Waves and Oscillations
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Lecture - 33
Dispersion of Waves

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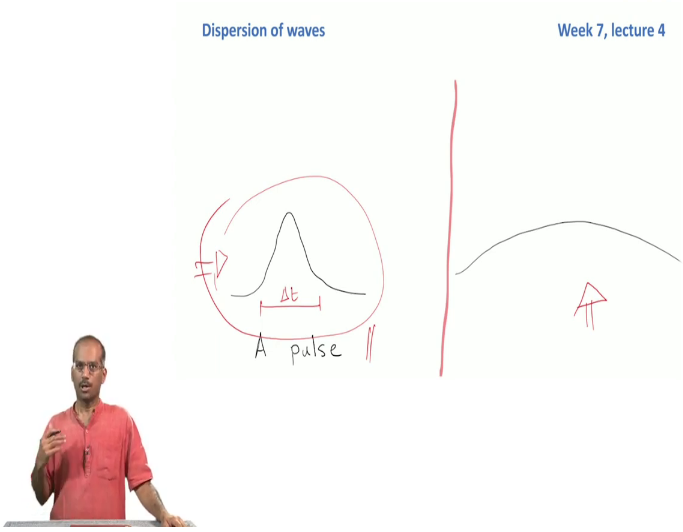

Welcome to this 4th lecture we are in the 7th week. Starting from now we are going to look at groups of waves, which is why the topic for today's lecture is dispersion of waves. So, let me spend a little while to explain first what is it that is different now compared to what we had been doing.

We were looking at still a single oscillation frequency which was call the normal mode frequency, but in general that is not the case often in real life the wave phenomena that you tend to encounter, does not really come with the single well defined frequency.

For example, the white light that we see for instance the light that comes from the sun is made of several frequencies that was the well known experiment that Newton had done more than 300 years back. So, you take a white light of that kind and pass it through a prism and on the other side what you see is, it splits into 7 colors.

The conclusion is that white light is made up of at least 7 different frequencies to put it in a very approximate sense. The sound waves that you typically hear are not made up of single frequency it is often combination of several frequencies.

So, in principle and in practice we should start looking at waveforms which are made up of several frequencies. A good example of that would be something like a wave pulse the kind that I have drawn here. So, a pulse is typically something that has a short let say time frame a short time frame like Δt over which the amplitude is sufficiently strong enough. Outside of that small range Δt it is amplitude is nearly 0. So, does not even exist. It is like starting a wave pulse in a string for example, the pulse alone can travel.

So, such pulses typically are made up of combining many plane waves each plane wave has a definite frequency, each frequency might travel at different speeds in which case the shape of the pulse that you had started with may not be maintained at later times because each component is trying to move at different speeds.

So, ultimately you will see that what started out as something like this might end up something like this. So, in general behavior of this kind where sharply peaked pulses finally, kind of become even doubt spaced out, it is in general called dispersion. I can give you another analogy for instance you imagine that there is a say a marathon race. Now, a nowadays there are many city marathons.

So, at the starting point you have a large number of runners who was starting at the same time but after sometime if all of them where exactly running at the same speed they would all reach the endpoint precisely at the same time. So, the crowd which started as a group would move together as a group and finally, reach the ending point as a group. So, there will not be a distortion in the shape of the crowd.

On the other hand what normally happens is that, everyone runs at different speeds there are people who run very fast tend to be winners and there are others who are slow and just manage to complete it. So, finally, in such a situation you will see that very soon the marathon group that started out actually spaces out. So, we had seen large number of results for what would be called monochromatic waves with single frequency.

So, now we are going to see results for what happens if your waveform is made up of many different frequencies. To understand it in a sort of simple framework, let us start by constructing a waveform which has only 2 frequencies. In that case it will be easy to understand what to expect and then we can extrapolate the result to larger number of frequencies.

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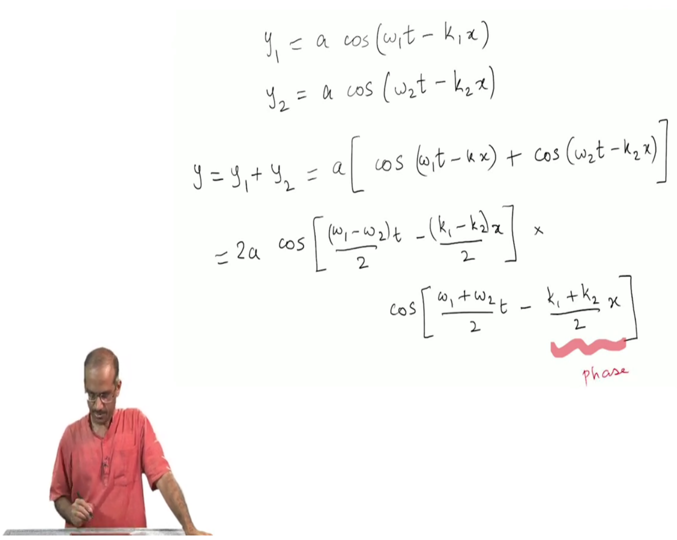

Let us assume that I have 2 waveforms of this type, y_1 which is $a \cos(\omega_1 t - k_1 x)$ and I have another one which is y_2, again $a \cos(\omega_2 t - k_2 x)$ and in general if you take a_1 and a_2 as 2 different amplitudes it becomes very hard to do the analysis.

So, for simplicity we will take both the amplitudes to be the same each of this is a monochromatic wave. So, you will see that y_1 is described by a single angular frequency ω_1, and y_2 is described by a single angular frequency ω_2. Now if I make a wave which is the superposition of both these waves which means that y will be a function of course, position and time as usual but this is going to be a superposition of y_1 and y_2.

So, it is $y_1 + y_2$ so, that would be something like this the net displacement is simply sum of the displacements of y_1 and y_2. Now all I need to do is to use this trigonometric formula for $\cos a + \cos b$. So, apply that here and we can rewrite this equation is $2a$ times.

So, this is simply straight forward, this is a kind of exercise we had done quite some time back when we looked at the beats phenomenon, it is exactly similar as of now. To understand what it says it is better to plot this. So, I am going to have especially if you assume that ω_1 and ω_2 are nearly equal to one another but not exactly equal.

So, in that case this number $\omega_1 - \omega_2$ will be small whereas, $\omega_1 + \omega_2$ will be a large number and of course, you can treat this $k_1 + k_2$, $\frac{k_1 + k_2}{2}x$ as some phase factor, similarly you can do that for $\frac{k_1 - k_2}{2}x$.

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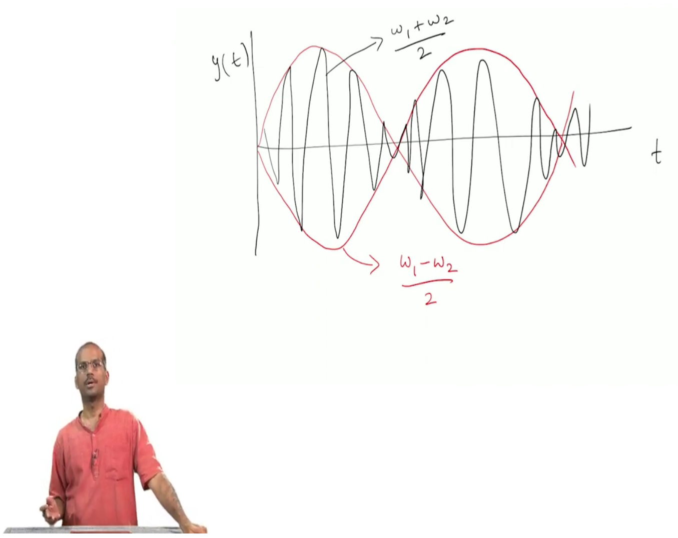

So, what you are going to have is an overall envelope of slower oscillations, which might look like this and of course, there is going to be fast oscillation something like this.

So, here I have plotted y as a function of x of course, is fixed. So, let us just take it as a function of time. Red curve here corresponds to a frequency of $\frac{\omega_1 - \omega_2}{2}$, given that ω_1 and ω_2 are nearly equal that is a slower of the 2 frequencies and the faster one which displays fast oscillations is $\frac{\omega_1 + \omega_2}{2}$. When you have a curve like this you would say that the phase velocity of this curve would be determined by the slower mode.

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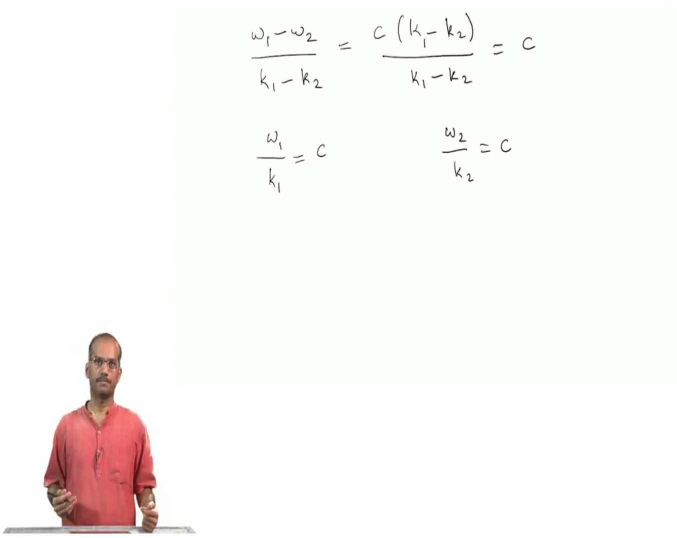

So, which means that the phase velocity of the slower one would be $\frac{\omega_1 - \omega_2}{k_1 - k_2}$.

So, if I assume that $\frac{\omega_1}{k_1}$ is equal to some velocity c and $\frac{\omega_2}{k_2}$ is also equal to c itself. So, here the medium is such that $\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = c$. So, in which case I could write this equation as a $\frac{c(k_1 - k_2)}{k_1 - k_2}$ and this is equal to c. So, each of the frequency component which in this case is ω_1 and ω_2, they would move together.

So, in that case they would maintain whatever the phase relationship that they had at initial time.

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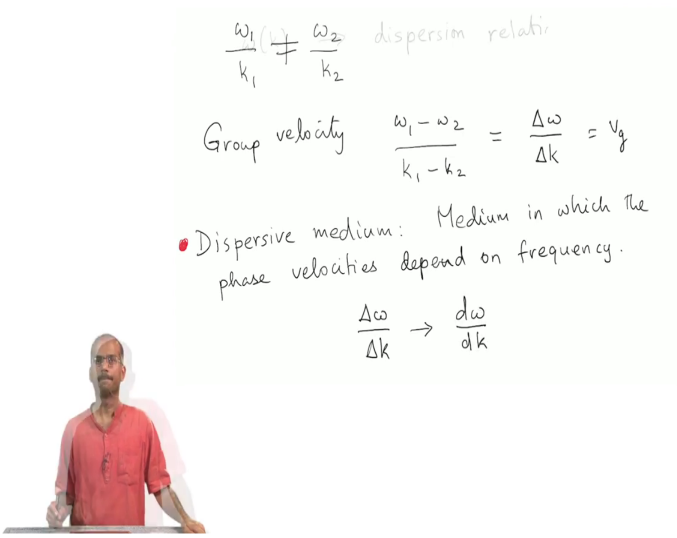

Now, let us consider the other case what happens if $\frac{\omega_1}{k_1} \neq \frac{\omega_2}{k_2}$? So, in that case we could define what is called a group velocity and we can indicated by Δw which implies the difference between the 2 angular frequencies divided by the difference in the wave numbers. Let me call it v_g. We have now 3 possible velocities one is $\frac{\omega_1}{k_1}$, other one is $\frac{\omega_2}{k_2}$ which are the velocities of the individual waves.

Now, we added these 2 waves and that has given rise to this third velocity which is the group velocity v_g. In this case what is going to happen is the 2 ways will not be in step with one another. So, there is going to be dispersion. So, the initial shape of the wave that you had started with is going to change as a function of time. So, now, we can of course, more clearly express this idea of dispersion in terms of what is called a dispersive medium.

So, dispersive medium is a medium in which the phase velocity is would depend on frequency. If phase velocity is depend on frequency is each component of the wave that makes up the group is going to travel at different speeds. So, it is going to disperse the initial profile that you had started out with. $\frac{\Delta w}{\Delta k}$ in the case where you have large number of such waves superpose together to form a group this would translate to become $\frac{d\omega}{dk}$.

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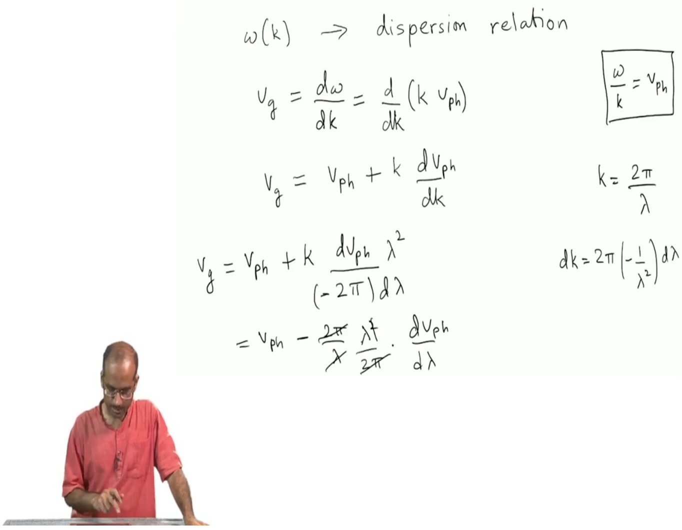

So, once you know $\omega(k)$ you could find $\frac{d\omega}{dk}$ find the group velocity, and determine if there is going to be a dispersion or not. Typically when we say energy is carried by the wave it is carried by the wave with the speed equal to the group velocity. It so, happens in some cases that phase velocity is sometimes larger than the velocity of light but the actual speed with which say a signal can be transmitted or information can be passed on this given by the group velocity.

So, given a dispersion relation we can write an expression for group velocity. So, let us start with v_g which is group velocity which is $d\omega$ by;

Now, notice that $\frac{\omega}{k}$ is equal to phase velocity, let me call it v_{ph}. So, I am going to substitute for ω from here in which case I will have $\frac{d}{dk}(k\, v_{ph})$. Now let us do this differentiation. So, if I differentiate this I will have and now this can also be written in terms of the wavelength sometimes it is convenient to work with wavelengths in which case one could just make the substitution that $k = \frac{2\pi}{\lambda}$.

So, if k is $\frac{2\pi}{\lambda}$, dk would be $2\pi \times \frac{-1}{\lambda^2} \times d\lambda$ and if I now substitute for k as $\frac{2\pi}{\lambda}$, I will get $v_{ph} - \frac{2\pi}{\lambda}\frac{\lambda^2}{2\pi}\frac{dv_{ph}}{d\lambda}$.

Now, of course, 2π and 2π will cancel this λ will cancelled with one of this. So, I am going to.

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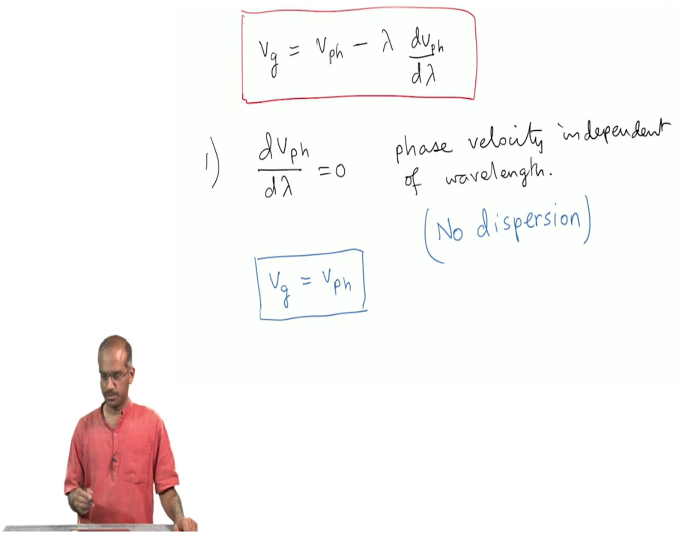

So, I am going to be left with the final expression which is $v_g = v_{ph} - \lambda\frac{dv_{ph}}{d\lambda}$. This is going to be another form of the relation between phase velocity and the group velocity and we had already seen that we had obtained one another relation between group velocity and phase velocity in terms of wave number which is this relation.

So, based on the relation that I have, I can identify 3 possibilities. So, the first one is when phase velocity is independent of the wavelength. So, in other words $\frac{dv_{ph}}{d\lambda} = 0$.

Phase velocity of each of the components that make up the group is exactly same and they would not show any dispersion.

So, clearly if you put $\frac{dv_{ph}}{d\lambda} = 0$ in this relation, you will get that group velocity is equal to phase velocity entire group travels with the same speed as any one member of that group.

Now, the second case could be when $\frac{dv_{ph}}{d\lambda}$ is positive $\frac{dv_{ph}}{d\lambda} > 0$. So, you can go back to this equation. Now if $\frac{dv_{ph}}{d\lambda} > 0$ or positive in that case v_g will be a number that is smaller than phase velocity.

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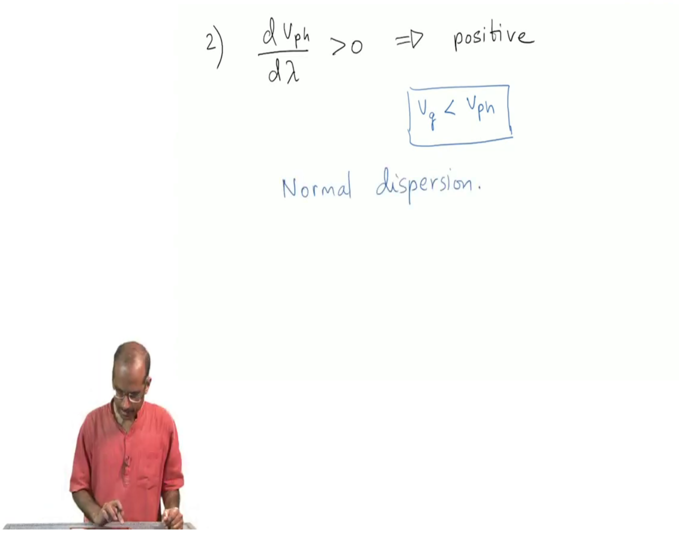

So, this corresponds to the case when v_g is smaller than phase velocity. So, this regime where the group velocity is smaller than the phase velocity is called the normal dispersion.

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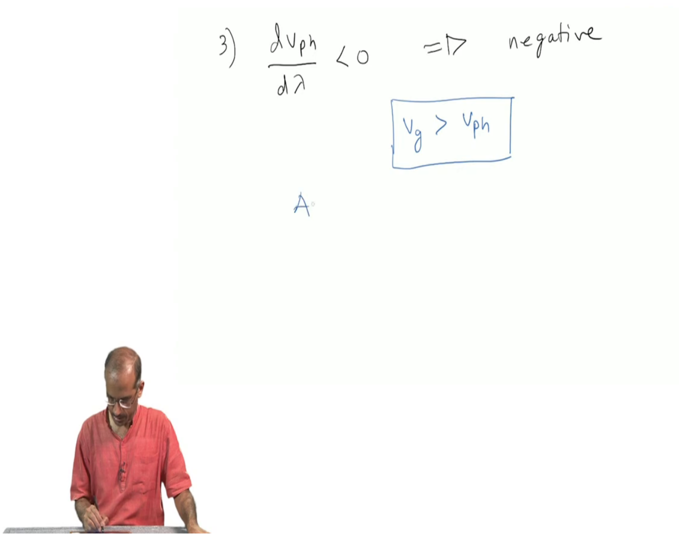

I can consider the case when $\frac{dv_{ph}}{d\lambda} < 0$.

So, in this case the group velocity will be larger than the phase velocity and this is what is called the anomalous dispersion.

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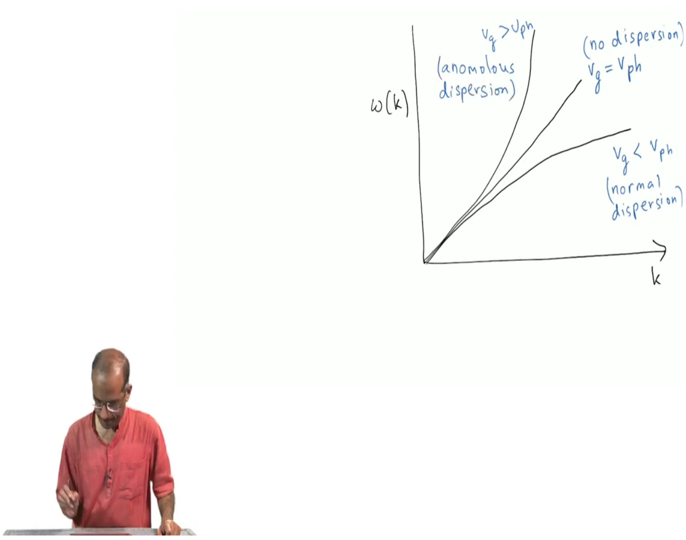

So, I am going to plot $\omega(k)$ as a function of k. If $\omega(k)$ is simply equal to k or some constant times k that is the case when group velocity will be equal to the phase velocity.

So, that is the case of no dispersion. So, in this case v_g is less than the phase velocity group velocity less than the phase velocity and this is of course, the normal dispersion case this is of course, the case of anomalous diffusion where v_g, group velocity is greater than the phase velocity.

So, you can imagine why it is called anomalous because it looks like the group of waves which make up your pulse or waveform, each of them has a phase velocity which is smaller than the velocity of the group as a whole that looks quite a unusual situation which is why it has a name called anomalous dispersion.

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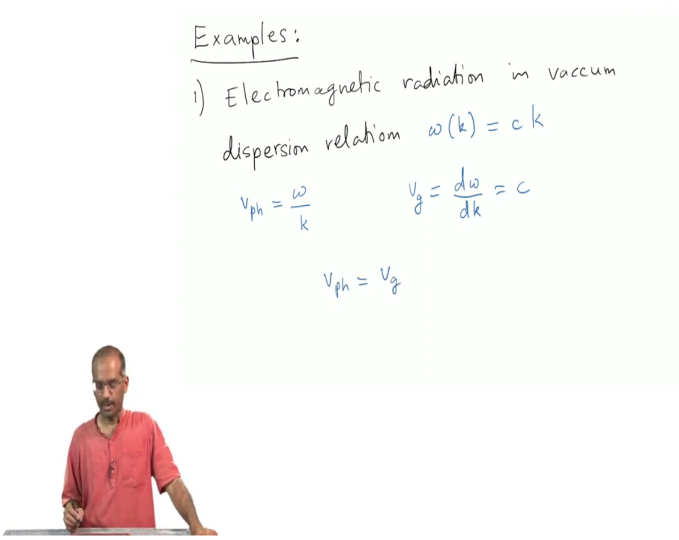

So, what happens electromagnetic radiation in vacuum? So, in this case the dispersion relation would be $\omega(k)$ is equal to $c \times k$, where c is the velocity of light and I need to find out what is the phase velocity. So, the phase velocity is of course, $\frac{\omega}{k}$ and what is a group velocity? So, the group velocity is $\frac{d\omega}{dk}$ which is equal to c. So, you can see that for electromagnetic radiation in vacuum phase velocity is equal to group velocity and hence there is no dispersion.

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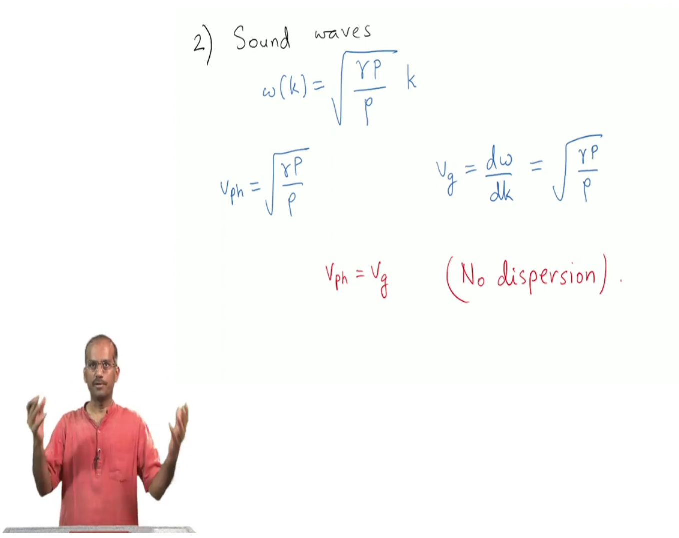

Let us look at a second example, sound waves at standard temperature and pressure let say and also in the audible range. So, here the dispersion relation is $\omega(k)$ is equal to γ P by ρ times k.

So, P is the pressure, ρ is the density of the gas and gamma is the ratio of specific heat capacities. Now again I can calculate what the phase and the group velocities are. So, the phase velocity is simply γ P by ρ and group velocity which is $\frac{d\omega}{dk}$ would also be equal to $\sqrt{\frac{\gamma P}{\rho}}$.

So, again we have a case of $v_{ph} = v_g$. So, the group velocity is equal to the phase velocity and again there is no dispersion. Now think about this result for a moment suppose there was dispersion in sound waves. So, the medium through which sound wave travels is simply the air around us. Now if that medium where dispersive to sound waves what would happen?

So, whatever I speak which is made up of many different frequency components would reach you at different times. So, it can be quite confusing.

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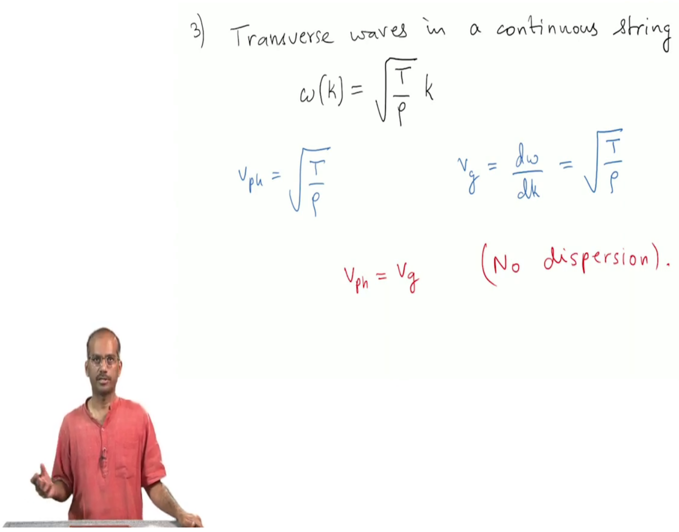

So, the third example is something that we have seen in some detail transverse waves in a continuous string. So, in this case the dispersion relation $\omega(k) = \sqrt{\frac{T}{\rho}}k$, T is the tension in the string and ρ is the linear density of the string. So, this is the case that we made we derived results for this case in last 2 weeks.

So, now I can ask what is the phase velocity. So, phase velocity is simply $\sqrt{\frac{T}{\rho}}$ what about the group velocity that would be $\frac{d\omega}{dk}$ which is equal to $\sqrt{\frac{T}{\rho}}$.

So, again here is one another case where phase velocity is equal to group velocity again there is no dispersion here. So, we saw 3 different cases where there was no dispersion. So, let us now look at a case where there is actually some dispersion.

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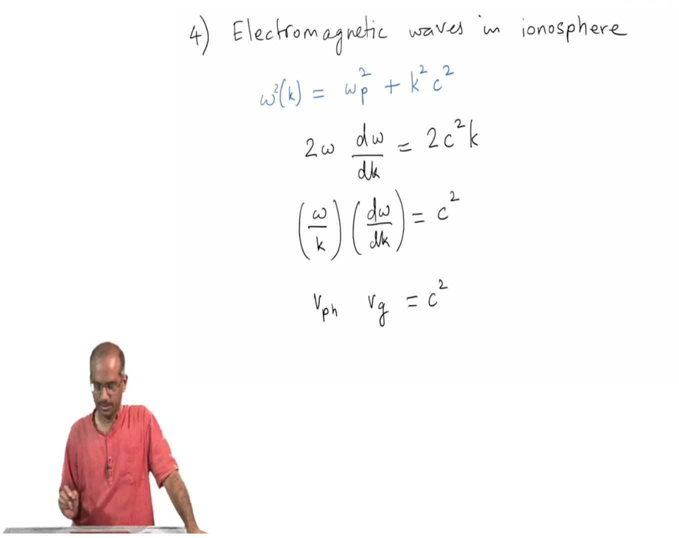

Electromagnetic waves in ionosphere. In this case the dispersion relation is given by the following relation.

So, this is the dispersion relation that is valid in some range of frequencies. So, right now let us not worry about what is the range of validity of this dispersion relation, in some limited range of frequency this is valid ω_P is the is what is call the plasma frequency.

Now let us differentiate this with respect to k and wish and we will get the following

$$2\omega\frac{d\omega}{dk} = 2c^2k$$

We have sort of differentiated with respect k and now let me make a small adjustments here.

So, 2 and 2 will cancel. So, I will have $\frac{\omega}{k}\frac{d\omega}{dk} = c^2$ and you can see that $\frac{\omega}{k}$ is of course, the phase velocity and $\frac{d\omega}{dk}$ is the group. So, the product of phase velocity and group velocity is equal to c^2.

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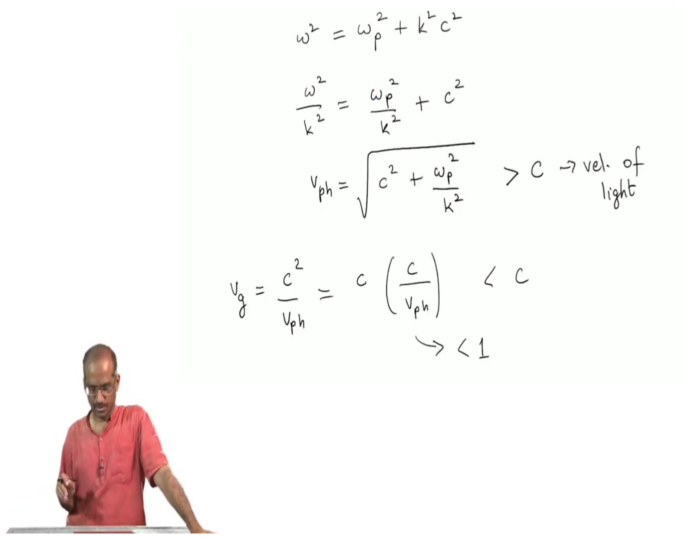

So, let me go back to my dispersion relation which would be $\omega^2 = \omega_P^2 + k^2c^2$.

So, I have $\frac{\omega^2}{k^2} = \frac{\omega_P^2}{k^2} + c^2$. So, here I have an expression for ω/k which is the phase velocity. So, my phase velocity is equal to $\sqrt{c^2 + \frac{\omega_P^2}{k^2}}$.

So, whatever be the values of ω_P^2 and k^2, they are all positive numbers whatever be that this quantity the phase velocity is greater than c greater than the velocity of light ok. That looks like it should not have been, but this is not the velocity at which with which the signals are transmitted.

So, let us look at the group velocity. So, group velocity would be of course, $\frac{c^2}{v_{ph}}$. So, that is what we get from this relation group velocity $\frac{c^2}{v_{ph}}$ which can be written as $c\left(\frac{c}{v_{ph}}\right)$ and we just saw that phase velocity is greater than the velocity of light.

So, this term would be less than one. In which case the group velocity as a whole will be less than velocity of light c. So, this is an example of a dispersion relation for the case of electromagnetic waves in ionosphere for which the phase velocity is larger than the velocity of light but the group velocity is smaller than the velocity of light.

Since all the signals and information can be transmitted only at the group velocity it does not violate any other principles of physics finally, I will leave you with one another example.

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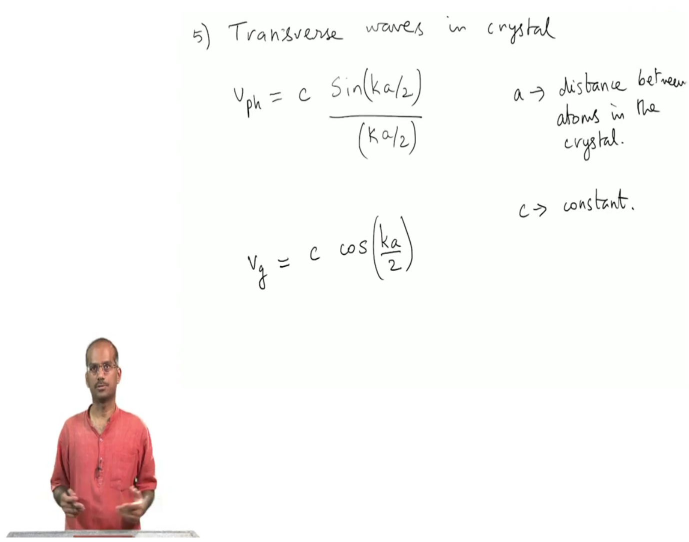

So, in this case let me give you directly the phase velocity which is given by$c\frac{\sin\frac{ka}{2}}{\frac{ka}{2}}$.ok.

It is a simple exercise I will leave you to do that the group velocity will be equal to $c\cos\frac{ka}{2}$.

So, this is another case in which you do see dispersion. As I close let me remind you once again that we looked at dispersion of waves and this is a property of not a single wave but this is a property for a group of waves in a particular medium. The key to understanding dispersion whether a particular pulse or a group of waves would be dispersive or not is to know what is it is dispersion relation.

So, once we know its dispersion relation we can figure out if it is a case of normal dispersion, anomalous dispersion or maybe there is no dispersion at all.