]Waves and Oscillations
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Lecture - 03
Superposition of Oscillations: Beats

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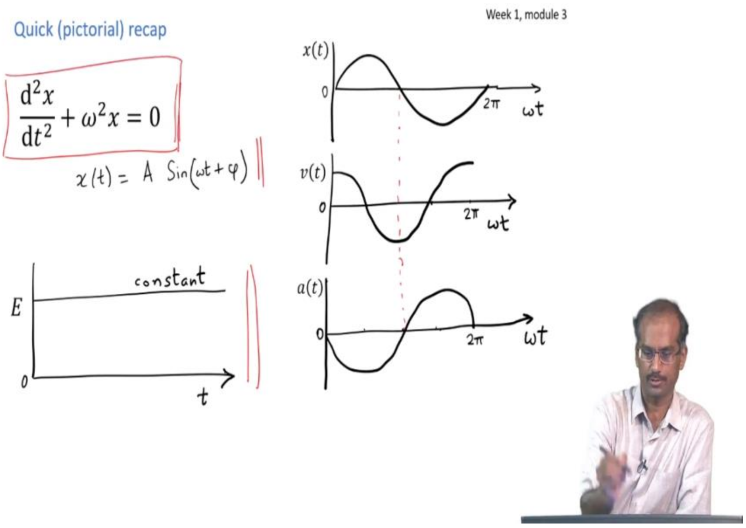

Welcome to the third module, so let us again as usual begin with the quick recap of what we had done in the last two modules. So, we started with our basic ansatz for an oscillating system. The oscillating system could be anything, but nevertheless the basic physics is that the restoring force is proportional to displacement with a negative sign and this is true in the limit of smaller displacements. Now, if you follow this thread of logic finally, you end up with an equation of motion, which is given here right in front of you and \omega here is the angular frequency.

So, in the last module we saw that given the displacement being a function of time and we were able to obtain the functional form which satisfied this equation. So, for example, we could one possible solution that we wrote down was to say that, displacement was some $Asin(\omega t + \phi)$, where ϕ in general is some phase. And if I try to plot these; sketch these functions that is how that is what I have here on the right hand side. So, displacement as a function of time is a sine curve just as I have written it down here and velocity is simply the first derivative of the displacement with respect to time, shown here and acceleration is the second derivative of displacement with respect to time.

So, all that is sketched here and we also saw that we could calculate the total energy of an oscillating system. It is made up of kinetic energy and the potential energy. So, the total energy is sum of these two energies. So, individually when you calculate the kinetic and potential energies, each one of them each of these components is a function of time. So, they vary with time, but quite miraculously they vary with time in such a way that the total energy is a constant. So, I were to sketch the total energy as a function of time, this is what I will get. Basically it is a constant flat line as a function of time and again we rationalized it by saying that when we started writing down the equation of motion, we did not allow any avenue for the initial energy that is given to the system to be dissipated.

So, when I start off an oscillation say in a pendulum or in some other oscillating object, I give it some energy and the energy does not really dissipate itself. Of course, in real life there would always be dissipation and we will deal with dissipation in one of the future modules. Today we will look at how we can combine more than one oscillations in the same dimensions as well as in perpendicular dimensions. So, this will be the remit of this module and the next one.

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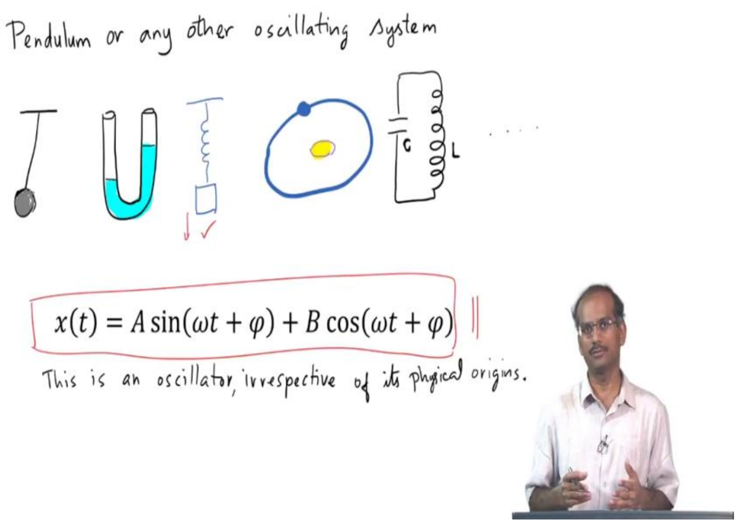

To be able to do that we need to slightly liberate our ideas, in the sense that you can have any sort of an oscillating system. So, I have a pendulum here, I have a U tube with some liquid in it and if I press the liquid in one of its arms is going to start oscillating up and down so, that is an oscillatory system. And I also have; I also have a mass which is hanging from a rigid support through a spring so, if I pull it down a little bit here and I can make it oscillate up and down so, that is an oscillatory system. And I have a two-body problem here, maybe you can think of it as a sun and a planet that is going around the sun for instance, this can be thought of as an oscillating system. Will see how rotations can also be thought of as oscillations in one of the later modules and I also have an L C circuit.

So, you can actually construct a whole lot of different systems, where you will see oscillating solutions or oscillations do happen physically in such systems. Now when you go to the limit of small oscillations in all these cases finally, you will end up with an equation of motion which looks exactly like this. So, in all these cases, the governing equation of motion is simply a simple harmonic oscillator and since we already know the equation of motion and we have already written the solution, which means that every time we come across a new oscillating system we do not have to reinvent the wheel in some sense.

So, we could basically straight away write down the solution, just like the way I have already written it down here. So, this simply says that it does not matter what the system is until this system actually shows oscillatory behavior and in the limit of small oscillations, I can always assume that the solution is of this form. So, now, when I am going to combine oscillations, two oscillations in the same dimension and so on I will not even specify a system. So, this implies that this could have come from any of these systems that I have here, a pendulum or a mass that is hanging from a spring and so on.

It does not matter all this details do not matter until it is an oscillating solution. So, all that would matter for is at least for this module and the next one is that we are dealing with a oscillator, which is described by an equation of the equation of motion that is shown here and its physical origin is more or less immaterial for our purposes.

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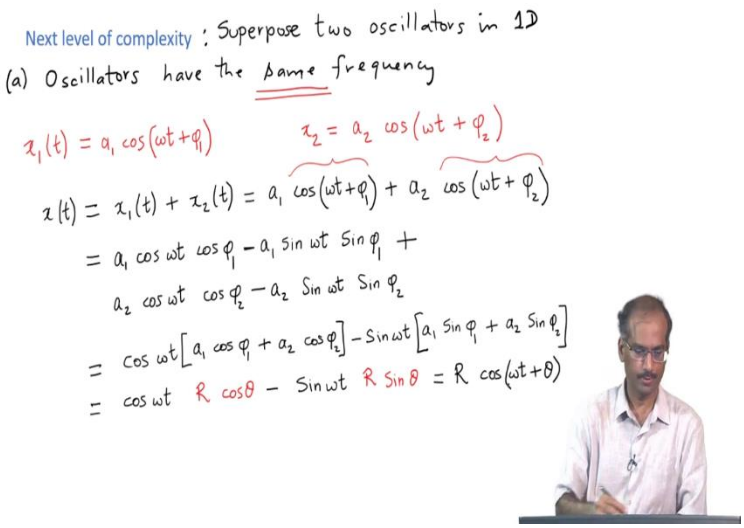

So, with this background, now let us get to the next level of complexity. So, the idea here is to super pose two oscillators in 1 dimensions ok. So, it is like having two pendula in the same dimension or may be in some way two oscillating systems, but essentially both of them are oscillating in 1 dimension and I would like to know what is the net oscillation, if I combine two oscillations in the same dimension and the restriction that I place is that I have two oscillators and they have same angular frequency. So, which means that I can start with the following answer.

So, I have two oscillators and without going through the grind that we went through in the first module and the second module, I will directly write down the solutions. So, that is there right in front of you x_1 represents the displacement of the first oscillator and a_1 represents the amplitude of the first oscillator and ϕ_1 is the phase of the first oscillator and x_2 is the displacement of the second oscillator and a_2 is the amplitude of the second oscillator and ϕ_2 is the phase of the second oscillator.

Now, I want to combine these two oscillations. So, which means that my combined solution $x(t)$ would simply be equal to $x_1(t) + x_2(t)$. And this sort of writing down the solution as super position of two solution is possible, simply because the equation of motion that we wrote down for the harmonic oscillations is what would be called a linear differential equation. So, in a sense all we are doing is to implement a mathematical property of linear differential equations, that if I have two different oscillations and their solutions a linear combination of those solutions would also be a solution.

Now, let me complete this. So,

$$x(t) = x_1 + x_2 = a_1\, cos(\omega t + \phi_1) + a_2\, cos(\omega t + \phi_2)$$

Now all I need to do is to use the $cos(A + B)$ formula and write it differently.

$$x(t) = a_1\, cos\, \omega\, t\, cos\, \phi_1 - a_1\, sin\, \omega\, t\, sin\, \phi_1 +$$

$$a_2\, cos\;\; \omega\, t\, cos\, \phi_2 - a_2\, sin\, \omega\, t\, sin\, \phi_2$$

Now, what I will do is separate out $\cos \omega\, t$ and $sin\ \omega t$. So, let me first take $\cos \omega\, t$ so, if I take $\cos \omega\, t$ out I will have $a_1 \cos \phi_1 + a_2 \cos \phi_2$ and now, I will take $\sin \omega\, \mathrm{t}$ out; in which case I will $a_1 \sin \phi_1 + a_2 \sin \phi_2$ So, I have done nothing more than rewriting the form an expression in a slightly different way.

Now, the next step is crucial, because even when I combine the two oscillations what I get is another oscillation. So, somehow the final result that I get should represent an oscillatory solution and that I can do by rewriting this quantity within the brackets as $cos\, \theta$ and this quantity here I will write it as $sin\, \theta$ In this case, this is again another expression for $\mathrm{co\,s(a + b)}$ So, I could write it as $\cos(\omega\, \mathrm{t} + \ \theta)$ So, I know that when I combine two oscillations, I need to get another oscillation.

So, looking for such a solution, let me do the following; it's $\cos \omega\, t$ here, but you will notice that the quantity is the quantity within the bracket is entirely a constant. a_1 is amplitude, a_2 is amplitude, both are constants ϕ_1 & ϕ_2 are two different phases both are constants. So, the entire quantity within the bracket is a constant. So, I could write it as some other constant or $\cos \theta$ minus again $\sin \omega\, t$ So, the quantity within the bracket here is also a constant. So, I will write it as $R \sin \theta$ now what I have is simply another $\cos(a + b)$ formula, this can be written as $R\cos(\omega \mathrm{t} + \theta)$

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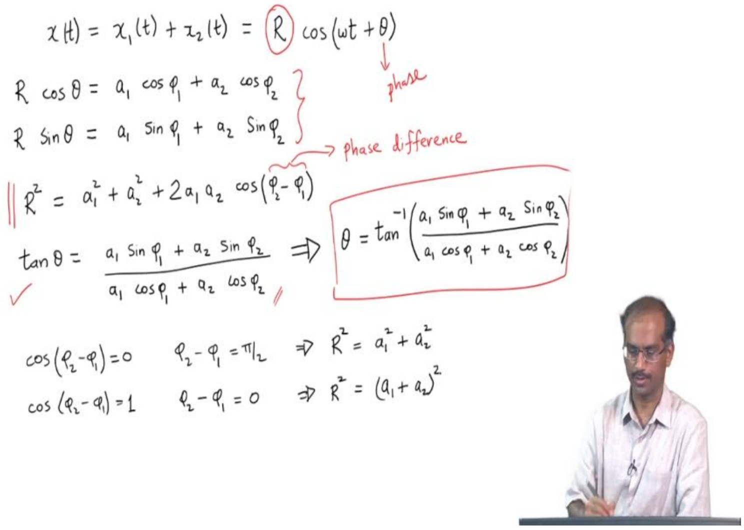

So, now I have the following result. So, $x_1(t)$ is a combination of two displacements of the two oscillators and that is simply equal to $R\,cos(\omega t + \theta)$ So, what we have obtained is what we were expecting to obtain namely, that when I combine two oscillations, I will get another oscillation. So, the solution that I have got is indeed oscillatory. The phase here is theta and the amplitude here is R.

So clearly, I have now got two different constants R and θ which at this stage I do not know what they are. So, they need to be determined from the properties of the original oscillation. In other words these two new numbers that I have introduced R and θ should somehow be related to $a_1 \phi_1$ and $a_2 \phi_2$ and it is not very difficult to obtain these relation if we simply write down what is $R \cos\theta$ and $R \sin\theta$

o, let me first write it down for your benefit. So, I have $R \cos\theta$ and $R \sin\theta$ written in front of me and now from here it is obvious what to do. So, if I want to find R all I need to do is to and add both these equations. If I do that, I am going to get the following result. If I and add these two equations I will get an expression for R which looks like this and if I divide $R \sin\theta$ by$R\,cos\,\theta$ I will get an expression for $\tan\theta$ and from this expression I can write down an expression for θ as tan inverse of these constants that I have here.

So, now, I have determined both R which is the new amplitude and the new phase θ, in terms of the original amplitudes and the phases. So, the original amplitudes and phases were $a_1\phi_1$ and $a_2\phi_2$ and the new amplitude R and the new amplitude phase is simply a function of $a_1\phi_1$ and $a_2\phi_2$. So, when I put together two oscillations in the same dimension, all I get is another oscillatory solution. The only difference is that the amplitude of the new oscillation and the phase of the new oscillation are dependent on the amplitudes and phases of the two sets of oscillations which were combined together. And if you notice in this expression for R^2, you notice that there is a term $\phi_2 - \phi_1$, which is simply the phase difference between the two oscillators.

So, the new amplitude is going to depend on the phase difference between these two oscillators. Let us say that, I choose $\phi_2 - \phi_1$ such that the term $\cos(\phi_2 - \phi_1) = 0$ and this would happen if for instance $\phi_2 - \phi_1 = \pi/2$ for example; that is one value of phase difference for which $\cos(\phi_2 - \phi_1)$ would be 0 in which case R^2 would simply be equal to $a_1^2 + a_2^2$.

On the other hand suppose, let me consider another case for which $\cos(\phi_2 - \phi_1) = 1$. So, this for instance would happen if the phases were the exactly the same. So $\phi_2 - \phi_1 = 0$. In such in that case $\cos\phi_2 - \phi_1 = 1$ and then, R^2 would simply be equal to $a_1^2 + a_2^2 + 2a_1a_2$ that would be equal to $(a_1 + a_2)^2$

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$$\cos(\varphi_2-\varphi_1)=0 \qquad \varphi_2-\varphi_1=\pi/2 \qquad \Rightarrow R^2 = a_1^2+a_2^2$$
$$\cos(\varphi_2-\varphi_1)=1 \qquad \varphi_2-\varphi_1=0 \qquad \Rightarrow R^2 = (a_1+a_2)^2$$
$$\cos(\varphi_2-\varphi_1)=-1 \qquad \varphi_2-\varphi_1=\pi \qquad \Rightarrow R^2 = (a_1-a_2)^2$$

We can look at another limiting case which is when $\cos(\phi_2 - \phi_1) = -1$ which could happen when $\phi_2 - \phi_1$ is equal to for instance π and this would give me the following relation that $R^2$is equal to $(a_1 - a_2)^2$ So, clearly the value of the resultant amplitude R depends on the phase difference and in fact, if you are combining two oscillations, in which $a_1 = a_2$ and there is a phase difference of π between the two oscillating systems, the resulting oscillation would have an amplitude equal to zero ok. That is something spectacular you have two individual oscillators you combine them and they do have a phase difference of π in which case interestingly the resulting oscillation has an amplitude 0.

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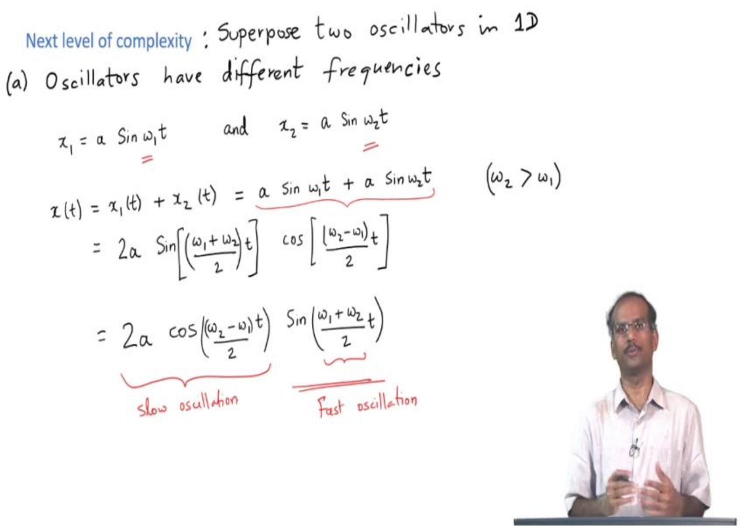

So, now let us go to the next level of sort of complexity. So, earlier we combined two oscillations in 1-dimension, which had same frequencies, but different phases. So, now let us do a problem where we have two oscillators, but they will have in general different frequencies. So, corresponding to this situation I can write down the following solutions, x_1 is equal to… so, I have directly written down the solution so you should immediately imagine that there are two oscillators, for which the solutions have been written down.

The first oscillator is oscillating with an angle of frequency ω_1 and the second oscillator is oscillating with an angle of frequency ω_2 and these two oscillators are oscillating in the same direction or in the same dimension and they have been combined together. Now, what is it that we can expect from these two oscillations. The general principle is the same if I combine two oscillating systems, I am going to get another oscillating system.

So, in fact, it is not restricted to just two oscillating systems; this way I could add any number of oscillating systems and the solution of the sum of all of them would also be an oscillating solution. So, as usual I have written down $x(t)$ is sum of $x_1(t)$ and $x_2(t)$ which in this case would be a $a \sin \omega_1 \mathrm{t} + \mathrm{a} \sin \omega_2 \mathrm{t}$ and I will also without loss of generality assume that $\omega_2$is greater than ω_1. It does not matter, but just makes life somewhat simpler for us.

Now, we will use again another trigonometric identity. So, this time we will use $\sin \mathrm{a} + \sin \mathrm{b}$ formula to resolve this. So, if I do that, I could rewrite this expression here differently. So, all I have done is to simply use the $\sin \mathrm{a} + \sin \mathrm{b}$ formula. So, what you see is that, I have again an oscillatory solution, if you like you could think of the solution to be some $sin\, \omega\, t$ where the amplitude itself is dependent on time.

So, to do that let me just rewrite it slightly differently. Now when you look at it in this form you could think of this as oscillation, whose frequency would be given by this quantity;$(\omega_1 + \omega_2)/2$, that is the frequency being the average of the two frequencies and this is this entire term could be thought of as the amplitude and the amplitude now depends on time. To understand how this basically plays out let us try and plot out each of these terms separately.

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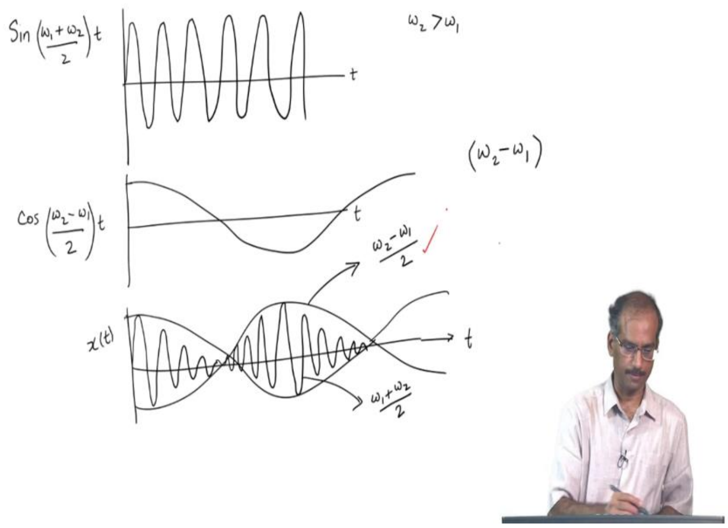

So, we keep in mind that $\omega_2 > \omega_1$. So, the quantity $(\omega_1 + \omega_2)/2$ is fairly large. So, you might actually get something like this and so on. Now let me sketch the cosine term. So, if we keep in mind that $\omega_2 > \omega_1$, you would notice that this number $\omega_2 - \omega_1$ is smaller than $\omega_1 + \omega_2$. So, clearly this is going to have a much smaller frequency oscillation. In other words, as a frequency $\omega_1 + \omega_2$ is larger than $\omega_2 - \omega_1$. So, what we are going to have is something that is going to have smaller frequency like this.

Now our solutions says that these two components needs to be multiplied ,and if I do that and sketch it here, what I would defectively see is that the cosine function would decrease in amplitude and so on. So, and then what you are left with this an envelope that goes like this.

So, here I have plotted $x(t)$ as a function of so, there is this envelope which comes from the cosine term whose frequency is $(\omega_2 - \omega_1)/2$ and then there is the fast oscillation, which comes from which happens with the frequency $(\omega_1 + \omega_2)/2$. And when we look at both of them happening together in the sense that what we are actually experiencing is product of these two terms and you get this a waxing and veining of oscillations periodically and this happens with the frequency which is equal to $\omega_2 - \omega_1$

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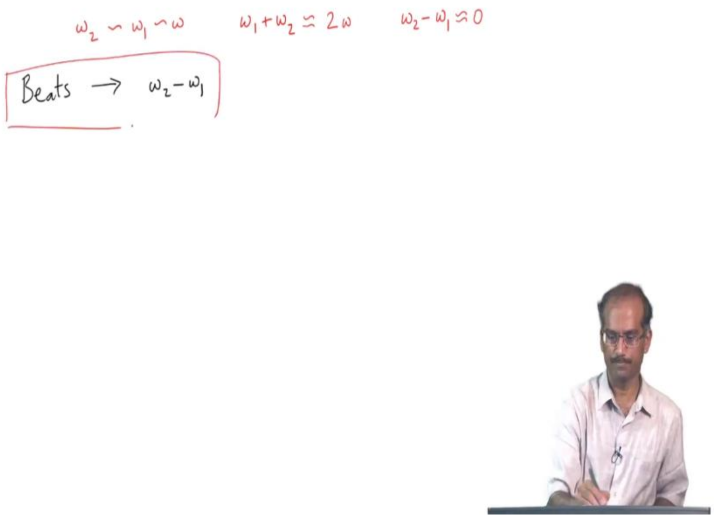

In the limiting case if I assume that ω_2 is approximately equal to ω_1, but not exactly equal in which case $\omega_1 + \omega_2$ is approximately equal to 2 times ω simply because, ω_2 is approximately equal to ω_1 which is approximately ω and $\omega_2 - \omega_1$ this is approximately equal to 0. In such case in such a situation the product of our two terms basically tell us that the \sin term this is going to do a fast oscillation, because $\omega_1 + \omega_2$ is large and the cosine term here is going to do a slow oscillation simply because $\omega_2 - \omega_1$ is very close to 0 very small number.

In such a situation you end up with a phenomenon of beats and beats have a frequency which is equal to $\omega_2 - \omega_1$. So, when you combine two oscillations which have different frequencies in such a situation and especially if you go to a limiting case where the two frequencies are very close together, in such a case you can experience the phenomena of beats.

You will see that displacement essentially increases goes to 0 and sort of oscillates in this fashion that is shown here and typically when you do it do this experiment with, with tuning forks what you actually listen or what you hear is simply the intensity of sound, which are related to squares of these quantities in which case the beats would have frequency equal to $\omega_2 - \omega_1$

Again to summarize this part of what we have been discussing, we looked at how to combine two oscillators in 1 dimension and the two oscillators have different frequencies. And when you assume that let us say that there are no phase difference between these two oscillators, it is very easy to write down the combined solution as usual. The principle is that the net displacement is sum of the two displacements, because they are in the same direction or in the same dimension and when you simplify it what you see is that the net displacement is a combination of two parts; one which is fast oscillation and the other one which displace slow oscillations.

And when you try to sketch it you see that the fast oscillation is being modulated by the slow oscillation, which shows up as the profile in the with the frequency $(\omega_2 - \omega_1)/2$ here and when you take this to the limiting case where ω_1 and ω_2 that is the two frequencies are nearly the same, but not quite exactly the same; in that case you get what are called beats there is strong waxing and veining of oscillations and when you listen to two tuning forks which are held close together and led to oscillate, you do get very strong sound intensity that is increasing decreasing and so on. So, that is the phenomena of beats.

To summarize what we have been saying, we started by saying that you could be dealing with any kind of oscillator in the limit of small displacement. In all such cases the governing equation of motion is the same and the governing solution is in general can be written as a combination of \sin and cosine with some face to it. Then we dealt with the problem of how to combine two oscillators in one dimension and here the restriction was at the two oscillators have the same frequency.

And the main lesson here is that whenever you combine two oscillators not just two oscillators any number of oscillators, the net result is that the net displacement would also be another oscillator, as shown here. And all you need to do is to work out the relation between the parameters of the original problem and the parameters of the combined solution, which is what we did here. The combined solution has an amplitude and phase which is related to the amplitude and phases that were originally specified for the individual oscillators here.

And then we looked at the problem of two oscillators, again in 1 dimension, but with different frequencies and here in this case then we looked at the problem of again two oscillators, but with different frequencies in 1 dimension. So, in this case we showed that there would be strong waxing and veining of the displacement curve and the limit when the two frequencies are nearly the same, it gives rise to the phenomena of beats.

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Let us see how beats can be demonstrated with the simple cell phone that both of us have here. So, we are using android app which will generate for us pretty much frequencies in wide range and he has set his app to 434 Hertz and as you see I have set it to 435 Hertz. So, let us begin by just generating 434 Hertz. So, this is just one wave form, whose frequency is 434 Hertz.

Now, I am going to generate 435 Hertz and bring them close together and hopefully will see or actually will hear the beats happening. So, I have started mine let me bring it closer, you should be able to hear the waxing and veining. So, this is an experiment you can do it for yourself and check the phenomena of beats. In the next module we will look at what happens when we combine two oscillators in perpendicular directions and that would be the subject of the next module.