Waves and Oscillations
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Lecture - 29
Wave Equation: Problems

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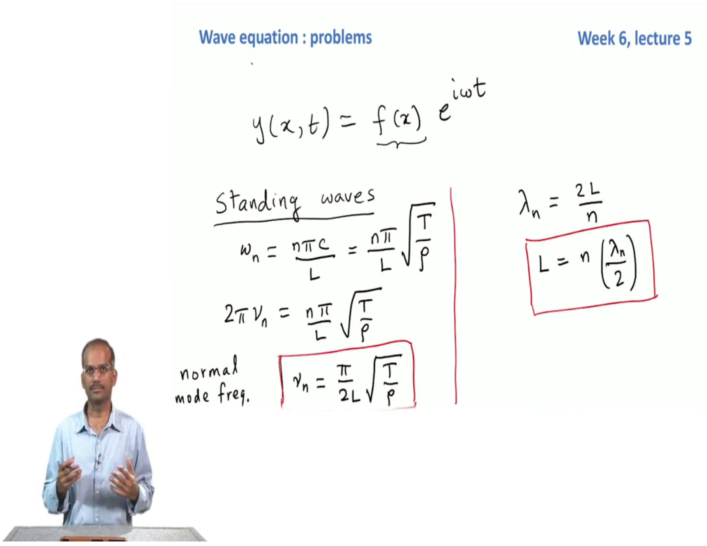

So, welcome to the last lecture of this week. So, all this week, we have been looking at the wave equation, solutions and then we put in boundary conditions, again obtain solutions and so on. So, in terms of phenomena that we saw, we saw progressive waves which did not have any boundary conditions and then we saw standing waves which were associated with boundary conditions. For a one-dimensional wave equation, there were two boundaries and the boundary conditions were such that it did produce for us standing waves.

So, as I said standing waves, usually would look like as though the wave is not moving which is why the name standing waves. So, let us quickly recap some of these results and then we will do some problems. So, the basic sort of machinery for all this is the wave equation and you could imagine that it is like a physically the problem is that of a string and you are setting it to oscillations maybe just by plucking it somewhere and it is going to oscillate and you are assuming that there is tension, uniform tension in the string and so on.

And, it is the tension in the string which provides the restoring force. So, the general solution for the wave equation would look like something like this. So, it would be like y, which is the displacement. So, now, the displacement is both a function of position along the string, as well as time. So, I indicate it by x and t. So, that is some function of position multiplied to $e^{i\omega t}$. So, this $f(x)$ contains all the information about the amplitude and amplitude itself is dependent on the position, so, all that is embedded inside this $f(x)$.

So, it is actually another sine or a cosine function and the amplitude put together. And, $f(x)$ could be chosen depending on boundary condition, depending on problem and so on and so forth. So, when we put in the boundary condition for the standing wave part, we obtained the following solution. So, we saw that the normal mode frequencies index by n, where n indexes the mode is ω_n is $\frac{n\pi c}{L}$, where L is the length of the string; c is the phase velocity of the wave.

So, I can rewrite this differently. So, this $\frac{n\pi}{L}$ and c let me replace by T by root of T by ρ, T is the uniform tension in the string and ρ is the linear density of the string. Now, let me rewrite ω_n in terms of frequency itself, ω_n is the angular frequency. So, that will be $2\pi\nu_n$ is of course, $\frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$. So, finally, I could write and equally important relation would be the expression for the wave length. So, here L is of course, the length of the string, does not change and this amounts to saying that, when you have standing waves, the entire length of the string has to be some integer times half wavelengths.

So, only those modes will be supported for which the integer times half wavelength is exactly equal to the length of the string. So, all other modes will not be supported, which is why any arbitrary solution of the wave equation, is not a solution for the standing waves. So, there are these constraints which come because, you have boundary conditions, the boundary condition need to be respected and the fact that you have conditions like this, on admissible values for wavelengths, admissible values for frequencies.

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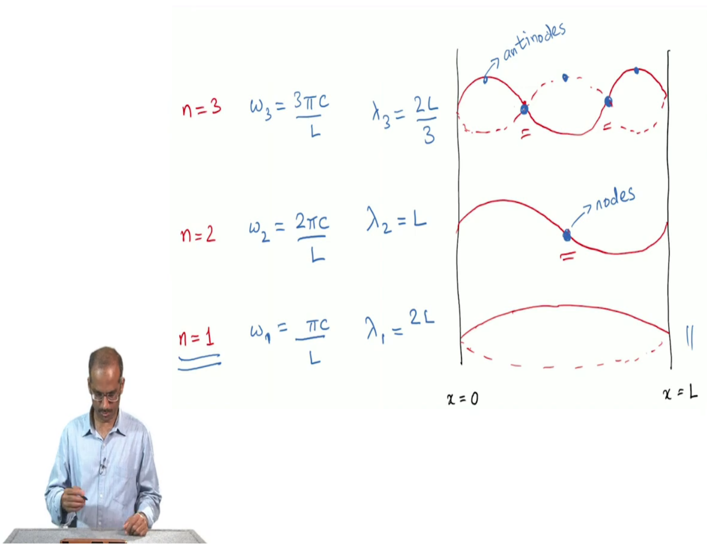

Basically, imply that these conditions are being respected and, indeed if you put in all this, as we saw in the previous lecture, when you draw out the patterns of oscillation of the fundamental mode. So, fundamental mode is the one for which $n = 1$, the 1 that is shown here this case. So, in this case the total length going from $x = 0$ to $x = L$, supports one half the wave length. On the other hand, when you go to $n = 2$, it supports two half wave lengths and as you as you can see based on the drawing that is shown here, for $n = 3$, you have three half wave lengths supported.

So, this pattern actually continues and $n = 1$ corresponds to lowest in energy as well. So, we will see in some of the later lectures, how to compute the energy of each of these modes and in fact as I pointed out in my earlier lecture, there is this very nice progression of the nodes or nodal points. So, at the nodal points the amplitude the displacement is zero. So, like you see here for instance, in the case of the fundamental mode, $n = 1$ there is no node at all, in the case of $n = 2$, there is only one mode, which is right at the centre here and in the case of $n = 3$, there are two such nodal points and of course, as you can see the progression the next one will have three such nodal points and then there will be 4 such nodal points and so on.

So, if I have a string let it oscillate and I press the string at some point like this, I am actually creating a nodal point there. So, a standing wave system will adjust itself such that that point becomes a node and another point to mention is that, these are like the natural frequencies of the system, it is equivalent to the kind of natural frequencies that we defined much earlier on in our course for just harmonic oscillations. So, if you do not provide for any damping mechanism, it is going to oscillate with some natural frequency and the normal mode frequencies that we have obtained are precisely the natural frequencies of the system.

If you allow external forcing to come into play then and if the frequency of the external forcing matches the one of these normal mode frequencies, you should expect to see resonance and you will see that in this case as well. So, with this background, let us do a 2 or 3 problems.

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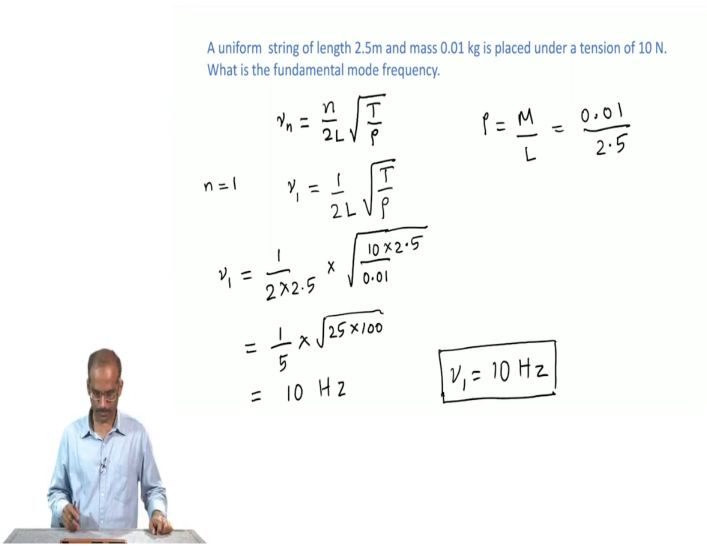

A uniform string of length 2.5 meter and mass is 0.01 kilogram is placed under a tension of 10 Newtons, what is the fundamental mode frequency and there is also a second part to the same problem.

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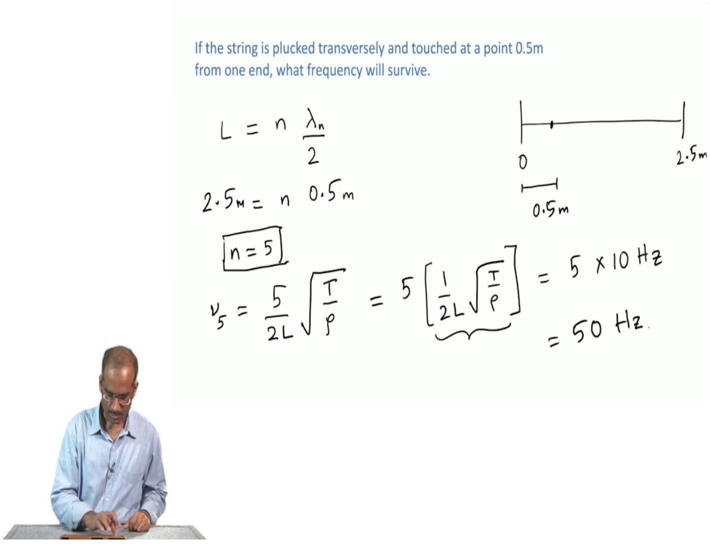

If the string is plucked transversely and touched at a point 0.5 meters from one end, what frequencies will survive ok. The relation that we need is the relation for the frequency which you have written down here, frequency of the nth mode and here the question is about the fundamental mode frequency. So, $n = 1$. So, I can say that ν_1 is $\frac{1}{2L}\sqrt{\frac{T}{\rho}}$, T is tension and it is given already, ρ is not given but, can be calculated from the data given here. So, ρ is mass divided by length, it is linear density and mass is 0.01 kilogram divided by 2.5 meter.

So, now I can go ahead and calculate everything that I need, 1 divided by 2 into length is 2.5 multiplied by tension is 10 Newton divided by ρ which is 0.01 divided by 2.5, it will multiply here. So, of course, once you take the square root out, that will be 5 into 10, 5 and 5 will cancel. So, the fundamental mode frequency is 10 hertz. So, ν_1 is 10 hertz. So, this is the required answer.

Now, let us address the second part of this problem. So, in this case, you have a string and let us say this is 0 and this is length L, which is given as 2.5 meters. Now, so the string is plucked to set it to oscillate and it is pressed at some point, which is 2 which is 0.5 meters from one end. So, now, if you do this the question is what frequency of oscillation would survive? The only frequencies that would be allowed are the ones for which that point is a node. So, here we assume that there is no node between the left end of this string and this 0.5 meter. So, which means that this is the first node the point where we press. So, I need to find out what frequencies will survive.

So, let us first find which mode will survive, for that I need to use this relation which I just pointed out, that length is equal to $\frac{n\lambda_n}{2}$. So, $\frac{\lambda_n}{2}$ for us is 0.5 meters. Therefore, and length of course, I know. So, this is 2.5, this is $\frac{n\lambda_n}{2}$ is 0.5. So, this is 0.5 meters, 2.5 meters. So, n is 2.5 divided by 0.5 that is equal to 5.

So, the fifth mode will survive. So, now, that I know that fifth mode will survive, all I need to do is to plug in the value of n in this relation and just get the frequency, which is what we will do now. So, that relevant frequency is a ν_5. So, this I can write it as $\frac{5}{2L}\sqrt{\frac{T}{\rho}}$ and this quantity here within the square bracket is the normal mode frequency or the fundamental mode frequency, which is given here we just calculated it, it is just 10 hertz. So, I just need to put in that number so, 5×10 hertz.

So, that will be 50 hertz. So, the answer is that fifth mode, whose frequency is 50 hertz will survive, if I press a point which is 0.5 meters away from one end. Now, let us look at the next problem.

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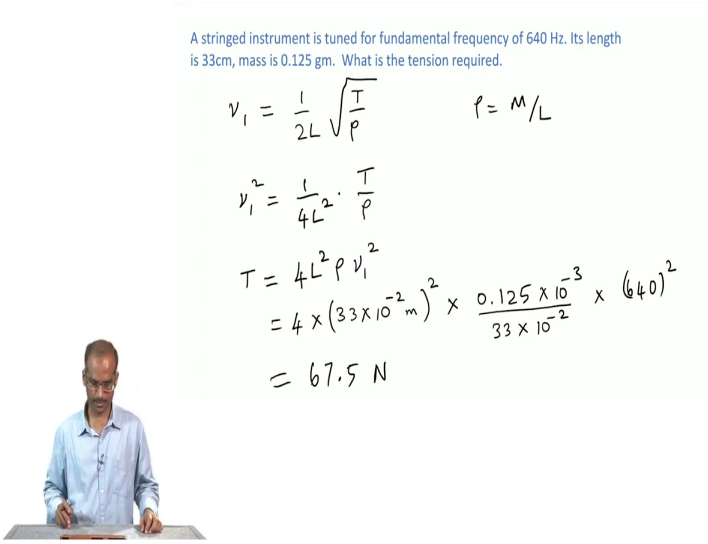

So, in this case it is a more simpler problem, a stringed instrument is tuned for a fundamental frequency of 600 and 40 hertz, it is length is given, it is 30 33 centimetres and mass is 0.125 grams, what is the tension required in this case. So, here again it is a straightforward application of the formula, I just need to put in the formula, put in all the data and get the result.

Again here 640 hertz is given as the fundamental frequency. So, I could write it as ν_1, which is $\frac{1}{2L}\sqrt{\frac{T}{\rho}}$. So, here length is given and mass is also given and rho is mass divided by length, T is what I need to calculate. So, if I square both sides of this equation, I will get and rearrange it, I should be able to get the following. So, $\nu_1^2 = \frac{1}{4L^2}\frac{T}{\rho}$ ok, I need T. So, $T = 4L^2\rho\nu_1^2$.

So, now it is just a question of substituting numbers, 4 multiplied by length is given as 33 centimetres. So, that is 33 into 10 power minus 2 meters and, whole square multiplied to rho which is mass by length, that is 0.125 into 10 power minus 3 kilogram, divided by length which is 33 into 10 power minus 2 meters, multiplied by nu square which is 640 whole square.

So, you put in all the numbers, if you calculate using a calculator or something, you should get the tension to be about 67.5 Newton's. So, the third problem is a variation on one of the things that we did earlier.

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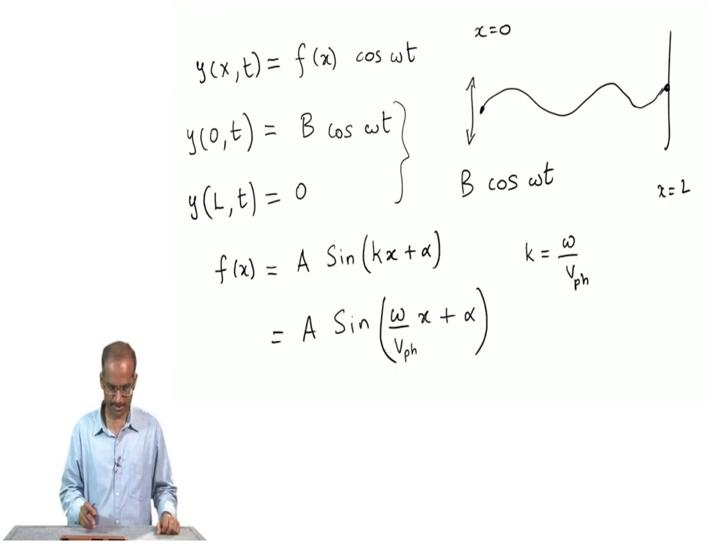

So, in this case again you have a string but, the string is tied at this point. So, it is something like this and here this point is a oscillated up and down and in this case the forcing is $B\cos\omega t$ and this point is fixed, there is a external forcing of $B\cos\omega t$ and at $x = L$ the string is fixed, let us see under what conditions we get, resonance in this case.

And, so again the equation of motion is same as before, which is the wave equation itself. So, solution it is something that we can straight away write which is. So, basically says that if you are going to like provide an external forcing oscillate, let us say a string. Ultimately the string will start oscillating with the same frequency as your forcing, which is why if the external forcing is $B\cos\omega t$, the final solution I have already factored in that and I have put my solution to be $f(x) = B\cos\omega t$. So, it is not going to be a different frequency. So, that is the first thing and then.

Now, the question of boundary condition so, this solution still has to respect the boundary condition. So, the first of the boundary condition is that, why evaluated at zero and t, should be $B\cos\omega t$ ok, this is obvious simply because at $x = 0$, we are providing a external forcing, which is $B\cos\omega t$. So, clearly the string at that point has to undergo exactly the same dynamics.

Now, at the other end it is fixed. So, the second boundary condition for me is y evaluated at L at any time t, should be equal to zero, let me choose $f(x)$ to be some $A\,\sin(kx+\alpha)$ and remember that k is equal to ω by the phase velocity, which I will indicate by v_{ph}. So, that is phase velocity. So, now, let me put this in that case it will be A. So, that is my $f(x)$, $x=L$, y is zero. So, I will have when will this sine function be equal to zero, that will be zero if $\dfrac{\omega L}{v_{ph}}+\alpha$ is equal to some $n\pi$.

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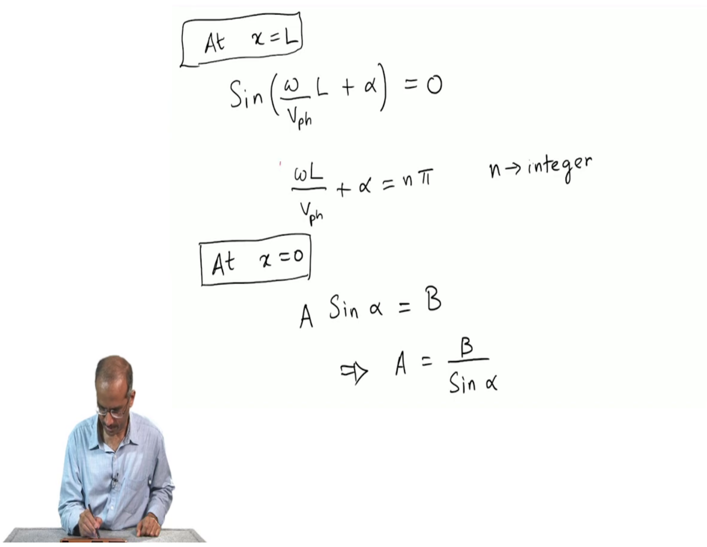

So, this is at $x=L$, if I put in the boundary condition at $x=0$, we will get that and from here, $A=\dfrac{B}{\sin\alpha}$. Now, all I need to do is to substitute for α from this equation.

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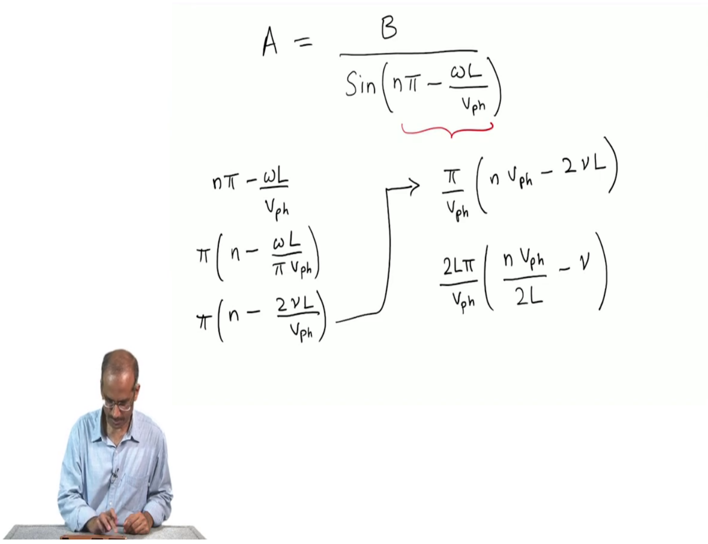

So, here I have the relation. So, clearly if the denominator is zero or it goes to zero or goes to very small number then of course, A is going to be very large, we can make it a little more clear by doing a small manipulation here. So, let me just take this part alone, let me just look at this one alone because, all we need is that this has to go to zero and it is clear that if $n\pi$ is equal to $\frac{\omega L}{v_{ph}}$ it will obviously, go to zero and you are going to see a large amplitude A but, to make it a little more intuitive, let us look at only this part.

So, that is $n\pi - \frac{\omega L}{v_{ph}}$, let me take π outside here. So, that would be $n - \frac{\omega L}{\pi v_{ph}}$ and if you now use the fact that $\omega = 2\pi\nu$, ω is the angular frequency and ν is the frequency in that case this will be π into n minus that will be $2\pi\nu$ so, that is $2\nu L$ divided by v_{ph}. And, from here now let me do one more manipulation. So, let me take v phase outside, that will give me $n v_{ph}$ minus 2ν into L we are nearly done. Now, let us take $2L$ outside. So, that will be $\frac{2L\pi}{v_{ph}}$ which will give me. So, this is what I have. So, which means that now, I can plug this in into this equation. So, I would get the following equation.

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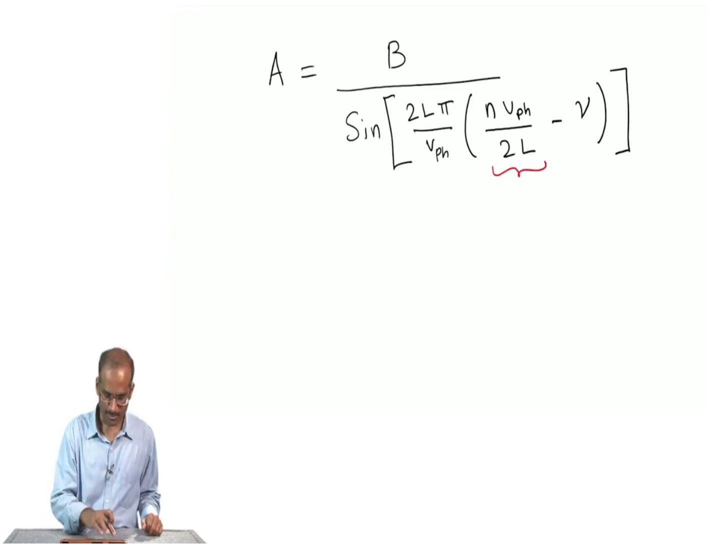

So, now when you look at the same equation in a slightly different form after manipulating it, you will see that this quantity here is simply equal to the normal mode frequency. So, whenever you are driving frequency is equal to the normal mode frequency, you should see A in principle becoming infinite but, of course, we know that physically there are no infinities and there would be other damping effects that would be proven that would be present and prevent a from becoming infinity but, nevertheless in response to a driving, which is given by $B \cos \omega t$, for certain values of the driving frequency, the response is going to be large.

So, this is very similar to the kind of resonances that we have seen for driven damped oscillator. So, in fact much of the phenomena that we saw for driven and damped oscillating system would also work in the case of coupled system, in the subsequent weeks we will try and look at computing the energy of these oscillating systems and we will also learn how to do Fourier decomposition to do to calculate these energies.