Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 30
Reflection and Transmission of Waves

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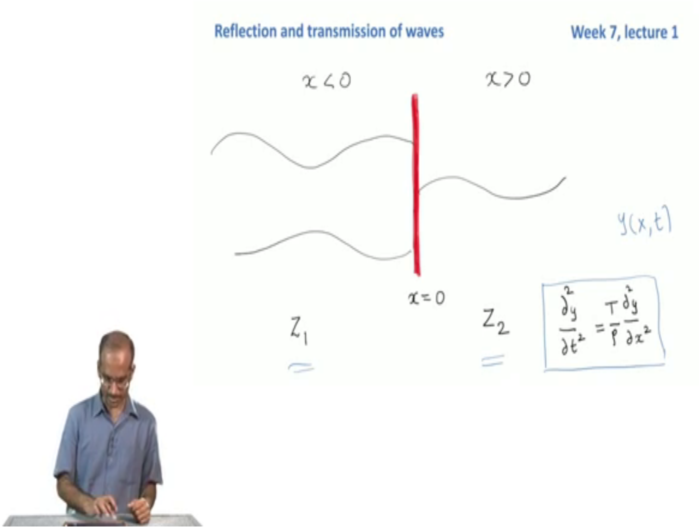

As we begin the 7th week we are going to now look at even more advanced dynamics that waveforms can show. For example, what we will be doing this entire week would be about Reflection and Transmission of Waves. I assume that there is some waveform that is propagating let say from the left hand side from the far left hand side and at $x = 0$ as it is shown in this figure there is a change of medium. So, when the waveform reaches this boundary part of it would be transmitted to the second medium.

So, when x is positive, $x > 0$ would correspond to second medium and there will be a part of it which will be reflected back into the first medium which corresponds to $x < 0$. So, x is of course, the position and since we are dealing with one dimensional wave forms this alone is sufficient to talk about; to talk about the dynamics of the waveforms. So, this is the standard scenario which we are going to work with.

So, let us look at some examples before we actually begin our calculations if we think of our standard example namely the string, its equivalent to saying that you know at $x = 0$ suddenly the material of the string has changed. Because in all these problems where a wave is launched on a string the medium is the string itself and in the medium of string there is some waveform that is travelling at $x = 0$ suddenly the string has changed. So, its possible that for example, you could have maybe a thin string for $x < 0$ and a much thicker string for $x > 0$.

So, suddenly the wave finds itself in a different medium when it crosses from $x < 0$ to $x > 0$ that is one sort of example you could keep in mind. But there are many examples for instance routinely we will have to discuss things like what happens when light travels from one medium to another or when sound waves travel from one medium to another.

So, these are typical examples of when waves do meet the boundary between two different media and they do travel from one medium to another medium and there is also often when such a thing happens a reflection back into the first medium itself. So, anything that I speak in this room a little bit of that is heard in the next room adjacent send to me. Simply because the wave is generated the sound waves they go hit the wall part of it is transmitted and part of it is of course, reflected back. So, I could say that in medium one corresponding to $x < 0$ the impedance of the string is Z_1 and in medium two corresponding to $x > 0$ impedance is Z_2.

So, when I says wave is going from medium one to medium two its actually going from a string which is characterized by impedance is Z_1 to another string which is characterized by impedance is Z_2. We will continue to work with this wave equation and the solutions of these wave equations which are simply y as a function of position and time. Now, the additional complication is that we are going from one kind of medium to another kind of medium. So, let me recap this idea of a impedances.

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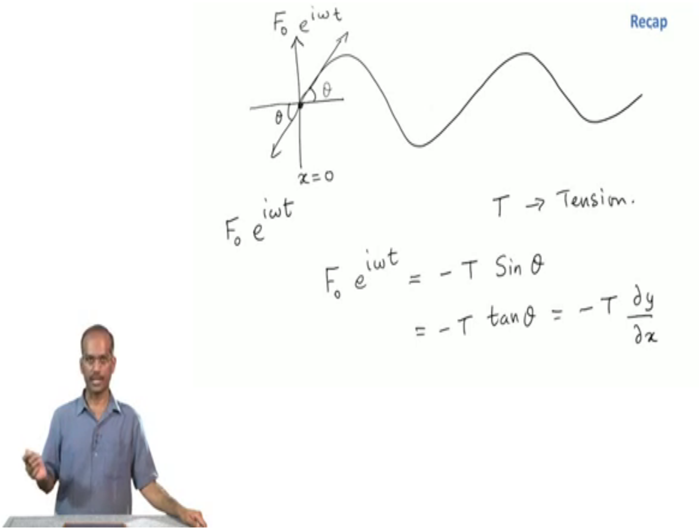

The problem was that I have a string I do not need to worry about what is a boundary condition on other end of the string, but at my end I am giving it oscillations and I am assuming that there is uniform tension in the string. So, the forcing that I am providing here at this point propagates as a wave through the string.

So, in such a situation we equated the forces at $x = 0$. So, you might recall this figure. So, at $x = 0$ we equated the forces. So, the forcing external forcing was $F_0 e^{i\omega t}$ and that is equated to the component of tension which is $-T \sin\theta$, where θ is the angle that this string makes with the horizontal. Now, of course, we work with the small angle approximation as usual. So, $\sin\theta$ is approximately $\tan\theta$ which is $\frac{\partial y}{\partial x}$ just the gradient of the displacement at $x = 0$.

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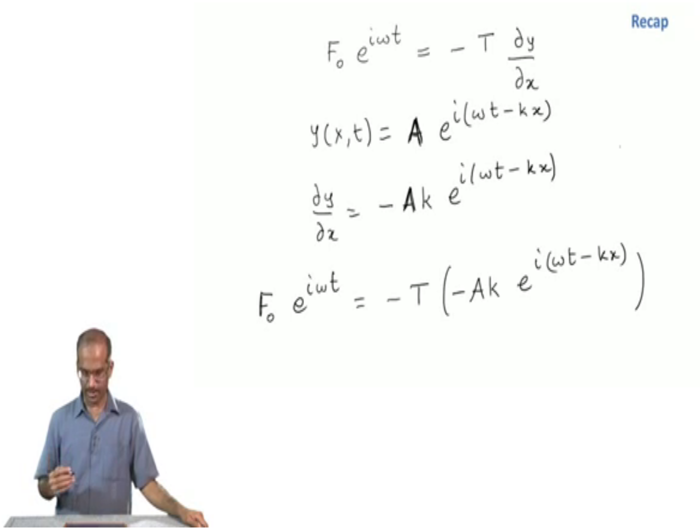

So, we go through this derivation we compute this $\frac{\partial y}{\partial x}$ at $x = 0$.

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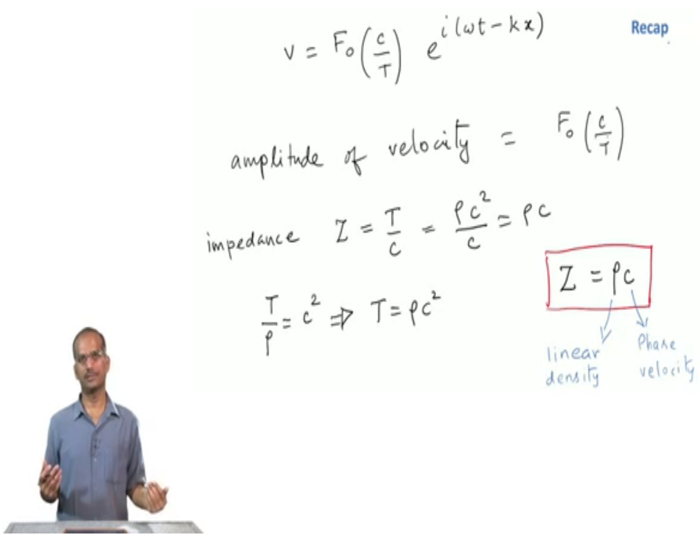

And finally, we recognize that there is a quantity impedance that is involved here and impedances simply ρc when we use impedance to characterize a medium what quantities are involved, one is the linear density and other is the phase velocity together these two quantities characterize the medium. Physically impedances the resistance that medium offers to the flow of waves at $x = 0$ I have my boundary.

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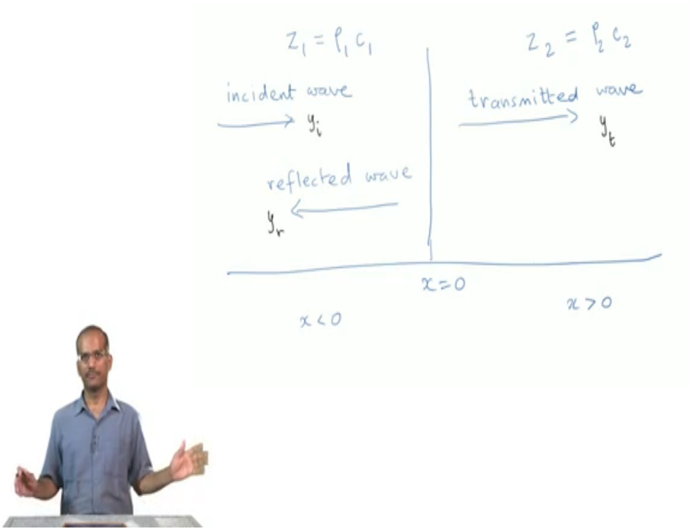

So, this is a boundary where there will be a change of media ok. So, for $x < 0$there is one media which is characterized by Z_1 and Z_1 from the formula that we made just now is $\rho_1 c_1$. ρ_1 is the linear density in that medium and c_1 is the phase velocity and similarly in the region when $x > 0$ the media is characterized by the impedance Z_2 which will be given by $\rho_2 c_2$ again ρ_2 is the linear density in the second medium.

So, I am going to have a wave that is coming in from far left. So, that would be my incident wave and what are the various things that this incident wave can do it comes from far left hits the boundary at $x = 0$ there are only two things that can happen one is there can be transmission; so there will be a transmitted wave. And another possibilities there can be reflection. So, you can have a reflected wave. So, its reflected back in the same medium.

And for each of these waves incident, reflected and transmitted wave each of them are solutions of our wave equation. So, we shall start by writing down the solutions for each of these components. For a string is the medium through which all this is happening we shall assume that there is uniform tension T in this string.

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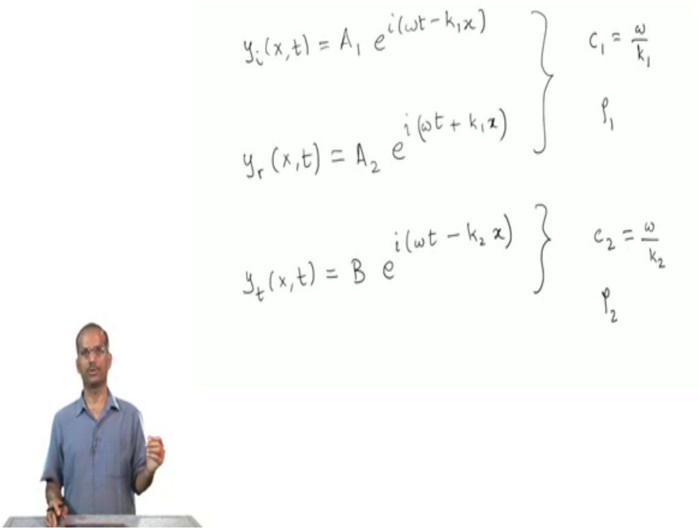

So, let me write the incident wave. So, you should remember that this displacement of the incident wave as usual will be a function of position and time. So, I should be able to write it as $A_1 e^{i(\omega t - kx)}$. Similarly, I can also write for the reflected wave which is y_r again a function of position and time. So, in the case of reflected wave it goes in the opposite direction. So, I need to change the sin x which is why I have $e^{i(\omega t + kx)}$. So, that tells me explicitly that its a wave that is going from right to the left and all these are happening in medium one corresponding to $x < 0$. So, now, let me also write the solution for the transmitted wave which will be as usual function of position and time.

Now, you will notice that I have used same value of ω for the incident for the reflected and the transmitted wave, but I have use different values of wave number k_1 and k_2; k_1 for the first medium and k_2 for the second medium. So, I should recall this idea that the frequency of oscillation is not going to change even across the boundary just going to remain the same. So, together as a coupled system ω is like your normal mode frequency, its not going to change. So, phase velocity for instance which I can indicate by c_1 would be $\frac{\omega}{k_1}$.

So, if you are in medium one both the incident and the reflected wave would have the same phase velocity simply because ω is same and k_1 is also same for both of them. On the other hand for the reflected wave the phase velocity would be a different value. So, that would be c_2 which is $\frac{\omega}{k_2}$. So, let me go back and write these things here in our figure. So, this is y_i and this is y_r the reflected wave and this is y_t which is the transmitted wave we need to put in some boundary conditions that would connect what happens at $x < 0$ to what happens at $x > 0$. So, typically when you see such things happening where a wave goes from one media to another media you would not see suddenly the displacement changing in value just because it is changed from one medium to another medium.

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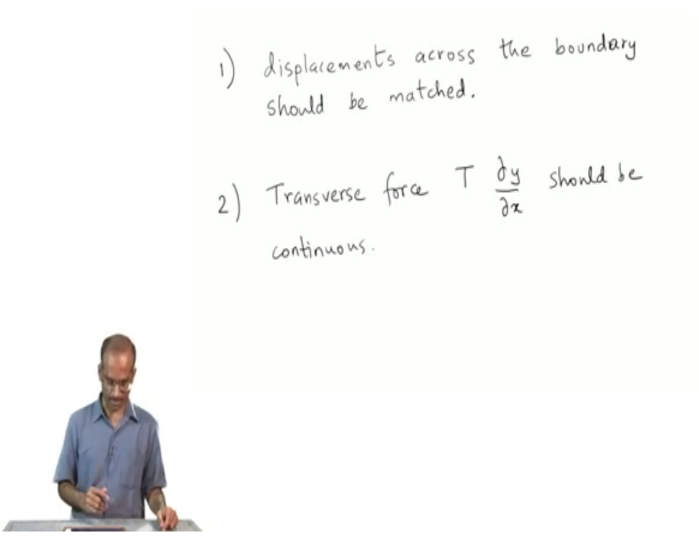

One of our first boundary conditions should be that displacement across the boundary should be matched, its another way of saying that displacement cannot be discontinuous at the boundary. You cannot have a small displacement of the wave that is incoming and suddenly that produces a huge displacement on the other side you know that cannot happened physically that is not feasible. Second condition is what can be called a dynamical condition namely that the transverse force which is given by T into the gradient $\frac{\partial y}{\partial x}$ should be continuous.

So, we have the physical system that is setup wave going from one media to another media there is part transmission part reflection and we have written down the solutions for each of those components you characterize each of the media we have written down the solutions. Now, to bring them all together we need some point of commonality in a sense and that is provided by these boundary conditions.

So, both boundary conditions of physically motivated the displacement across the boundary should not be discontinuous. So, we put in condition the displacement should be continuous and transverse force should not be discontinuous if it does there would be net force. So, we avoid that situation and we put in a condition that transverse force should be continuous as well ok. Now, we are set up to apply these conditions on our equations. So, first of that says that displacement should be matched which implies the following.

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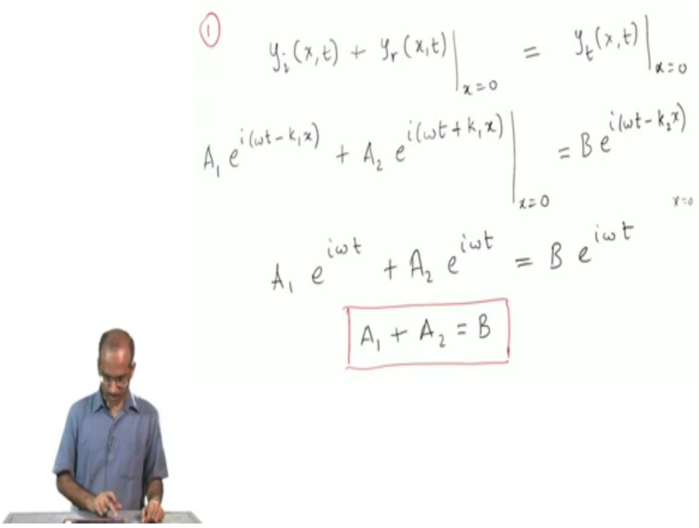

So, I just need to put in $x = 0$ in the in these equations and as you can see it will give me $A_1 e^{i\omega t} + A_2 e^{i\omega t} = B e^{i\omega t}$ and of course, $e^{i\omega t}$ can be cancelled that is one way of saying that $e^{i\omega t}$ cannot go to 0. Hence it must be true that $A_1 + A_2 - B$ should be equal to zero which is equivalent to saying that $A_1 + A_2 = B$.

So, it connects the amplitudes of the incoming wave, the reflected wave and the transmitted wave. So, that is the result of putting in the first boundary condition. So, I have put in the first boundary condition. Now, let us use the second boundary condition namely that the transverse force should be continuous at the boundary which means at $x = 0$.

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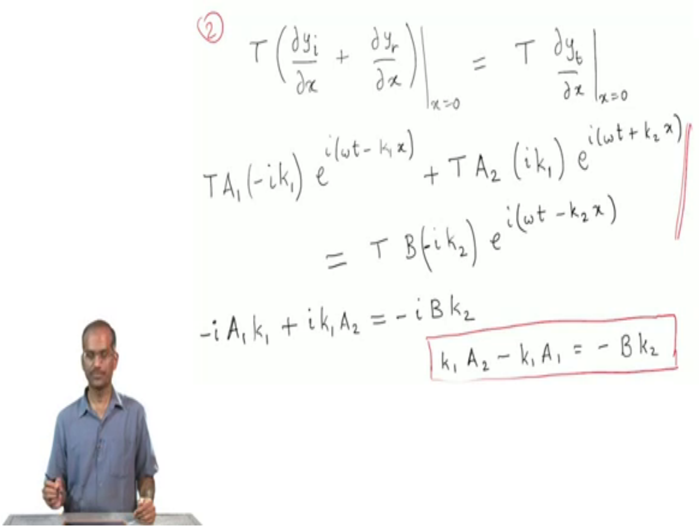

I can express this condition as now again since we already know y_i, y_r and y_t I just need to differentiate all three of them with respect to x I will get the following expression. So, I have the complete expression in front of me right now it just requires one differentiation with respect to x to be done. Now, we need to just substitute $x = 0$ throughout. So, this is again second expression that we have which again connects A_1, A_2 and B and there is also k_1 and k_2 involved. Now, if you look at what is unknown in this there are three quantities A_1, A_2 and B.

But we seem to have only two equations. So, it appears as though that we will have to determine three unknowns from only two equations its true in a sense, but we really do not need to know all the three quantities, its enough if we know each of these quantities as a ratio with respect to A_1. So, now let me write both these equations in that form.

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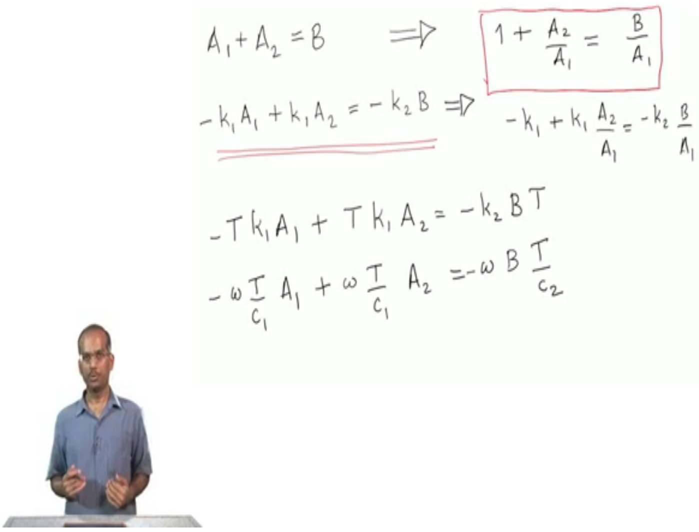

So, I had $A_1 + A_2 = B$ this would be, can be written as $1 + \dfrac{A_2}{A_1}$ which is equal to $\dfrac{B}{A_1}$. Second of the equations is that $-k_1 A_1 + k_1 A_2 = -k_2 B$ and that will give me $-k_1 + k_1 \dfrac{A_2}{A_1}$ which is equal to $-\dfrac{k_2 B}{A_1}$ ok.

So, we will keep these two equation aside for the moment we will come back to this, but let me focus for a moment on this second equation. Now, what I want to do is to introduce impedances Z_1 and Z_2. So, you will remember that we said impedance Z characterizes the medium. So, somewhere the impedances will have to enter this equation.

So in fact, we should what we should do is go back and insert the T which was which we actually cancelled off from this equation. So, let me rewrite this equation slightly differently. So, I am going to multiply it by T throughout which is the uniform tension in the string and this k_1 we will substitute by $\dfrac{\omega}{c_1}$ and k_2 will be substituted by $\dfrac{\omega}{c_2}$. So, Z_1 and Z_2 are the impedances in the two media.

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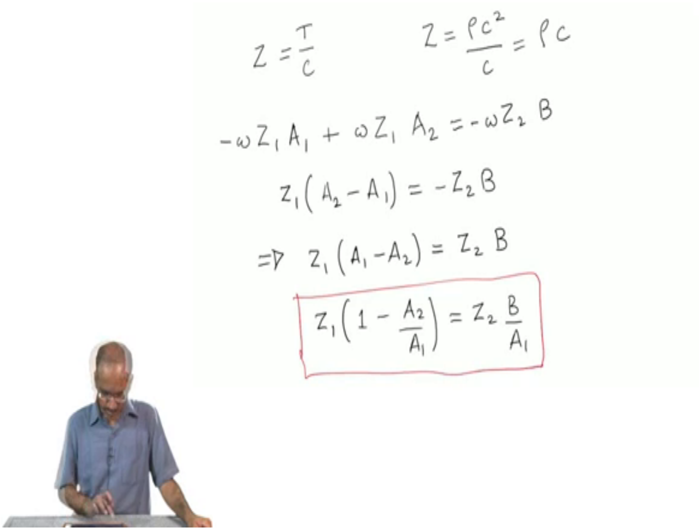

Now, I can cancel ω throughout and rewrite it slightly differently I can take Z_1 common. So, that would be $A_2 - A_1$is equal to $-Z_2B$. So, now, again I divide throughout by A_1 in which case this equation would become $Z_1\left(1-\frac{A_2}{A_1}\right)$ into $\frac{Z_2B}{A_1}$. Now, I have this equation and from the other condition I have this equation. From this point onwards I just need to solve for $\frac{A_2}{A_1}$ and $\frac{B}{A_1}$.

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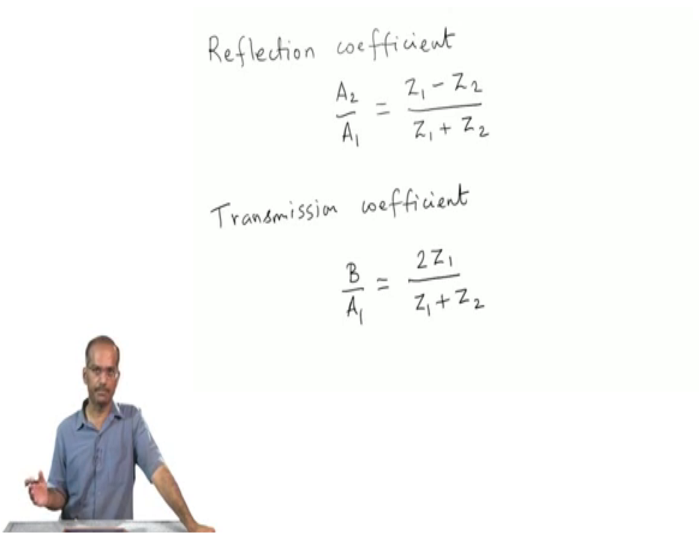

One thing to notice with these results is that this ratios $\frac{A_2}{A_1}$ which is a reflection coefficient and the transmission coefficient let us $\frac{B}{A_1}$ both are independent of ω it does not matter what your normal frequencies or what your frequencies what if the second medium offers a huge resistance which is equivalent to saying that the impedance of the second medium is infinity extremely large.

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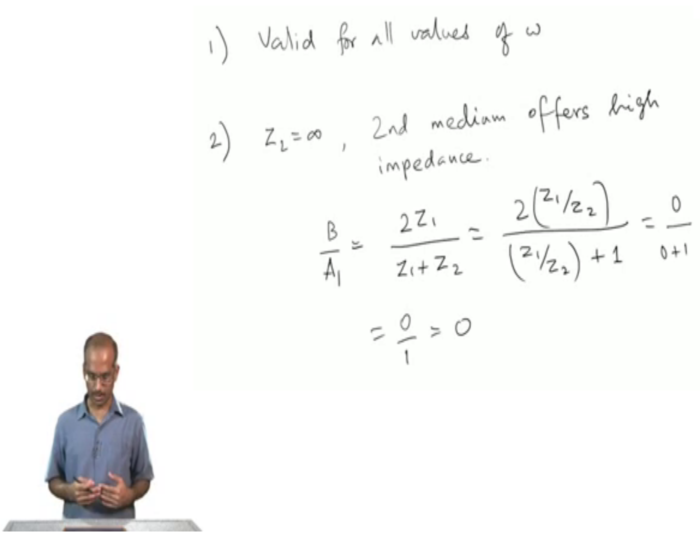

So, your intuition should tell you that if its going to offer huge impedance the wave should not be able to pass through into the second medium we need to check if that is true coming out of the formulas. So, here if I go back to this formula which is $\frac{B}{A_1}$ the transmission coefficient, if I divide the numerator and denominator by Z_2 I will have. So, $\frac{B}{A_1}$ would be $\frac{2Z_1}{Z_1+Z_2}$ let me divide throughout by Z_2. So, I will have $\frac{\frac{2Z_1}{Z_2}}{\frac{Z_1}{Z_2}+1}$.

Now, if you take the limit $Z_2 \to \infty$ I am going to get a zero in the numerator and $\frac{Z_1}{Z_2}$ will be zero plus one. So, I am going to get zero by one which is equal to zero. So, that tells me that the transmission coefficient is zero if Z_2 is infinity, it very much kind of agrees with our intuition that there should not be a transmitted wave if the second medium offers infinite impedance.

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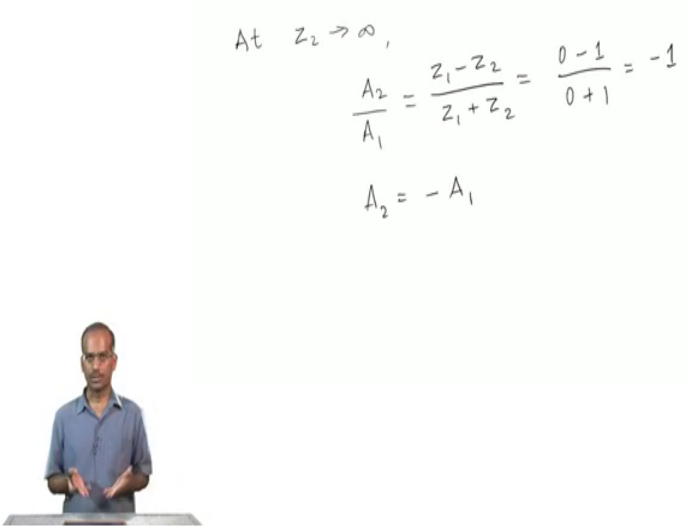

At Z_2 going to infinity its a simple exercise simply divide by Z_2 in the numerator and denominator $\frac{A_2}{A_1}$ will give you let say $\frac{Z_1 - Z_2}{Z_1 + Z_2}$ and if I divide by Z_2 that would be $\frac{0-1}{0+1}$ and that is going to give me -1.

So, A_2 is equal to $-A_1$. So, A_2 is the reflected amplitude and A_1 is the amplitude of the incoming wave. So, here nothing is transmitted everything is reflected and when such a reflection takes place there is a phase change of π. So, that is what we see here in this problem, so the entire wave is reflected nothing is transmitted. In the limit when $Z \to \infty$ R is -1 and T is 0.

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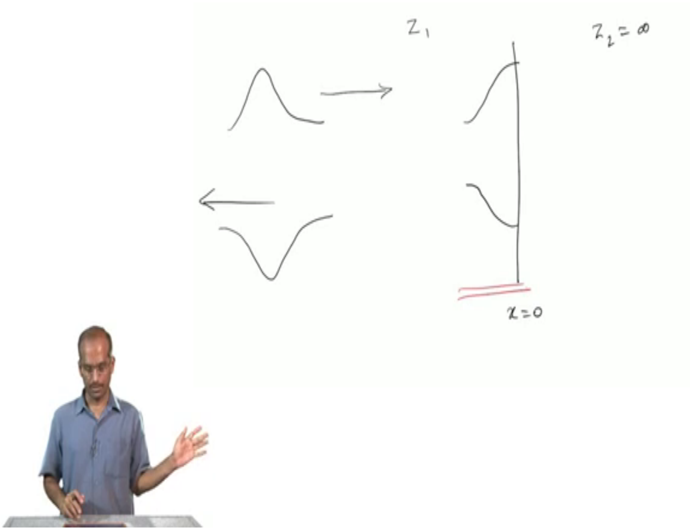

Say for example, this is my boundary at $x = 0$ and I have an arbitrary wave packet something that looks like this which is going in this direction that is my incident wave. And when it hits the boundary if assuming that Z_2 is infinity and Z_1 is some value of impedance what we will see is that. So, part of this wave has already been reflected and it comes out with the minus sign and long after it has interacted with the boundary the wave would look something like this and it would be travelling in this direction now.

So, what I have drawn here is what happens when it interacts with the boundary and long after all this is happened you will notice that there is only one outgoing wave which is exactly like the incoming wave, but sign has sign of amplitude has changed. So, that is what happens if Z_2 is infinity, if Z_2 was not infinity a part of this wave would have gone into the second region.

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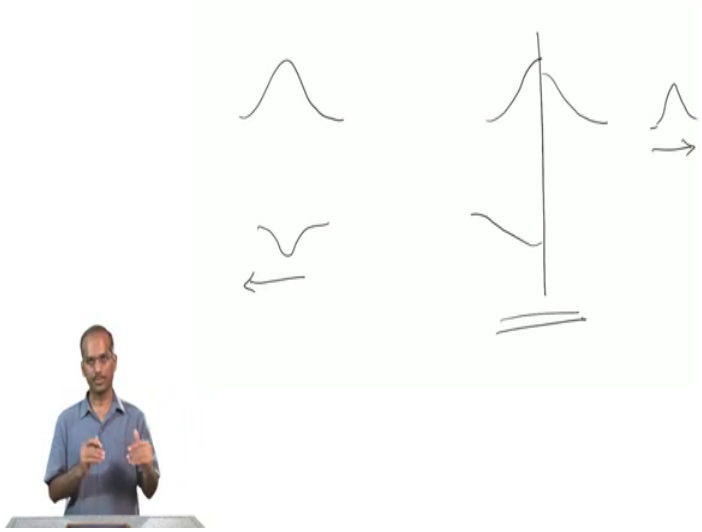

In other words the figure should have been drawn something like this; this is my incoming wave and maybe when it interacts with boundary there is a part that has crossed over with the different amplitude and a part that has gone like this. So, there would be a smaller amplitude wave here and another smaller amplitude wave here. So, long after it interacts with this boundary here there is one part which will be moving in this direction which is the transmitted wave with possibly smaller amplitude and there is another one which is reflected which is moving in this direction probably smaller than the incident one.