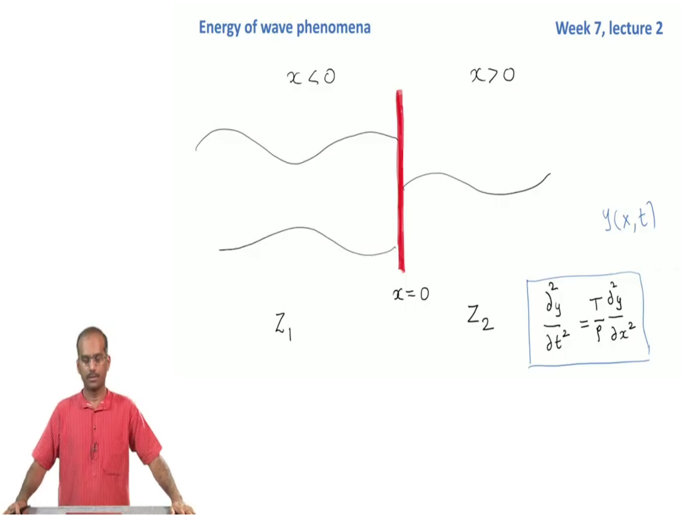
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

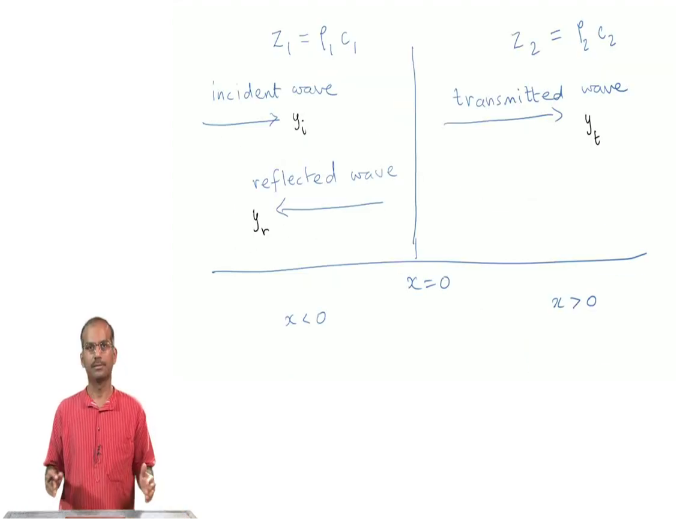
Lecture – 31
Energy of Wave Phenomena

(Refer Slide Time: 00:15)



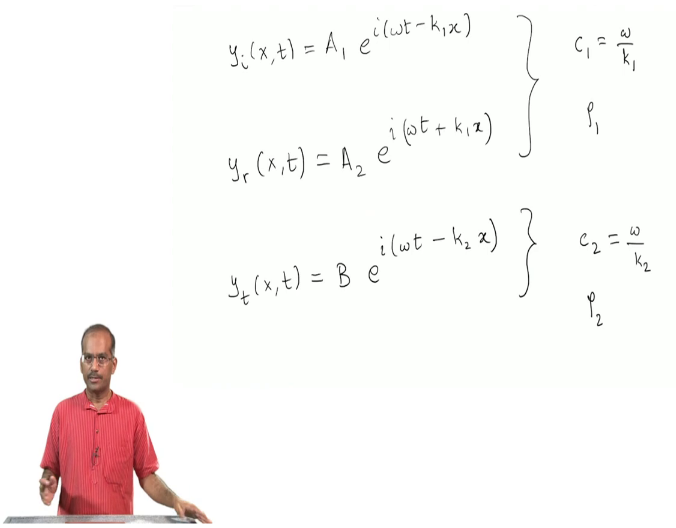
Welcome, this will be the 2nd lecture of 7th week. So, here is a picture that summarizes what we did in the last module. So, our standard scenario is that the wave comes from the extreme left and hits the boundary at $x = 0$. So, there is change of media a part of the wave gets transmitted into the second medium and another part gets reflected back into the first medium. So, we can identify an incident wave; for instance as it is shown in this schematic diagram.

(Refer Slide Time: 00:53)



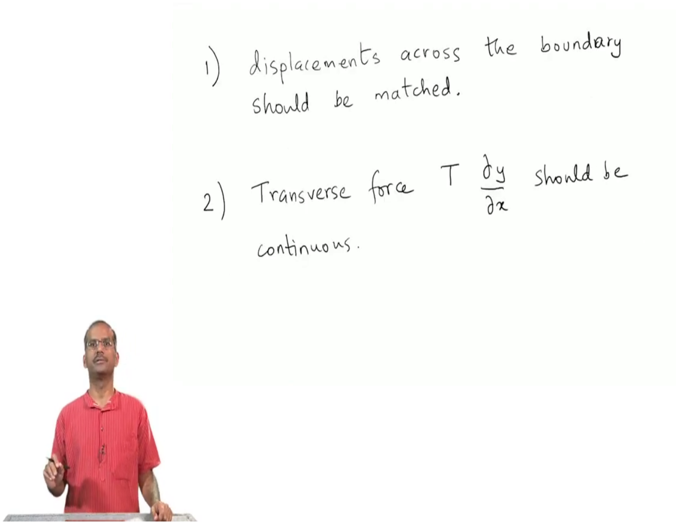
So, there is an incident wave, which I call y_i y again is the displacement and this displacement of the particles that constitute the wave would be a function of both space and time. So, y_i will be a function of x and t. So, there is a transmitted wave which we have designated as y_t and then there is a reflected wave which we have called y_r and importantly these two media one that is $x < 0$ and $x > 0$ they are characterized by different values of impedances.

(Refer Slide Time: 01:36)



So, we can write y_i as a function of x and t its all written here, we went through this in the last module ok. So, I will not repeat everything once again. So, all this constitutes the basic setting for the problem.

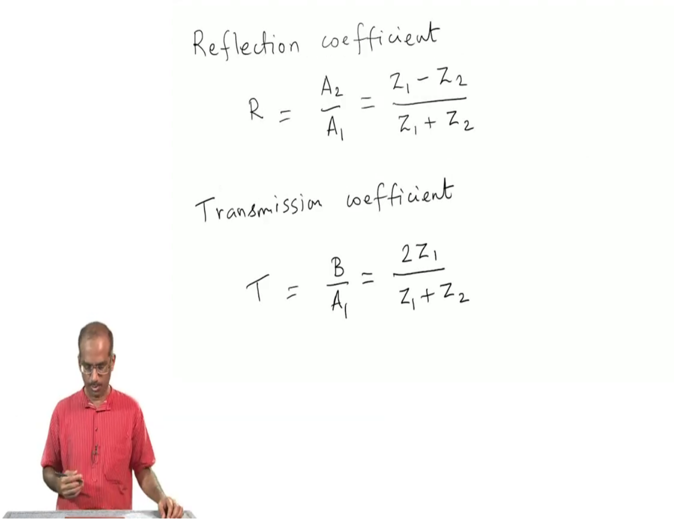
(Refer Slide Time: 01:54)



So, the important physics actually comes in what happens at the boundary? So, at the boundary, we said that there has to be a matching of the displacements. In other words, the displacements to the left side and to the right side of the boundary at $x = 0$should be precisely matched. In other words, the displacements must be continuous again you can see why it has to be so, simply because purely from very physical considerations you can imagine that you cannot have very large displacement on one side and just a little bit on the other side you cannot have very small displacement that is untenable such a thing cannot happen, purely from physical considerations unless you have some external forces acting on it to do such things.

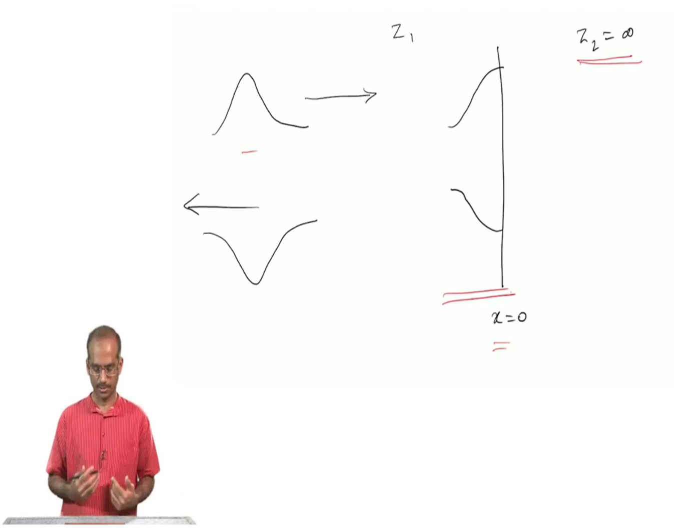
So, we do not have any such possibilities in this problem. So, the displacements will have to be continuous across the boundary. The second condition is that the transverse force; the force that acts in the vertical direction, the transverse force has to be continuous because if there is a discontinuity in the force, there will be a net force in some direction and it will just carry the whole string in that direction. So, again that does not happen in practice without any external force being applied and so on.

(Refer Slide Time: 03:26)



In the previous module, we were able to show that the reflection coefficient and the transmission coefficient are simply functions of Z_1 and Z_2

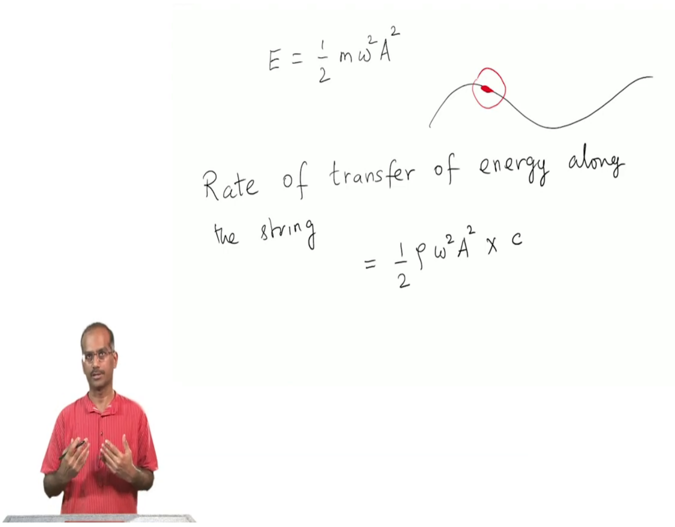
(Refer Slide Time: 03:39)



So, pictorially what we have is that there is an incoming wave as you can see which is this. It comes and hits the boundary here at $x = 0$ and what I have tried to show here is when part of the wave has already been reflected of the boundary. And here I have assume that Z_2 is infinity which means that second medium offers infinite resistance to the passage of wave. Which means the wave cannot actually penetrate through the second medium. So, it will have to be reflected back into the first medium which is what sort of happens in this case.

Let us look at reflection and transmission in terms of the energies because these waves also carry energy. So, when a wave comes and hits the boundary, its depositing some energy at the boundary. So, part of that energy is getting transmitted into the second boundary and part of that energy is being reflected back as well. So, today let us go over the same sort of phenomena, transmission and reflection of waves at a boundary, but in terms of energies. I am going to start by recalling our basic formula for the energy of a single simple harmonic oscillator.

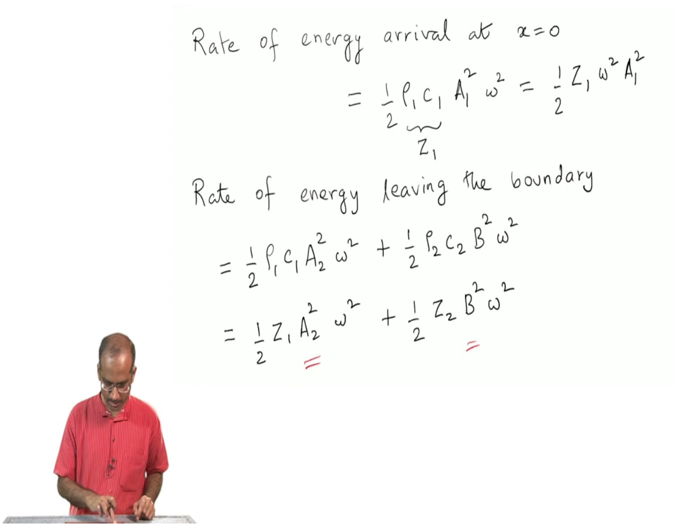
(Refer Slide Time: 05:06)



So, here m is mass of the particle, ω is the angular frequency A is the amplitude. So, this is the formula for energy valid for a single simple harmonic oscillator. Now I am looking at the case of a string that is oscillating. So, if I think about this small segment of that string, all that this small segment is doing is to simply oscillate up and down. Which means, that its essentially just doing the work of a single simple harmonic oscillator. This also implies that I can simply use this energy formula of a simple harmonic oscillator straight away. Not only that this is oscillating, but the disturbance is actually propagating through the string that will be equal to half.

Now, m is has to be replaced by ρ, linear density. So, if I am considering, let say unit a small segment of unit length ρ would simply be the mass. So, I am replacing m by ρ here ω^2 into A^2 multiplied by the phase velocity which I will write it as c. So, this gives me a formula for rate of transfer of energy along the string. So, this is going to be my basic formula that I will use to discuss, how much of energy is transferred along the string? So, remember that this is for a wave that is propagating along the string. So, the first question that I would like to know is how much of energy is let say, coming to the point $x = 0$, which is the interface between medium one and medium two.

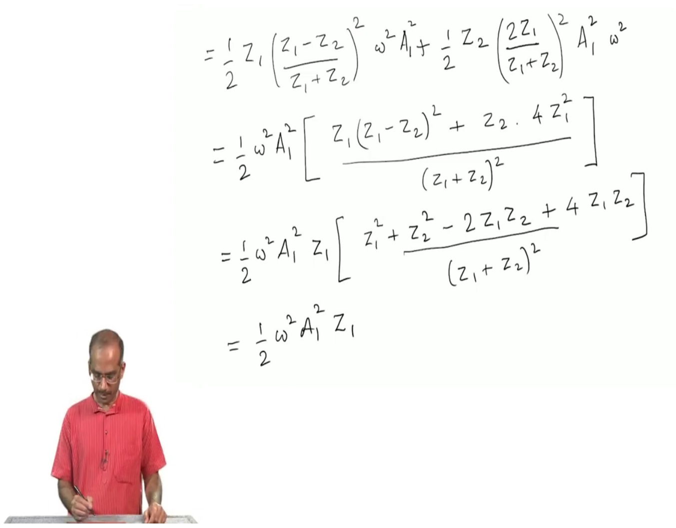
(Refer Slide Time: 06:58)



To this quantity, the rate at which energy arrives would be equal to $\frac{1}{2}\rho_1, c_1$ which is the phase velocity of the incoming wave in the first medium. Multiplied by A_1 that is the amplitude of the incoming wave multiplied by ω^2. Now, the energy that is coming to $x = 0$ is being split between two parts; one is part of that energy is sent into the transmitted wave and part of that energy is put back in the reflected wave. So, this total energy has to be now distributed among these two components. So, then I should be able to calculate the rate at which energy leaves the boundary and following the formula that we would written down earlier that would be $\frac{1}{2}\rho_1 c_1 A_1^2 \omega^2$ plus of course, this part was for the reflected component.

Now, I will write the energy for the transmitted component. So, that would be $\frac{1}{2}\rho_1 c_2 B^2 \omega^2$. The next thing I want to do is to replace this A_2 and B in terms of A_1 and we can do that because, we already have this relation that connects A_2 and A_1 and B and A_1. So, from here A_2 will be $\frac{Z_1 - Z_2}{Z_1 + Z_2} A_1$. So, I am going to use these relations.

(Refer Slide Time: 08:56)



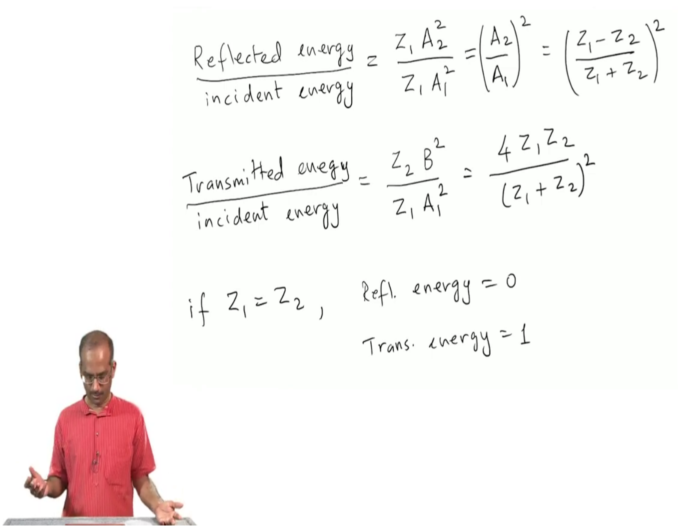
Now A_2 and B_1 have been replaced in favor of A_1. So, let us simplify this. This is going to give me $\frac{1}{2}\omega^2 A_1^2$ that is common which I can take out. And I will be left with $\frac{Z_1(Z_1 - Z_2)^2 + Z_2 4 Z_1^2}{(Z_1 + Z_2)^2}$

So, if I take out Z_1 again. So, I should have $\frac{1}{2}\omega^2 A_1^2 Z_1 \left[\frac{Z_1^2 + Z_2^2 - 2Z_1 Z_2 + 4Z_1 Z_2}{(Z_1 + Z_2)^2}\right]$.

Now if you look at the numerator, this is simply$Z_1^2 + Z_2^2 + 2Z_1 Z_2$ because this last two terms would partially cancel. So, the and that is simply $(Z_1 + Z_2)^2$.

So, the numerator and denominator is going to cancel out. So, its going to finally, give me just this quantity $\frac{1}{2}\omega^2 A_1^2 Z_1$. So, this is the energy that is expanded in the transmitted and the reflected wave. Now, if you compare this final relation with what we obtain here for the rate at which the energy arrives at $x = 0$ at the boundary. We will see that they are exactly equal. That should not be surprising because the rate at which the energy is arriving at $x = 0$ is the rate at which the energy is being pumped into the transmitter and the reflected waves. So, clearly there is energy balance because we have assume that there is no dissipation.

(Refer Slide Time: 11:33)

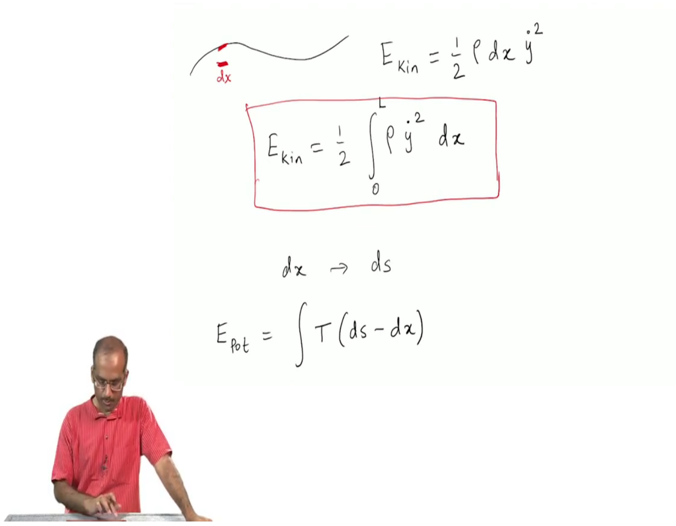


Now, we are ready to write everything in terms of energies. So, for example, so, the reflected energy would be and we already know this quantity $\frac{A_2}{A_1}$ from here. So, clearly that is $\left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$. So, that is the fraction of energy that is reflected back. Now what about the transmitted energy?

So, now let us write a simpler expression for the fraction of transmitted energy. So, clearly now, I have these two quantities which are ratios of reflected energy to incident energy and transmitted energy to incident energy. So, with this, we can make a few simple checks. So, for instance, what happens if $Z_1 = Z_2$. So, in that case, this ratio the reflected energy by incident energy would go to zero because you have the term Z $Z_1 - Z_2$, if both are equal that is going to be zero. And what will happen to transmitted energy, if Z_1 and Z_2 are equal. So, in that case this term would be 1 because on the denominator we will have 2 let say Z_1^2 and that will cancel with $4Z_1^2$ in the numerator.

So, if Z_1 is equal to Z_2, we will get that fraction of reflected energy would be zero and the fraction of transmitted energy would be one. That should not be surprising because after all $Z_1 = Z_2$ implies that, both the media the one on the left side and one on the right side across the boundary have exactly the same impedance which means that, as far as the wave is concerned it does not see a change in media at all, if the impedances are exactly equal. Until now, we were calculating the energy of a wave that is propagating. So, it is the energy for a propagating wave. Now, what about energy of a standing wave? Broadly speaking in such a scenario, the total energy is simply the sum of kinetic and the potential energies.

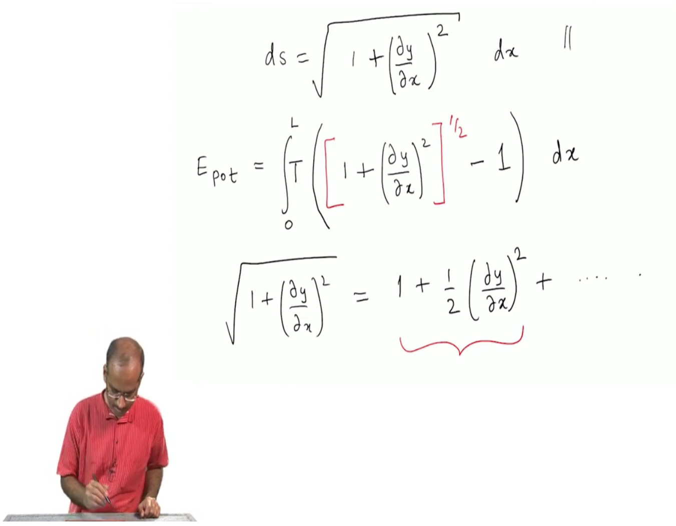
(Refer Slide Time: 14:18)



So, let say that, I have this string and I am looking at some small segment. Whose length is dx. So, in such a case the kinetic energy would be mass times velocity square. So, that is half mass here is the linear density multiplied by dx times velocity square, which is this the particle velocity. Now, I want to calculate the energy over a larger length in which case I could write E_{kin} to be. So, this is my expression for kinetic energy.

So, the potential energy, let us say in some deformed position would be simply equal to the amount of work done to deform the string to bring it to that position. So, the work is done by the tension in the string. So, if I have uniform tension T in the string and let us say that a small segment dx gets deformed and length becomes ds. Now under this transformation, how much of energy is stored as potential energy. So, here I have my expression for the potential energy which is the uniform tension T multiplied by this deformation which is $ds - dx$.

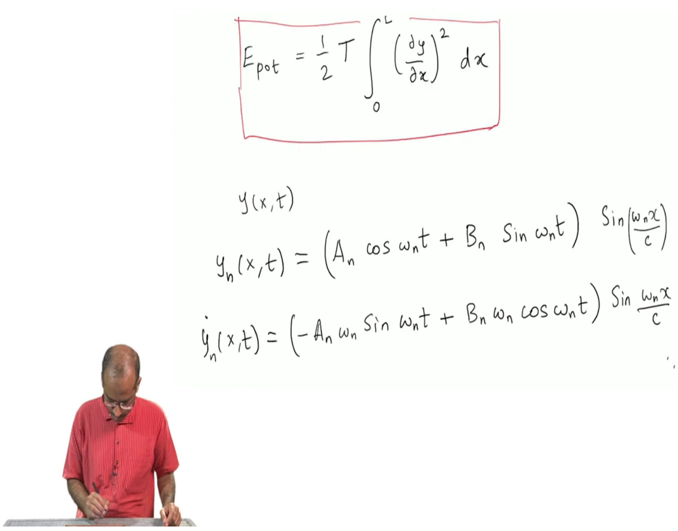
(Refer Slide Time: 16:03)



So, if you remember some modules back may be a week or too earlier we wrote down an expression for ds. So, I even give a short derivation for this segment of length ds in terms of dx. So, this is again under the assumption that the its all within the ambit of small oscillations. So, now, that we have these expression for ds, we just need to plug in this expression here and if you do that this is what I get E_{pot} is an integral over $ds - dx$; ds can be written in terms of dx like I have done here. So, everything fits in into this equation.

So, I can simplify this further within again the ambit of small oscillation when I say that $\frac{\partial y}{\partial x}$ is small enough. In which case, the quantity inside this red brackets would become the following. Plus there are higher order terms which we shall ignore because $\frac{\partial y}{\partial x}$ is a small enough. So, if I substitute this within this red brackets this is 1 and -1 would cancelled.

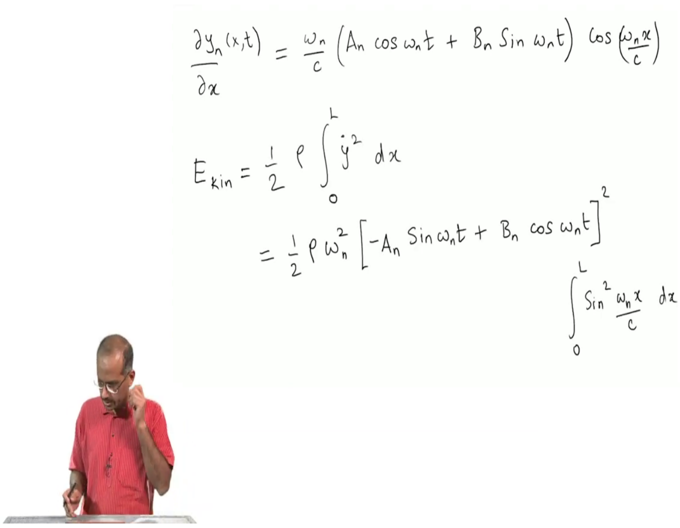
(Refer Slide Time: 17:30)



So, my final expression for the potential energy would turn out to be the following. So, all I need to do is to simply assume a solution for $y(x,t)$ and I need to compute $\frac{\partial y}{\partial x}$ and $\dot{y}$ substitute them here do the integral and get the answer. So, let me start by assuming that, the nth normal mode the displacement which is a function of x and t is given by the following expression. So, in these expression n denotes the index for the normal mode that we are considering A_n and B_n or the amplitudes of the two super post components. And ω_n is the normal mode frequency.

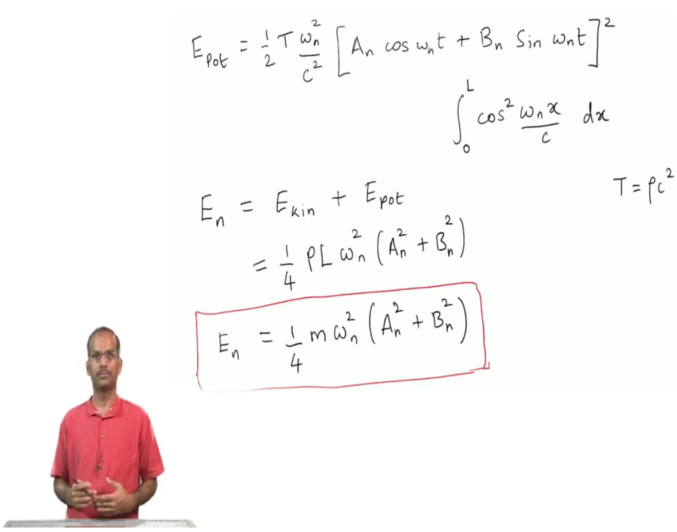
Now to compute the kinetic and potential energy I need $\dot{y}$ and $\frac{\partial y}{\partial x}$ ok. So, $\dot{y}$ would be $y_n(x,t)$ that would be. So, I have computed $\dot{y}(x,t)$. So, similarly I also need an expression for $\frac{\partial y}{\partial x}$ ok.

(Refer Slide Time: 18:47)



So, with these two expressions now rest of the work is simply substitute it in our kinetic and potential energy expressions. So, let us first calculate the kinetic energy by substituting $\dot{y}$ in this. So, E_{kin} is; so, I have this, what looks like a longish expression for the kinetic energy.

(Refer Slide Time: 19:16)



Similarly, I will have another one for potential energy. So, remember that these are all for the *n*th normal mode. So, now, we have these two expressions these are simple integrals to do. So, I urge you to do it plug in the answer you should be able to finally, get this expression. So, let me indicate by E_n, the total energy. So, that is the energy in the nth normal mode. So, that will be the sum of kinetic and potential energies. Of course, ρ into the total length is simply the total mass of the string. So, that will be $\frac{1}{4} m\omega_n^2 \left(A_n^2 + B_n^2\right)$ and each of these integral will contribute a value of half. So, here then we have an expression for the energy of nth normal mode. So, I will close this module with this result for the total energy and in the next module, we will look at what is called an impedance matching condition.