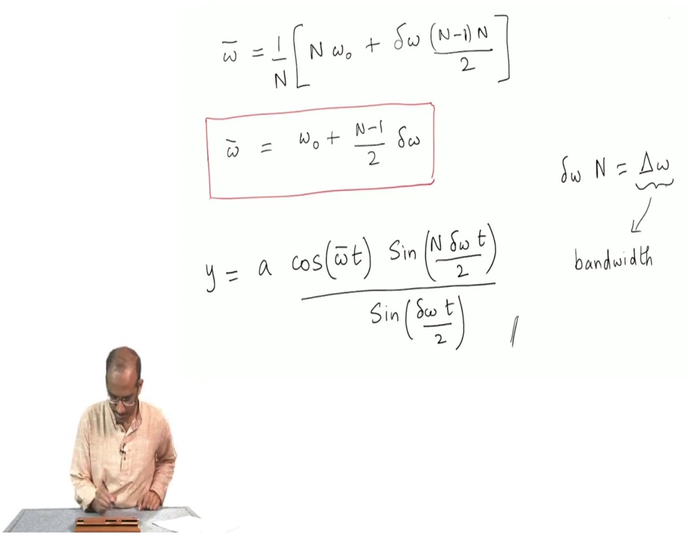

So, $\bar{\omega}$ will be $\frac{1}{N}$ into N times ω_0 plus $\delta\omega$ multiplied to that summation, it is going to be $\frac{(N-1)N}{2}$. So, this is the expression for average frequency. Now we will use this in our expression for y that we just got here. So, you will notice that this $\omega_0 + \frac{N-1}{2}\delta\omega$ is precisely this quantity, which means that that leads to a great simplification in terms of notation and also for understanding. Let us call this quantity which is $\delta\omega N$ as the bandwidth.

So, I will give a different notation, so it is $\Delta\omega$ which is this. So, all the frequency components that we are using to construct our wave form will fall within this bandwidth. Now I will write this equation using this $\Delta\omega$.

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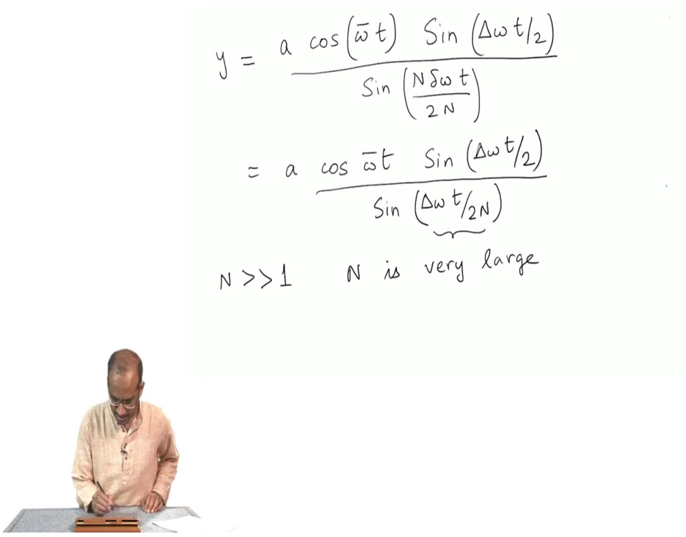

So, now I have written the same expression in terms of $\Delta\omega$ and in the denominator you will see that I have multiplied and divided by N. So now, I can complete this exercise. So, a cos $\bar{\omega}t$ into sin $\frac{\Delta\omega t}{2}$ and here.

So, this $N\delta\omega$ will be $\frac{\Delta\omega t}{2N}$. Now we will take the limit that N is very large that is, so if N is very large this quantity which is the argument of the sine function will become very small. So, in that limit you could always take sin θ to be θ itself.

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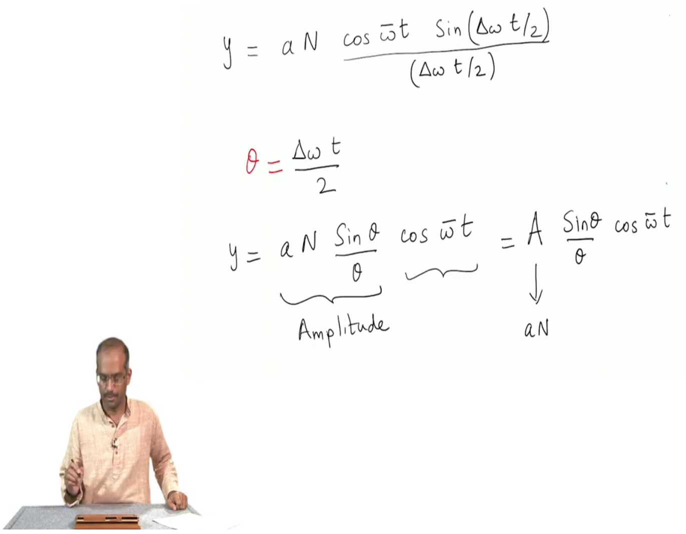

So, that is our next step I have done another small manipulation we take in the Nin the denominator to the numerator. Now, let me identify this quantity $\frac{\Delta\omega t}{2}$ as θ.

So, which means that I am going to have $\frac{\sin\theta}{\theta}$ and you can sort of see where we are headed but here we have $\Delta\omega$, so it is actually the half the phase difference between the first and the last waveform. So, the time dependence comes from this $\cos\bar{\omega}t$ term and this is the amplitude now. Amplitude itself is time dependent because θ is simply $\Delta\omega t$.

So, there is t which is hidden in θ and amplitude is time dependent and we have seen this kind of scenarios before when you add two waves. You see that the amplitude itself becomes a time dependent, so it is not quite surprising that it is happened here as well and let me also simplify it even further by writing it as $A\frac{\sin\theta}{\theta}\cos\bar{\omega}t$ and obviously this $A = aN$.

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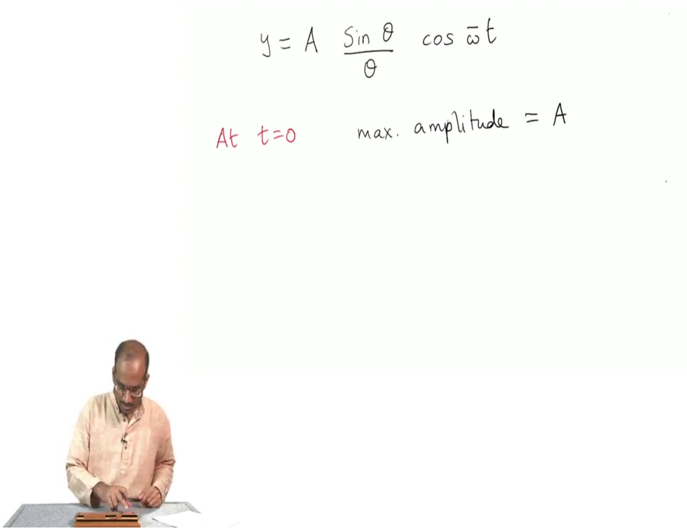

Now, let us analyze what happens to this result at $t = 0$. So, as $t \to 0$, $\frac{\sin\theta}{\theta}$ will get to one θ is time dependent it gets to one and hence the maximum amplitude would be A itself. This is the limit that $t \to 0$ or $t = 0$. We can draw the profile for y let us do that first before we go ahead further.

So, I have two parts to this y. one is $\cos\bar{\omega}t$ which is simply a cosine function. So, would oscillate at frequency $\bar{\omega}$. On the other hand I have this amplitude which itself is time dependent and this average frequency is of course the slower component and this is going to oscillate much faster.

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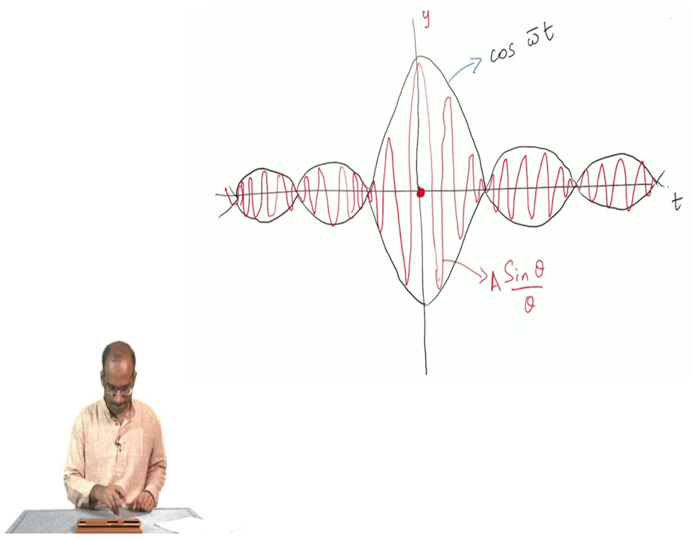

So now, if I put all these information together this is what I will get. So, this black curve that you see here would correspond to the $\cos\bar{\omega}t$ and the; and that is the slower component and the faster component is the one shown in red which would correspond to $\frac{\sin\theta}{\theta}$ part of course you can multiply A. If you like while the cos function does not decay $\frac{\sin\theta}{\theta}$ does decay which is why you see the amplitude at time $t = 0$ which is this point at the center is maximum, but after that the amplitude keeps decreasing.

So, at $t = 0$ the amplitude is A this point. So, you will notice that after some amount of time the amplitude becomes 0 at this point let us say here. Now let us calculate this time interval after which the amplitude goes to 0.

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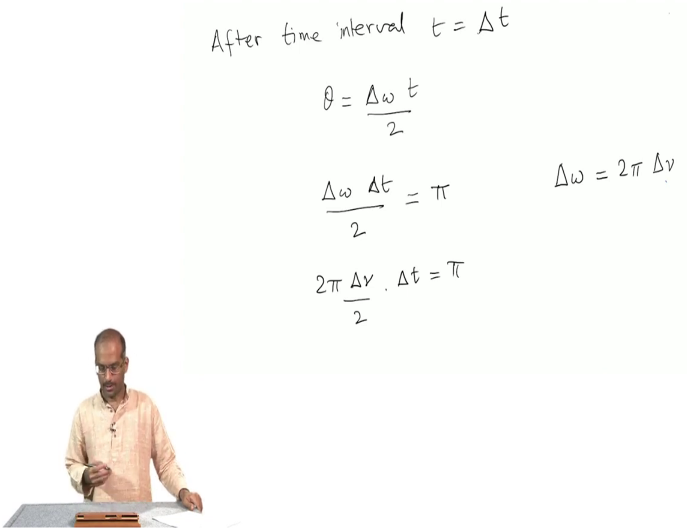

So, let us say that after a time interval of t equal to let us say some Δt the amplitude goes to 0 and this will happen provided that $\sin\theta = 0$. Which means that basically a statement saying that $\sin\pi$ is 0, $\sin 0$ is 0, but we want the next point where sine function goes to 0 that is when θ is π, so that is exactly what I have written here.

So, this can be rewritten slightly differently. So, $\Delta\omega = 2\pi\Delta\nu$. So, if I substitute that here will be $\frac{2\pi\Delta\nu}{2}\Delta t = \pi$ and of course, π; π and 2 will cancel giving me this expression that $\Delta\nu\Delta t = 1$.

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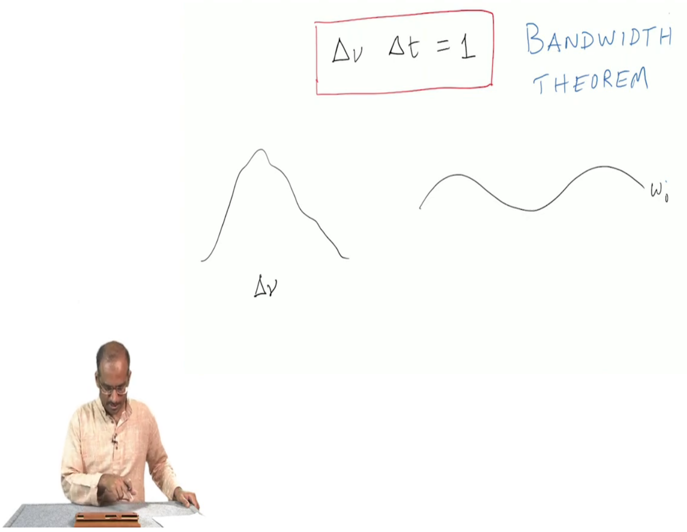

So, this is the all important bandwidth theorem. So, how do we physically interpret this relation in the following way? Suppose I constructed a profile let us say some arbitrary profile like this a wave profile by making by putting together a large number of plane waves of frequencies which have a bandwidth $\Delta\nu$.

The question is how long will it maintain it is profile, it is probably going to disperse but it will maintain it is profile for a time interval of Δt, $\Delta\nu$ bandwidth and the time interval Δt are inversely related. So, if you are putting together a large number of waves with bandwidth being very large, then the time interval over which you can maintain it is profile is small.

If my waveform is just made up of one frequency there is no group involved here in that case it is going to live long enough in principle for infinite times. I said it is all important because it is validity is not just about waves and oscillations. It is valid throughout physics you can regard it as a fundamental law in many areas of physics it appears in different form for instance, in quantum physics Heisenberg uncertainty principle is one manifestation of this bandwidth theorem.

Now, that we are done with the bandwidth theorem let us do few simple problems related to some of the things that we learnt in this week. So, the first of this is the summation which I said I will do as part of the problem.

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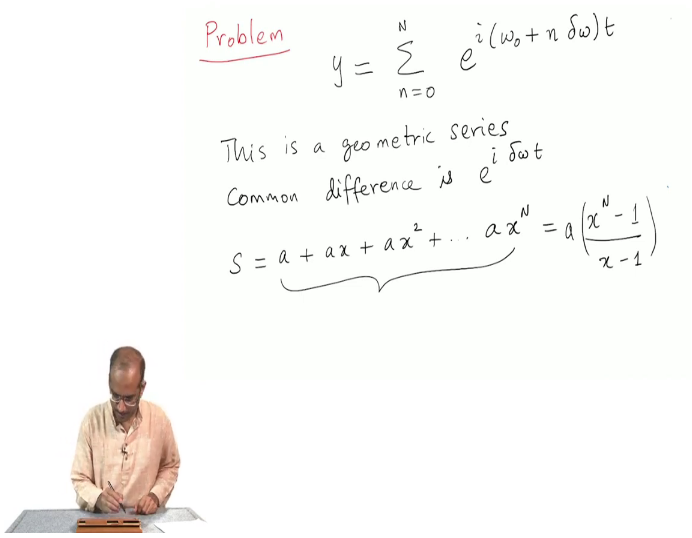

When we are looking at the bandwidth theorem we were trying to some $\cos((\omega_0 + n\,\delta\omega)t)$ but here I have just taken it as $e^{i(\omega_0 + n\delta\omega)t}$.

So finally, at some point we will take the real and imaginary parts and from there we can extract the result that we want. Geometric series and the common difference is $e^{i\delta\omega t}$. So, what I have here is the standard geometric series and of course this is the result which can be obtained from very simple consideration.

So, I am not going to derive that result we are going to use this result sum to N terms of a geometric series written, the result of summation using the formula for N terms of a geometric series.

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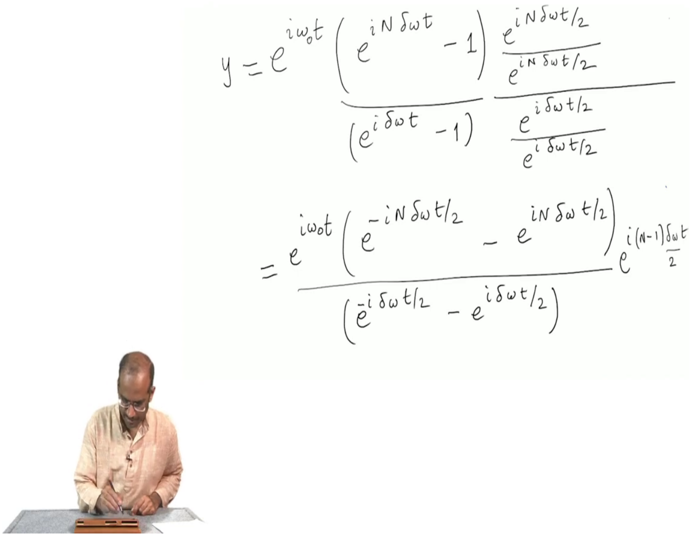

So, now what we shall do is to do simple manipulations on this. So, first half that would be I will multiply and divide by $e^{i\frac{N\delta\omega t}{2}}$ and similarly in the denominator I am going to do a similar manipulation. So, I will multiply and divide by $e^{\frac{i\delta\omega t}{2}}$ divided by $e^{\frac{i\delta\omega t}{2}}$. So now, you can see what will happen. So, if I take this denominator inside this brackets I will get the following $e^{i\omega_0 t}$.

So, it is going to give me $e^{-\frac{iN\delta\omega t}{2}} - e^{\frac{iN\delta\omega t}{2}}$; whole thing is divided by $e^{-\frac{i\delta\omega t}{2}}$ minus e power i, of course that is going to be a minus sign here $i\ \delta\ \omega\ t$ by 2 and of course, the numerators here are still there, so they will generate for me $e^{\frac{i(N-1)\delta\omega t}{2}}$. So, this is the expression that I have.

Now, it is clear what to do especially from the structure of the summation that we have multiply and divide by $2i$, these two terms within the brackets in the numerator and the denominator would become a sine function.

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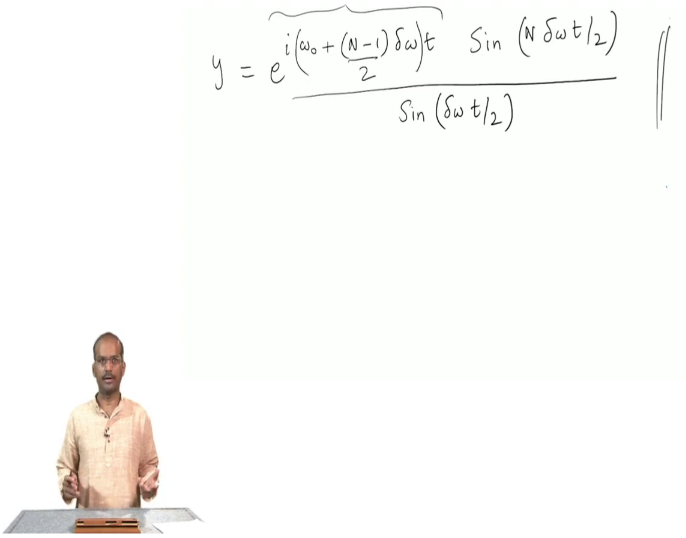

So, if I factor in that I can simplify this expression. So, I have the final result in front of me. So, this is the full result of the summation and if we need the real part alone you will notice that the only thing that would get affected is this first term here.

I just need to take the real part of it which should be cos $\left(\left(\omega_0 + \frac{N-1}{2}\delta\omega\right) t\right)$ and if you remember when we are driving the bandwidth theorem this is exactly what we called as the average frequency.

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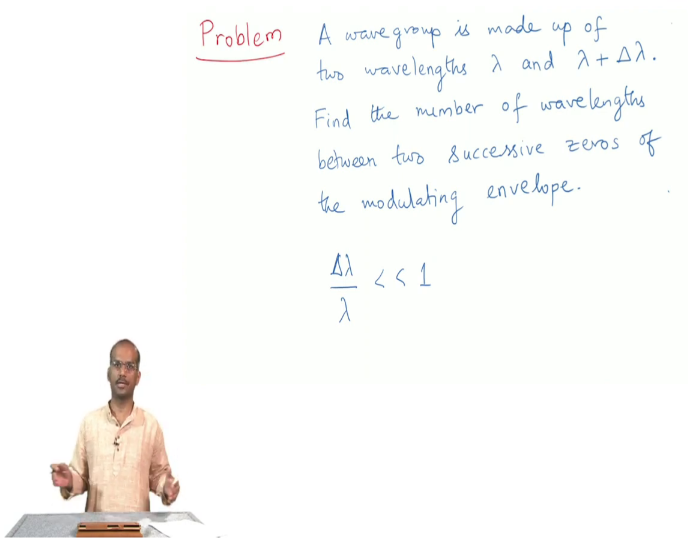

A wave group is made up of two wavelengths. So, clearly there are only 2 not N the wavelengths are λ and $\lambda + \Delta\lambda$ the question is find the number of wavelengths between two successive zeros of the modulating in envelope. And here one of the assumptions is that $\frac{\Delta\lambda}{\lambda}$ is small which means that it is much less than one. So, within the ambit of this assumption we need to find out how many wavelengths are contained between the modulating envelope.

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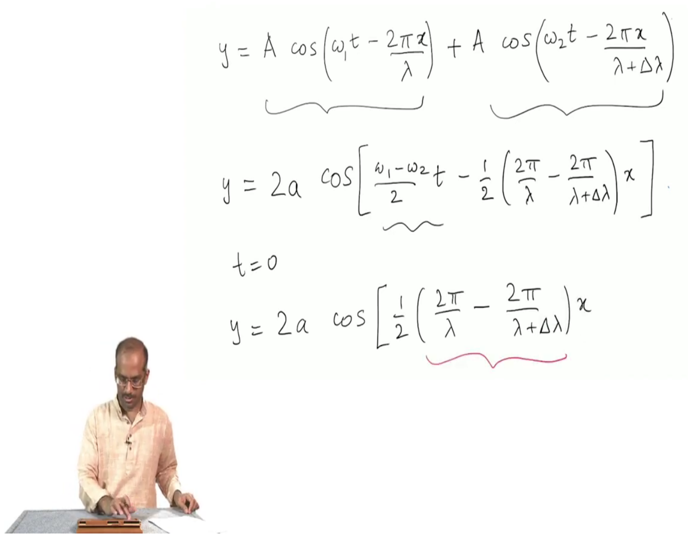

Let us start by writing down the resultant equation. So now, I have a resultant or the sum of two waves, the first one has wavelength λ second one has wavelength $\lambda + \Delta\lambda$. The result really does not depend on whether I choose cosine function here or a sine function here or even a combination of sine and cosine function.

So, all that would lead to some phase differences and so on, but the final result will not depend on what choice I make for the solutions of the individual components. So, I have two components here one is this second one is this first one has wavelength λ second one has wavelength $\lambda + \Delta\lambda$.

So, what we are interested in is in the slower oscillation, because that is the modulating envelope. So, I am not going to write out the faster oscillation part. So, let me just write about the slower one just worry about that.

So, I have written down the resultant and as I said I have kept only the modulating envelope part, there is one more cosine term which I have not written down here we do not need it for the way we are solving it and also I do not need the time part, so let me choose t to be equal to 0 because the result is valid for all the time. So, it does not matter what time I choose. So, if I choose $t = 0$ this part will simply to 0.

So, it is cos of minus some quantity which will be cos of that quantity, because $\cos(-\theta)$ is $\cos\theta$ itself. So, I am going to be left with $2a \cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda} - \frac{2\pi}{\lambda + \Delta\lambda}\right)\right]$. So, I need to crucially use this information that $\frac{\Delta\lambda}{\lambda}$ is smaller than 1.

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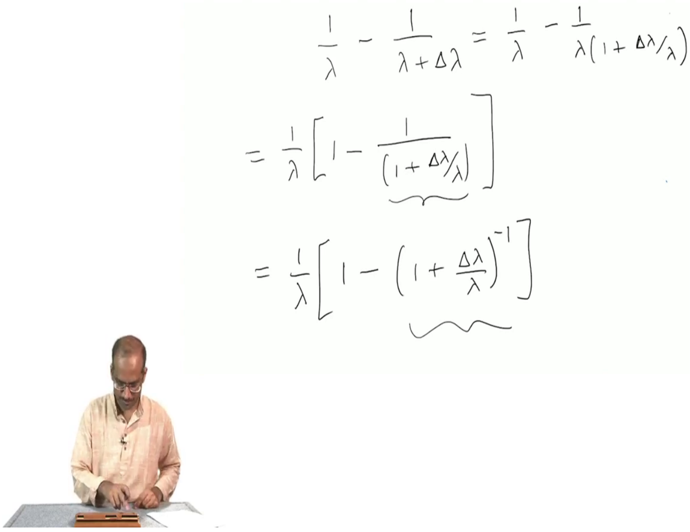

So, for that I need to simplify this. So, this will be $\frac{1}{\lambda}$ minus 1 by take λ outside that will be $1+\frac{\Delta\lambda}{\lambda}$ and now you can take $\frac{1}{\lambda}$ outside. So, that is going to be 1 minus 1 by 1 plus $\frac{\Delta\lambda}{\lambda}$ and this can be this one can be taken in the numerator with the power minus 1. So, that is going to be $\frac{1}{\lambda}\left[1-\left(1+\frac{\Delta\lambda}{\lambda}\right)^{-1}\right]$.

And now this one we will use the fact that $\frac{\Delta\lambda}{\lambda}$ is smaller than 1 and do an expansion binomial expansion I will keep only the first term.

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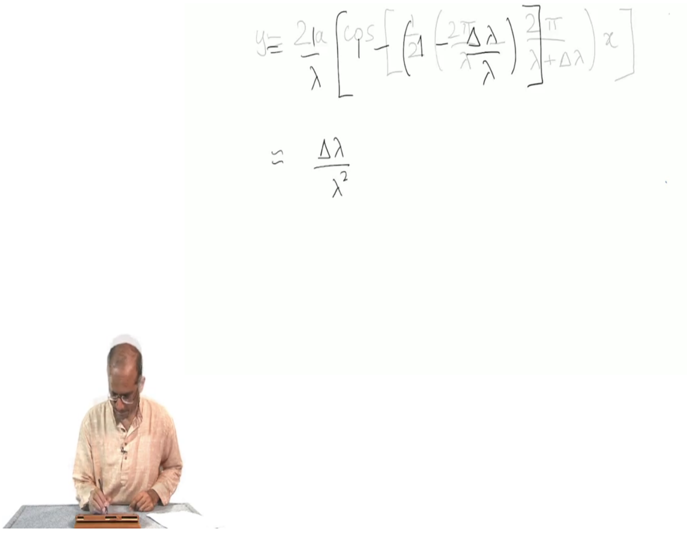

So, if I do that I am going to get the following that is going to be $\frac{1}{\lambda}$ multiplied by 1 minus. So, this will become $1 - \frac{\Delta\lambda}{\lambda}$. So now, it is an approximation the approximation is $\frac{\Delta\lambda}{\lambda}$ is lesser than 1. So, 1 and 1 would cancel so I am going to have $\frac{\Delta\lambda}{\lambda^2}$ ok.

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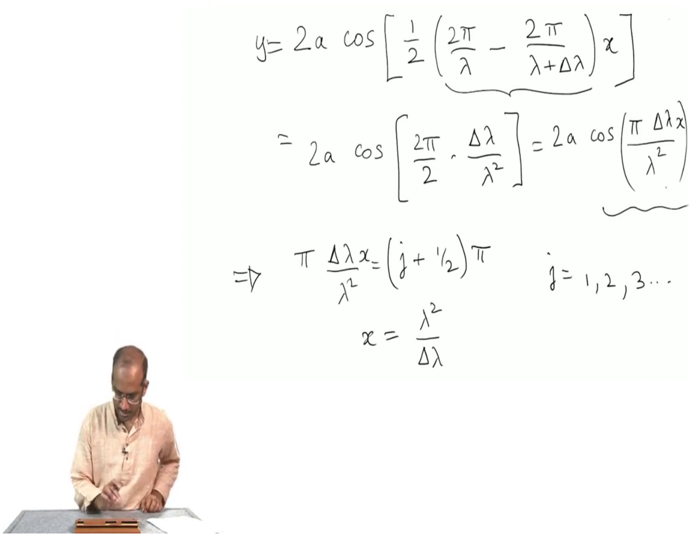

I will now substitute this expression for this quantity that I have just derived. So, it is going to be$2a$ cos $\left[\frac{2\pi}{2}\frac{\Delta\lambda}{\lambda}\right]$ and that would be $2a$ into cos of 2 and 2 would cancel. So, I am going to have $\frac{\pi\Delta\lambda}{\lambda^2}$, at what points zeros of this function occurs.

So, cosine function would go to 0 whenever the argument which has $\frac{\pi\Delta\lambda}{\lambda^2}$ is equal to some integer plus $\frac{1}{2}\pi$. So, this function which is simply the modulating envelope this will go to 0 at certain values of the argument of the cosine function and that happens at these values when $\frac{\pi\Delta\lambda}{\lambda^2} = \left(j + \frac{1}{2}\right)\pi$ and j is simply integers. Of course, there would be negative integers as well, but it is not of interest.

In fact, we are just interested in finding out the value of x which I seem to have forgotten here let me put that in here. So, I just wanted to find out the value of x between two such integers and if you do that it is very simple. So, I will get this relation that $x = \frac{\lambda^2}{\delta\lambda}$.

So, all it tells me is that the distance between two successive zeros is x and that is equal to $\frac{\lambda^2}{\Delta\lambda}$, but I; but what I want to know is how many full wavelengths are contained in it and that would be simply equal to $\frac{x}{\lambda}$.

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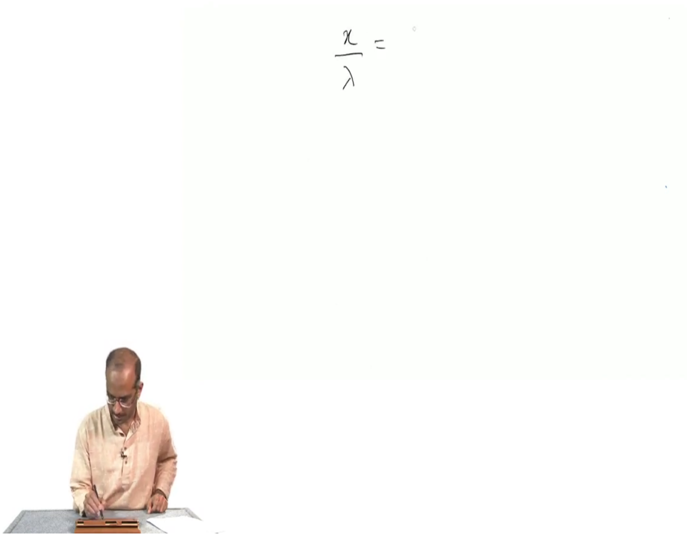

Where λ is the wavelength, so $\frac{x}{\lambda}$ will give me a number which will be the number of wavelengths contained in it and in terms of the equation that I have here $\frac{x}{\lambda}$ will be equal to $\frac{\lambda}{\Delta\lambda}$.