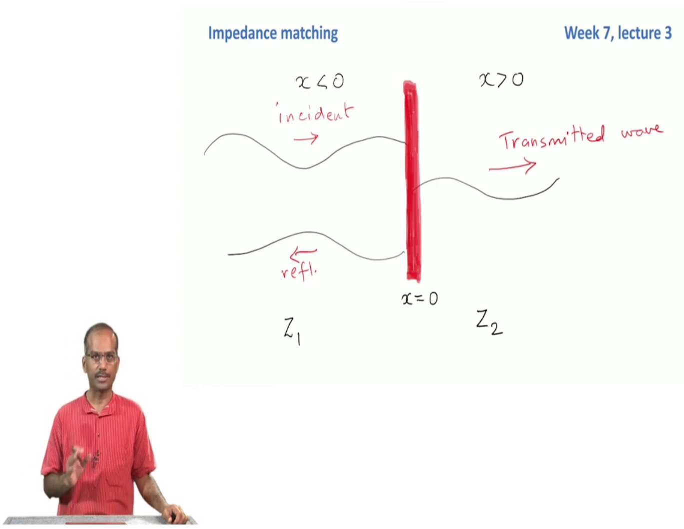
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 32
Impedance Matching

(Refer Slide Time: 00:16)



In this third lecture of seventh week, now we will use all the machinery that we learnt in the last two lectures. We will use everything that we know about impedances everything that we know about transfer of energy to address an important problem. So, what is that problem? Problem of transferring energy from one place to another and waves are routinely used in many practical applications to transfer energy and even signals.

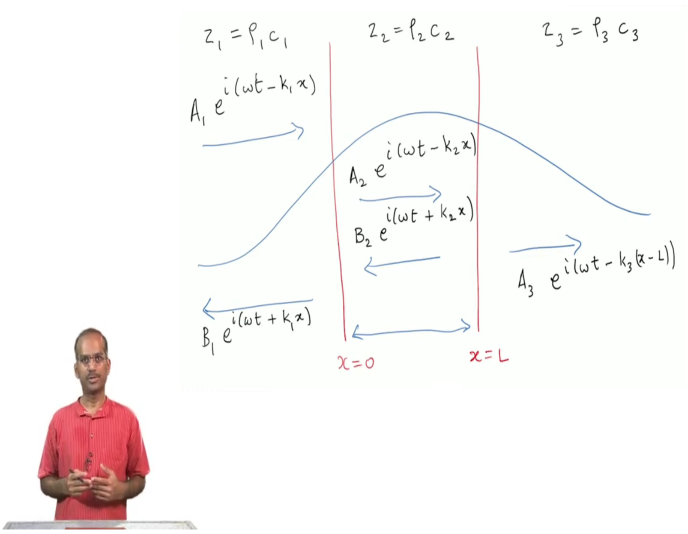
For instance every time you get internet in your home, so somewhere optical fiber cables are used. And one of the practical problems is that every now and then such cables might get cut and what is normally done is what is called splicing. So, it is like a patchwork, so you just put them together by some means and again get the cable up and running.

In some sense what use to be originally a long cable, now got cut and you have introduced a third body in it. What is a guarantee that all the energy that is coming in from your wave on one side is going to be properly passed on to the second medium? In fact in general this not going to happen. The wave that is coming in from the left we will carry some energy with it and when you have something like a coupler in between these two media there is going to be a loss of energy.

So, the practical problem is the following. So, in all such situations how do we ensure that the entire energy that is coming in from the left side of the medium through a propagating wave is carried without any loss into the medium that is on the right side? In other words we want complete transfer of energy from left to right with nothing being reflected. So, that is the practical problem and here in our physics terms this is how we sort of envision this problem. Here I have a wave that is incoming in this direction, in general you are going to have a reflected wave and then there is going to be a transmitted wave.

Now, I have this coupler in between, what I want to derive is a condition such that the entire energy goes from left side to the right side with a with nothing being reflected. So, which means that it is going to put in some constraint on some parameters of this coupler, that is exactly what I want to derive.

(Refer Slide Time: 03:04)



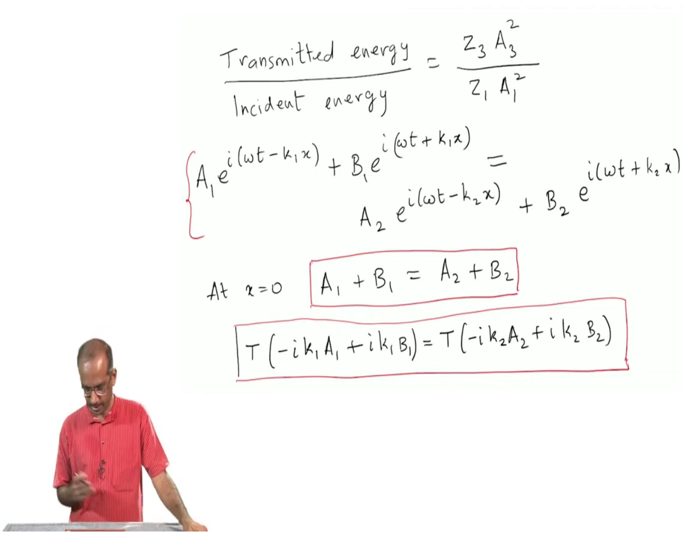
Let me say that I have two boundaries like this. So, one at $x = 0$ and other at $x = L$ and this divides this region into three media. So, I am going to have the first one whose impedance is Z_1 which is nothing but $\rho_1 c_1$, c_1 is the phase velocity of the wave in that medium.

And similarly I am going to have Z_2 which is $\rho_2 c_2$ all the ρ's are the linear densities and Z_3 is $\rho_3 c_3$. This distance between these two boundaries is length L and let me just picturize the wave. So, being something like this.

So, the incoming wave is $A_1 e^{i(\omega t - k_1 x)}$ following our usual convention and this one would be $B_1 e^{i(\omega t + k_1 x)}$. So, notice that there is a difference in sign there is $-k_1 x$ and $+k_1 x$, reflecting the fact that that directions are opposite to one another. And here for the wave that got transmitted into region two, I have $A_2 e^{i(\omega t - k_2 x)}$. And similarly I have a waveform that is getting reflected from the boundary at $x = L$ that would be; and lastly I have a transmitted wave.

And if you think carefully about it you will notice that I will have to write my transmitted wave in this form $e^{i(\omega t - k_3(x-L))}$. Clearly this is a wave that originates at $x = L$ and that would correspond to like the origin for this wave. So, at $x = L$ this would simply become $A_3 e^{i\omega t}$.

(Refer Slide Time: 06:04)



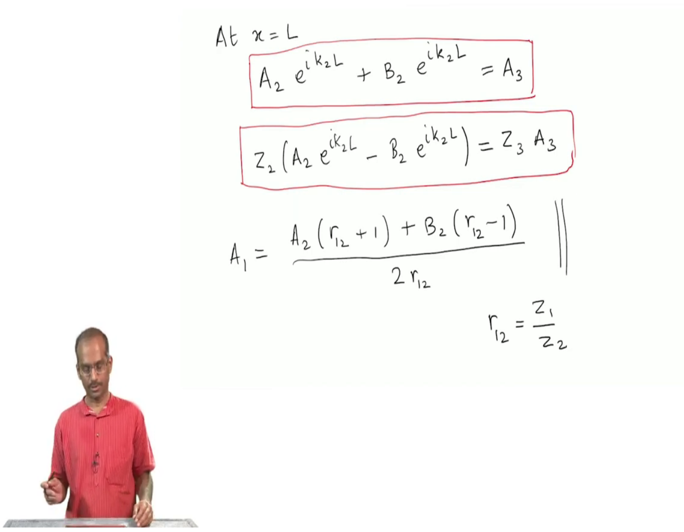
So, ultimately what I want to do is to get an expression for this quantity, first of the boundary conditions is that the displacements across the boundary shall be continuous and second is that the transverse force across the boundary should be continuous as well.

So, we will apply these two sets of boundary conditions that both the boundaries $x = 0$ and $x = L$. And before we go ahead I mean you can see that there are several amplitudes defined here there is A_1, A_2, A_3, B_1 and B_2. But as you can notice finally what I want is it is just a relation between A_3 and A_1 which means that as I proceed I will have to eliminate everything else basically A_2, B_2 and B_1 we will have to be eliminated.

So, let me first apply the continuity for the displacement at the boundary $x = 0$. So, if I do that I am going to get the following expression. I have written the full expression now you put in the condition that this needs to be matched at $x = 0$ and if you put that in then you will be left with $e^{i\omega}$ term in each of these and all of that would cancel and finally at $x = 0$ I am going to get um. So, this should have been A_1 and this should have been B_1, so I am going to get $A_1 + B_1 = A_2 + B_2$.

So, this is one relation that we have got by requiring that the displacement at $x = 0$be continuous. Now if I require that the transverse force at $x = 0$ be continuous I will get one more relation. So, all it requires you start from this relation which is given here and take the derivative with respect to x and then substitute $x = 0$. So, we did this procedure in the earlier modules. So, I will not repeat it in detail. So, I will directly write down the result that we will get.

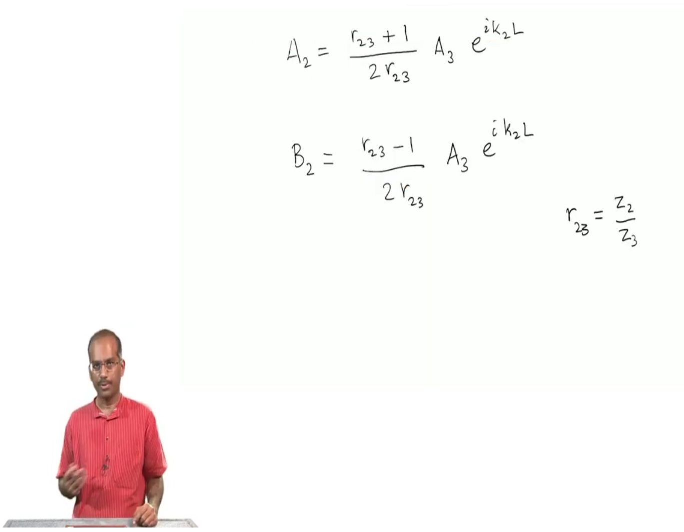
(Refer Slide Time: 08:40)



So, I have the second relation connecting these constants I need to do a similar exercise at $x = L$, you should be able to get the following relations. So, if you put in the condition that the displacements should be continuous you will get this relation. So, if you put in the condition that the transverse force should be continuous you will get this second relation. So, now, I have these four relations all I need to do is to manipulate in such a way that I can relate A_3 and A_1 which means that I need to eliminate B_1, B_2 and A_2 from these four equations that is what we need to do.

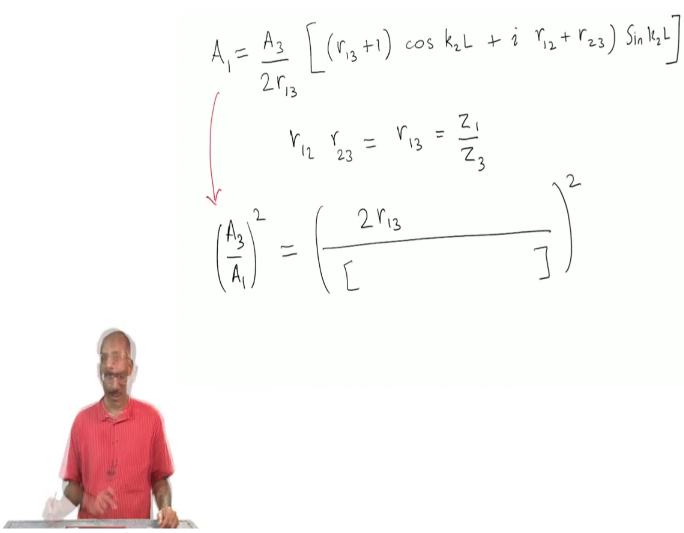
So, to start with you write these two equations eliminate say B_1 and write it in terms of A_2 and B_2 and similarly you go to the second set of equations write A_3 in terms of A_2 and B_2 and finally we shall of course relate A_1 and A_3. So, if I carry out this procedure this is how it would work out. So, for instance eliminating B1 from these two set of equations we will give me the following equation. So, I urge you to do it on your own this is simple straight forward manipulations no tricks involved ok. So, now I have this equation which relates A_1, A_2 and B_2 and just a short hand notation is that $r_{12} = \frac{Z_1}{Z_2}$.

(Refer Slide Time: 10:20)



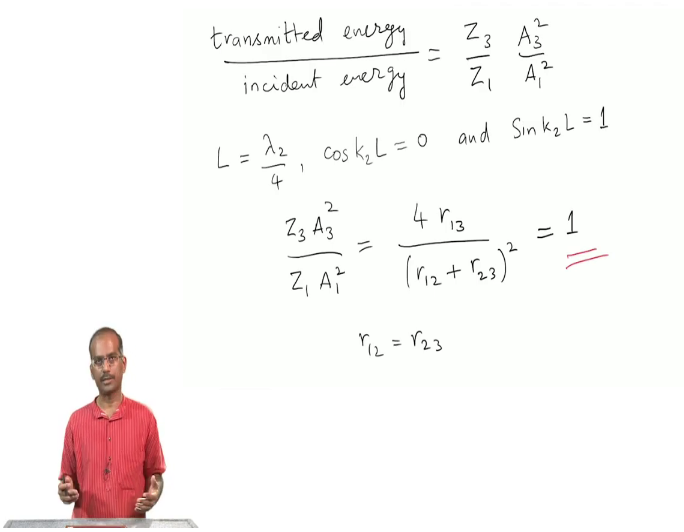
As you can see B_1 is eliminated and we get these set of equations.

(Refer Slide Time: 10:28)



So, this is the expression finally, I get which relates A_1 and A_3 and here $r_{12}r_{23}$ will be equal to r_{13} and simply substitute the values for r_{12} and r_{23}. You should be able to get this as $\frac{Z_1}{Z_3}$ the quantity that we need is $\left(\frac{A_3}{A_1}\right)^2$. So, it directly follows from here, so I should be able to write $\left(\frac{A_3}{A_1}\right)^2$ and that would be simply equal to $2r_{13}$. Of course, all this is whole square $2r_{13}$ divided by this quantity within this square bracket here and all that whole square.

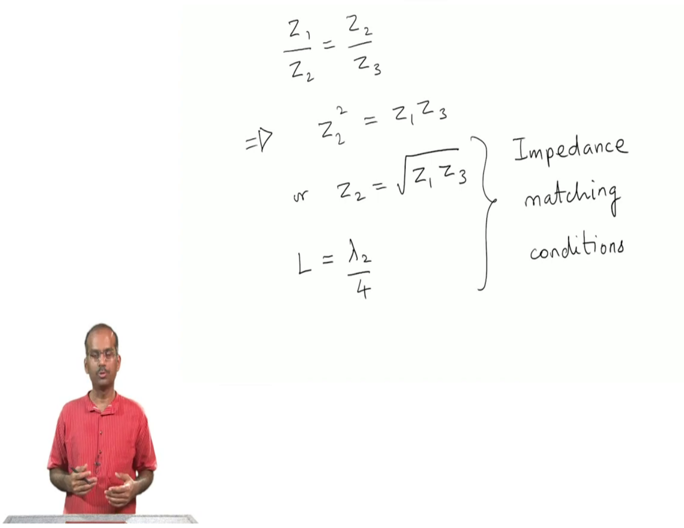
(Refer Slide Time: 11:34)



So, now I can go back to what I wanted to obtain from here and that is equal to $\frac{Z_3}{Z_1}\frac{A_3^2}{A_1^2}$. And now we have this expression for A_3^2/A_1^2 ok. So, we can substitute for $\left(\frac{A_3}{A_1}\right)^2$ from what we have obtain here, but before we do that I would like to invite your attention to what would come here in the denominator here, which would just be this quantity. So, you have a cosine term and a sine term there ok.

Now, if I make the following choices if I take L to be $\lambda_2/4$, $\cos(k_2L) = 0$ and $\sin k_2L$ would be equal to one. In that case what we have is $\frac{Z_3A_3^2}{Z_1A_1^2} = \frac{4r_{13}}{(r_{12}+r_{23})^2}$ and this needs to be equal to one. This is the condition that we have put that the transmitted energy should be equal to incident energy and this one reflects that condition and this happens when $r_{12} = r_{23}$.

(Refer Slide Time: 13:32)



So, r_{12} equal to r_{23} if you go back and see what is r_{12} and r_{23} is going to tell us that this condition needs to be satisfied $\frac{Z_1}{Z_2}$ should be equal to $\frac{Z_2}{Z_3}$. Equivalently $Z_2^2 = Z_1 Z_3$ or Z_2 should be equal to $\sqrt{Z_1 Z_3}$ along with this the length of the coupling medium should be equal to $\frac{\lambda_2}{4}$. So, these two conditions together would be call the Impedance matching conditions.

So, it tells me that the impedance of the middle medium or the central medium. So, you had three media there is one on the left one on the right and between these two was a middle one whose impedance was Z_2. So, if you choose Z_2 to be $\sqrt{Z_1 Z_3}$and you choose the length of that medium to be $\frac{\lambda_2}{4}$ in that case you can match the impedance. So, in practice this means that all the energy that is coming from the left would be completely transmitted without any loss. So, there would not be a reflected component in such a case. So, with this I will stop this module and we will continue with some problems in the next two modules.