Waves and Oscillations
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Lecture - 28
Transverse Waves in Periodic Structures

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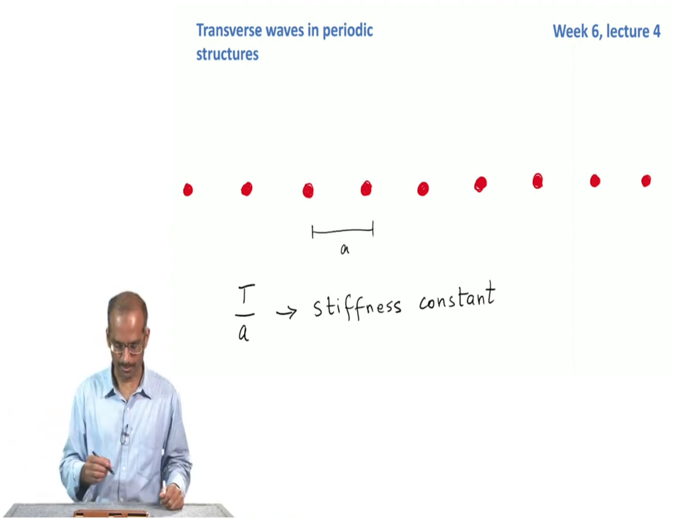

Welcome to the 4th lecture of the 6th week. One we looked at discrete collection of particles which were coupled together and then we took the limit that the distance between the particles goes to zero. So, they come close together and form a continuum, in which case it becomes like a string. So, we looked at the normal mode oscillations of a string.

So, here the title says that it is about transverse waves in periodic structure. So, you can think of each of these red dots maybe an atom and the distance between them is equal. So, one could think of the distance between these as a and each one is periodically placed in a sort of lattice. And we are going to look at what happens to transverse waves in this sort of a system, here there are no strings, it is actually atoms or ions. So, instead of the tension what we have here are the forces between these atoms, which are responsible for providing the restoring force. So, in other words, the stiffness constant actually arises from the forces between these atoms or ions.

So, you could say that a quantity like $\frac{T}{a}$, where now T does not represent tension, but the force between these atoms that would represent something like a stiffness constant for this problem; so which is why, I have not drawn the lines between these red dots that you see here. And this is going to be a finite crystal in the sense that it is not infinite in extent, so it is going to stop somewhere. And here we cannot take the wave equation, because our system is explicitly discrete; it has these discrete set of atoms. We cannot take the continuum limit here. So, we shall work with the discrete version that we had seen earlier. So, let me first begin by writing that equation again.

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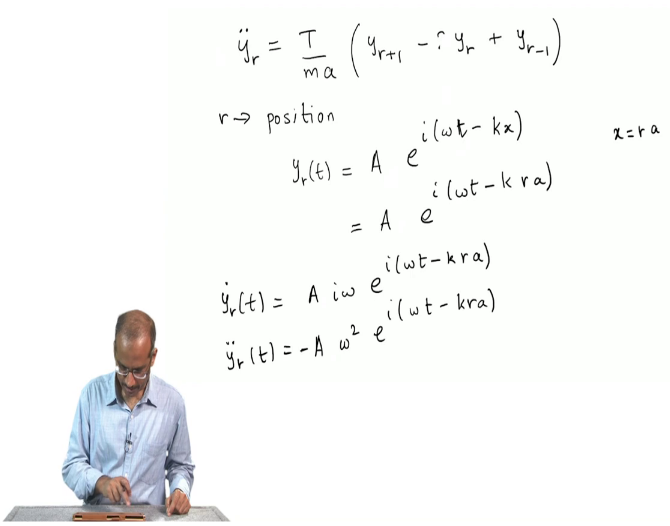

So, m is mass of the atom and y is of course the displacement. It depends both on time and position and the position is not a continuum, it is indexed by this quantity r. So, this tells you the position along the line. So, r would mean the rth atom for example. Let me assume that y displacement which is a function of position and t this, $A_r e^{i(\omega t - kx)}$, but here x is discrete. So, let me put that information as well, so $A_r e^{i(\omega t - kra)}$. So, I have said that x which could be a continuous position is ra; a is of course, the distance between two successive atoms. So, now all we need to do is to compute the second derivative and plug it into our equation of motion here.

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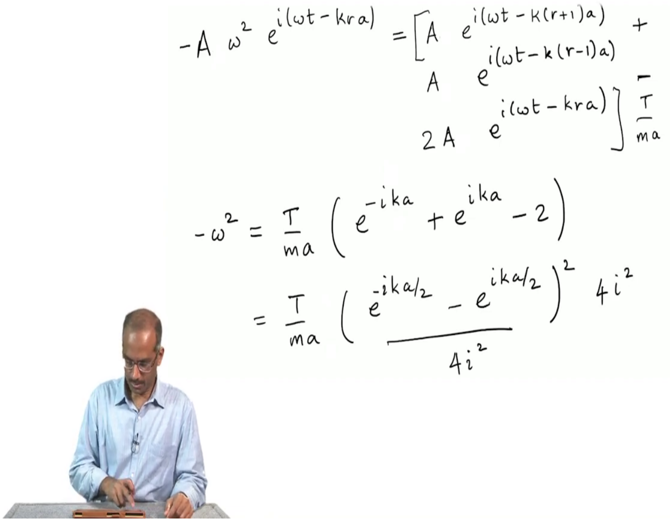

So, I have this $\dot{y}$ and $\ddot{y}$, if I plug in these in my equation of motion, I will get this and of course, there is an overall $\frac{T}{ma}$. So, this would now simplify quite a bit, so I will be able to get after cancelling of terms, I will get $-\omega^2$ is equal to $\frac{T}{ma}$, so, here I will get $e^{-ika} + e^{ika} - 2$. And it is easy to see that as far as the quantity inside this bracket is concerned, this whole thing can be written as. And if you have a structure like $e^{\frac{ika}{2}} - e^{-\frac{ika}{2}}$, clearly you can write it as a sine function provided of course I multiply and divided by some quantity like this.

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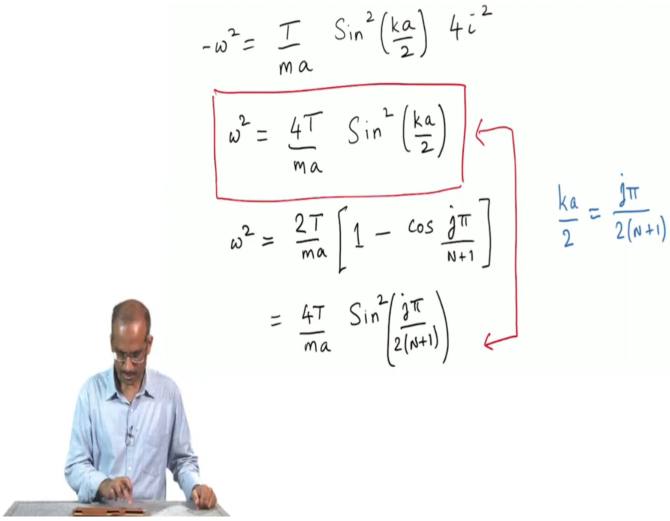

So, now we have everything that we need $-\omega^2$ is equal to $\frac{T}{ma}$. And this is the quantity inside this bracket along with $4i^2$ will be a sine function. So, I would get $\sin^2\frac{ka}{2}$ multiplied by $4i^2$, i^2 is -1 and would cancel with this i, -1 on the left hand side. So, the final normal mode frequencies would turn out to be

$$\frac{4T}{ma}\sin^2\frac{ka}{2}.$$

So, these are the normal mode frequencies that we require for this problem.

So, now if you recollect we already had obtained an equation for the normal mode frequencies for a similar case, for the case of particles which are connected by a spring and the particles are distant a apart. So, in that case let me rewrite the normal mode frequencies for that case, so that would correspond to,

$$\omega^2 = \frac{2T}{ma}\left[1 - \cos\frac{j\pi}{N = 1}\right],$$

where j is the index of the normal mode and N is the number of particles. These two equations this and this would agree with one another provided, we make the identification that $\frac{ka}{2} = \frac{j\pi}{2(N+1)}$.

So, we are just comparing the two sets of equation we obtain for normal mode frequencies. One for the problem that we are currently doing, which is a collection of atoms in a crystal and the other one was set of particles connected by a string. So, now we have a condition that we can use.

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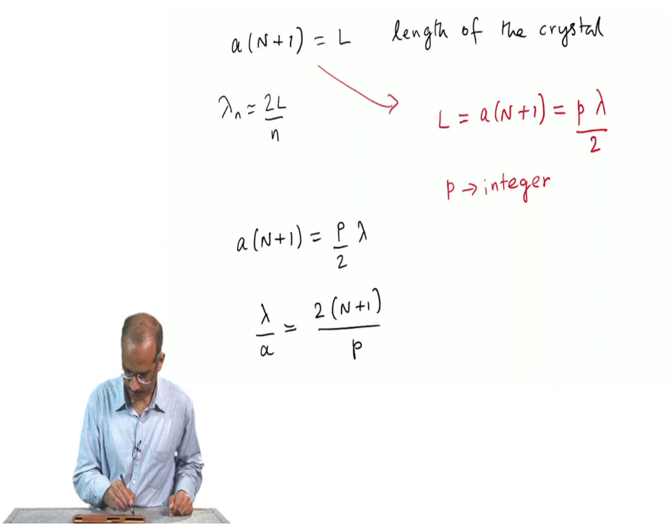

To go further we also know that $a(N+1)$. So, assuming that we have N atoms in the crystal, $a(N+1) = L$ which is the total dimension or length of the crystal, we can directly equate this using this condition. So, remember that this is the condition that we had obtained in the previous module for the allowed wavelengths of standing waves. Here the current problem that we are doing is equivalent to a system of N atoms, which are arranged in a linear lattice.

And in this case instead of two walls at two ends, you have the system terminating and of course, there is nothing beyond that, so you could expect that standing waves would be set up in this case. So, we can directly use the results that we got for the standing waves. So, here this result tells me that λ_n, which is the wavelength of the nth mode is $\frac{2L}{n}$.

So, I am going to use the use this result that $\lambda_n = \frac{2L}{n}$. So, this would imply that this quantity which is L can be related to the wavelength of the normal modes as follows. So, L which is equal to a into $N+1$ would be equal to some p times λ divided by two. So, all it tells me is that the total length of the crystal that I have should be an integer times half wavelength. So, this p is integer, but then this is the result that we had already seen for the case of standing waves in the earlier module.

So, the result itself is nothing new, but we are simply using the same result that total length has to support precisely integer number of half wave lengths. From this condition alone we can find out what is the largest and the smallest wavelength that will be supported. For example, let me rewrite this relation. So, I have

$$a(N+1) = \frac{p}{2}\lambda.$$

Let me write an expression for $\frac{\lambda}{a}$. So, $\frac{\lambda}{a}$ would be $\frac{2(N+1)}{p}$. So, p here of course, corresponds to the it is an integer, it corresponds to the index of the normal mode.

So, $N = 1$ would correspond to the fundamental mode and that would be the lowest in energy and that also corresponds to the largest wave length. So, here the wavelength would be large if p is minimum and the minimum p that you can have is one.

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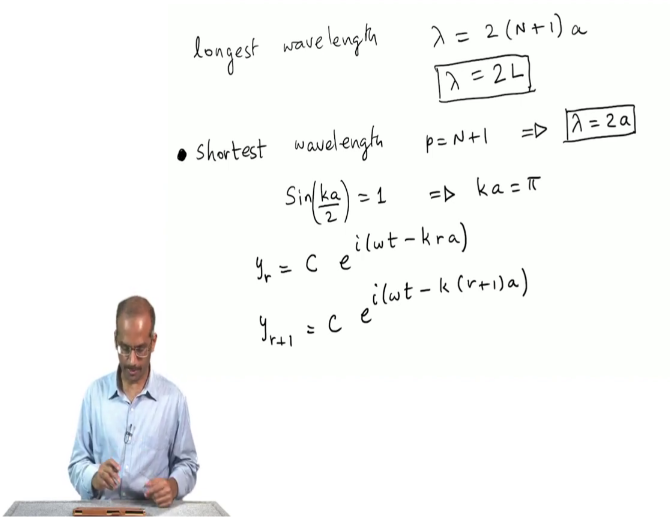

Which would mean that largest or longest wavelength that you can possibly set up in this crystal would correspond to $\lambda = 2(N+1)a$. And $(N+1)a$ is of course the length of the crystal, so that is $2L$. So, clearly this is the longest wavelength that you can set up. Similarly, we can argue and obtain what is the shortest possible wavelength that can be set up in this case, so that would happen if for the largest possible p value.

And in this case, the largest possible p value would be simply $N+1$, so this is the shortest wavelength possible and this is the longest wavelength possible. Let us now focus on this case. It is easy to verify that $\sin\frac{ka}{2} = 1$ and this is because ka is equal to π. Now, for this particular case of shortest wavelength where λ is equal to $2a$, I would like to know what is the ratio of $\frac{y_r}{y_{r+1}}$.

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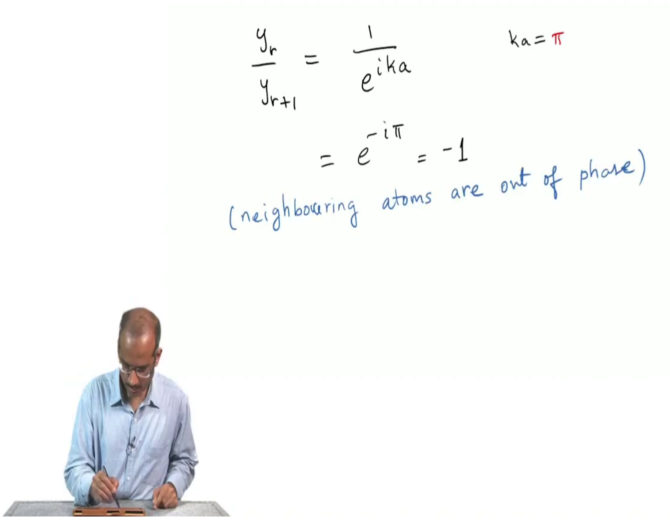

So, $\frac{y_r}{y_{r+1}}$ from the equation that we wrote down previously, $e^{i\omega t}$ would cancel, so would e power $-kra$. So, the ratio of $\frac{y_r}{y_{r+1}}$ would be $\frac{1}{e^{ika}}$. And we know that ka for this case of shortest wavelength, ka is equal to π we just saw that. So, if I plug in this value here, I would get $e^{i\pi}$ which is equal to -1 .

So, it tells me the displacements of the neighbouring atoms are opposite in sign for the largest mode. Again if you remember this is the kind of result that we had obtained for the case of particles connected together by a string ok, it is exactly the same result that the largest mode the neighbouring particles are off by a phase of π. Now, let us go back to our equation for the normal mode a frequency. So, let me write it again.

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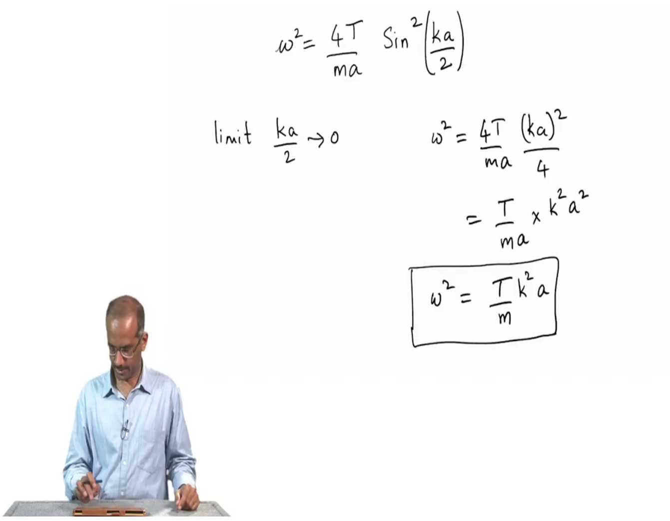

Now, let us consider the limit when $\frac{ka}{2}$ is small or when k is small really, a is a constant, so the only variable there is sort of k and let us take the limit $\frac{ka}{2} \to 0$; in that case in this limit the normal mode frequencies would be $\frac{4T}{ma}$. So, $\sin\theta$ can be replaced as θ in the limit of θ being very small, so this would give me $\frac{(ka)^2}{4}$.

So, $\omega^2 = \frac{T}{ma} k^2 a^2$; so the final result will be $\frac{Tk^2 a}{m}$. So, this would be the frequency in the limit, when $\frac{ka}{2}$ is small enough. Now, that I have this expression for ω^2 in the limit of $\frac{ka}{2}$ being small. I can now obtain the expression for phase velocity if we remember that c is equal to $\frac{\omega}{k}$, where c is the phase velocity.

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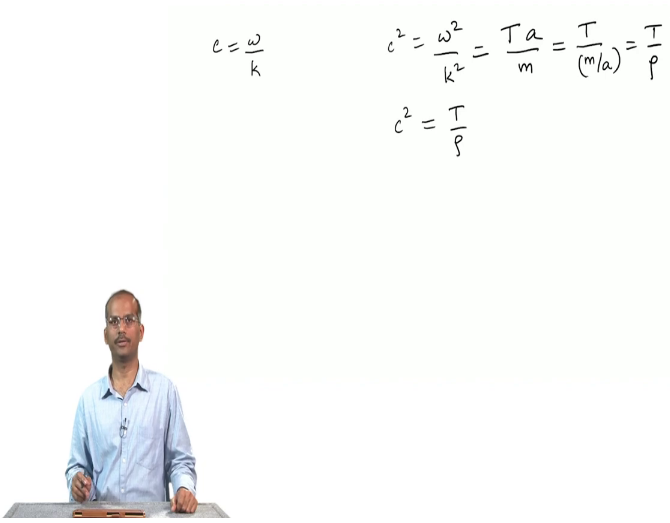

So, if I go back to my expression here, I can write an expression for $\frac{\omega^2}{k^2}$. So, c^2 would be $\frac{\omega^2}{k^2}$ and that would be equal to $\frac{Ta}{m}$, which would be equal to $\frac{T}{\rho}$. So, the phase velocity in the limit when $\frac{ka}{2}$ is small would be $\sqrt{\frac{T}{\rho}}$. Now, let us go to the other case when $\frac{ka}{2}$ is not really small and in this case we will have to work with the full equation.

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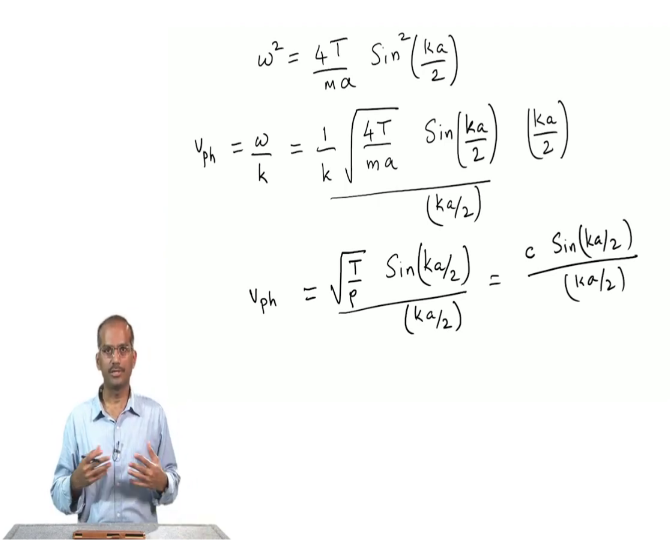

So, I will have my usual normal mode frequency which is $\frac{4T}{ma}\sin^2\frac{ka}{2}$. So, in general the phase velocity c is $\frac{\omega}{k}$. So, in anticipation of what is to come, let me divide this by $\frac{ka}{2}$ and also multiply by $\frac{ka}{2}$. So, now if you simplify this expression, I am going to get the following result. So, here I will have $\sqrt{\frac{T}{\rho}}$ multiplied to $\sin\frac{ka}{2}$ divided by $\frac{ka}{2}$. So, I could write it as c, keeping in mind that c is the phase velocity for small k, so that would be $\frac{c\sin\frac{ka}{2}}{\frac{ka}{2}}$.

So, phase velocity in the general case where k is large depends on k, which means k depends on λ, so phase velocity actually depends on λ. So, which means that different wavelengths are going to be propagated at different speeds, so you will see a phenomenon called dispersion happening.

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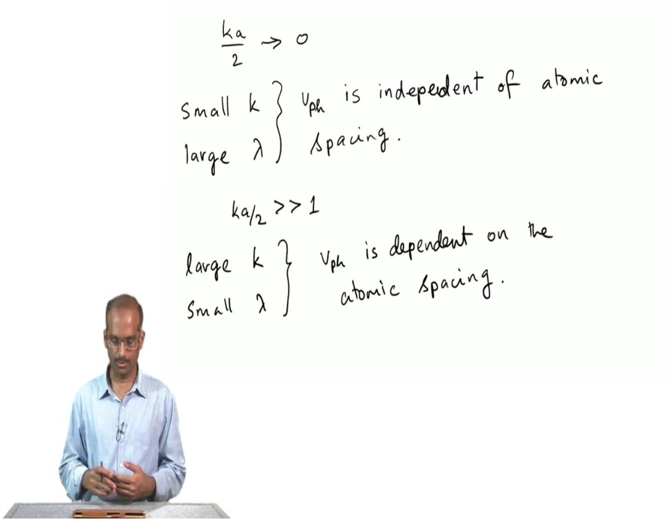

So, let us summarize this part. So, in this limit atomic spacing a does not figure in the phase velocity. So, in this limit it does not matter what your atomic spacing is, phase velocity is independent of that quantity. On the other hand, so in the limit of large k, the spacing between the atoms matters. The standing wave setup in such a crystal, their wavelengths do determine the phase velocity of the waves that are supported. So, with this lesson, let me close this module. And in the next one, we will do few more problems.