Waves and Oscillations
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Lecture – 27
Standing Waves

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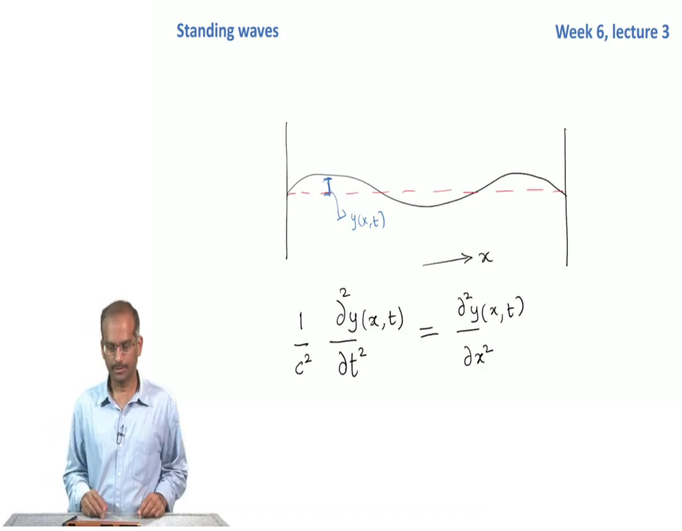

So, welcome to the 3rd lecture of 6th week. So, till now we say that you could start with a coupled system like for instance; a collection of particles which we were tied together by strings, and the particles were positioned at equal distances from one another. And by taking the limit that these particles come closer and closer or we said that the distance between them *A* tends to zero and in that limit we derive an operational equation which is call the wave equation. And we also rederive the same equation by considering a small segment of an oscillating string.

So, there is some string which is oscillating for whatever reason. And you consider a small segment of the string, look at the kind of forces acting on that small segment of the string. And from that we are able to derive an equation of motion, which is called the wave equation. And important input there is that we are still considering small amplitude oscillations. So, we will continue to look at small amplitude oscillations.

Then, we went ahead and looked at the solutions of these wave equations. So, we saw that physically it corresponds to a wave that is travelling in either say positive x direction or negative x direction. So, these are the two possible kinds of solutions that wave equation would ultimately lead to in one dimension. So, these are called progressive waves, because they simply keep propagating in one direction, ok.

On the other hand, there is another class that you can sort of manufacture by using this progressive waves. Suppose, let us see you put in a boundary at the end so that the wave that you produced cannot escape off to infinity, but will have to be reflected off from some let us say rigid wall. You can easily do this by let us say tying string between two walls and create a small disturbance somewhere in between; the disturbance would travel let say in one direction or even both the direction it will go and hit the wall.

Clearly, a small disturbance like this cannot penetrate the heavy wall so it would be reflected and soon you would see what would typically be called standing waves. They are called standing simply because on the phase of it visually when you look at it would appear as though the waves are not really moving. So, physically what you need to create standing waves or these two foundries at two ends in one dimension, so that the reflected component and the incoming component of the wave can together finally lead to standing wave pattern.

In today's lecture we will look at the Standing Wave Pattern or what it means to say that a wave is standing.

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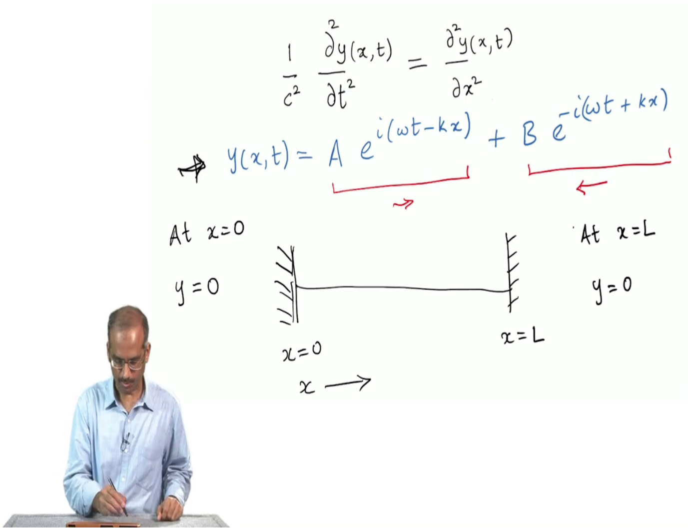

So, the solution that I have written actually consist of two parts. There is this part which corresponds to a plane wave that is travelling in the positive x direction, let us say in this direction positive x direction. And the second part is this, where it is a plane wave again but travelling in the negative x direction; in this direction.

So, you do not know in which direction your wave might be travelling if you make an arbitrary disturbance. So, in general you assume that your waveform is something that travels in both the directions, A and B are two amplitudes. They would continue to remain unspecified. So, this comes out as the solution of your wave equation and view superposed two possible solutions.

Now, the next point is to put in the boundary conditions. We said that to create a standing wave you need to have may be two walls between which you might possibly tie your string and create a disturbance. In other words, one way of expressing this idea is to say that; of course string is your medium in which the wave is going to travel and at this point let us call this x axis at $x = 0$ and $x = L$ there is a change of medium. So what was string; is now suddenly becoming a hard wall. So, that is a case at $x = L$ and $x = 0$.

So, which means if you tied your string at these two positions then the usual boundary conditions would apply, which would be a statement that at $x = 0$ your y which is the displacement would be zero. And similarly at $x = L$ displacement is zero.

So, we have a general solution here which is given by this equation, and now we want to obtain a solution which respects these two boundary conditions. So if you get that we would solve the problem. So, the next step obviously, is to plug-in these boundary conditions into the general solution that we have.

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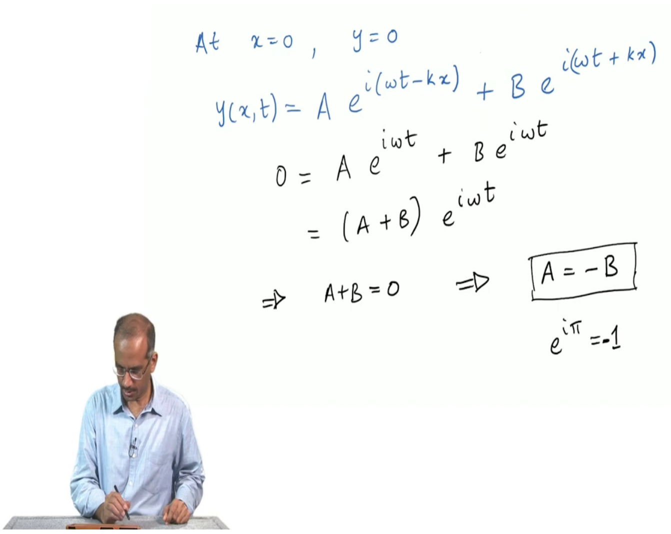

So, let us start with the first boundary condition which is that; at $x = 0$, $y = 0$. So now, I am going to apply this boundary condition.

So, if I say that at $x = 0$, $y = 0$ I would get the following. So, zero is equal to $Ae^{\iota\omega t} + Be^{\iota\omega t}$; which can be written as $(A + B)e^{\iota\omega t}$. And for this to be equal to zero it is $A + B$ which has to go to zero, because $e^{\iota\omega t}$ is not identically equal to zero for all arbitrary values of ω and t which means that this equation implies that $A + B$ is equal to zero which implies that $A = -B$.

So the two amplitudes, the amplitude of the left going wave and the amplitude of the right going wave are equal in magnitude, but opposite in signs. Should not be too surprising, because all it tells us is that if you have a wave which is going towards the right and there is a hard boundary at the right hand; which means that you have something like a wall there and it is going to get reflected there is a phase change of π upon deflection. So, this essentially is reflection of that fact which means that now we can incorporate this condition into our solution.

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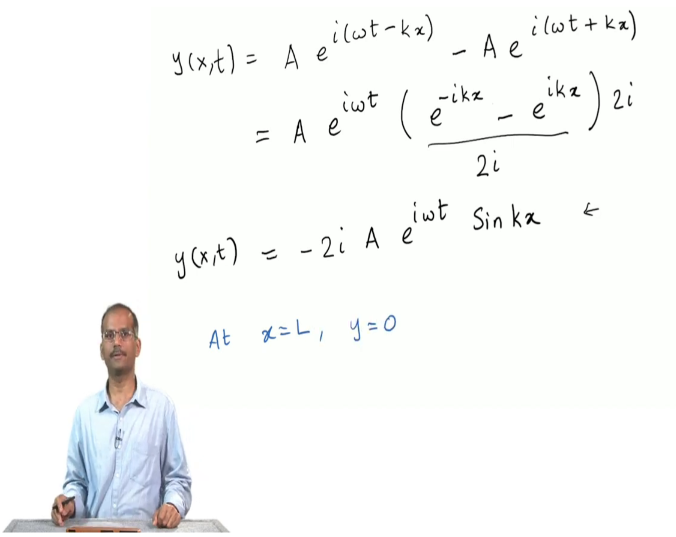

This can be rewritten; of course I should put A here. This can be simplified in the following way, $Ae^{i\omega t}(e^{-ikx} - e^{ikx})$. Now the next the step is somewhat obvious you can divide it by $2i$ and multiply by $2i$. And if you do that this quantity can be written as $-2iAe^{i\omega t}\sin kx$.

So, this is my solution that respects the boundary condition that at $x = 0$, $y = 0$. And that clearly comes out here, because if I put $x = 0$ in this equation y is indeed equal to zero, because $\sin kx$ will be equal to zero in that case, ok.

Now we had one more boundary condition as well, namely that at $x = L$, $y = 0$ as well.

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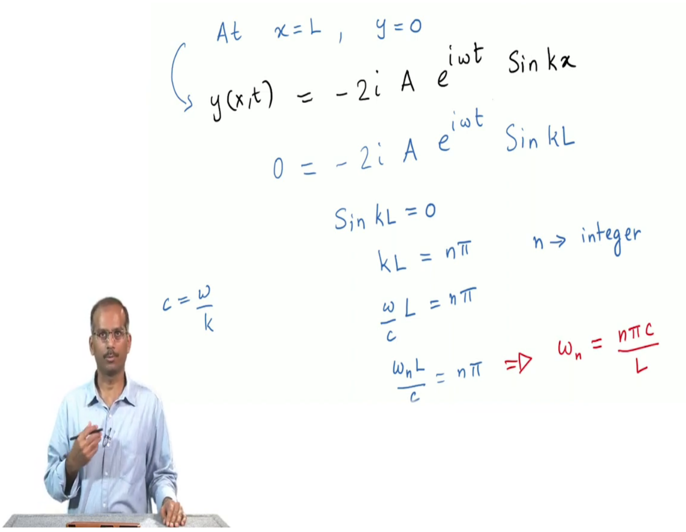

Now of course, amplitude A is not equal to zero in general and $e^{i\omega t}$ is not equal to zero in general and $2i$ is a constant. So, the only way in which this could generate zero is if $\sin kL = 0$. And if $\sin kL = 0$ this is going to give me the following condition, because this would imply that kL should be equal to $n\pi$.

And since ω now depends on n, remember that ω is the angular frequency more correctly the normal mode frequency for our string. So, it depends on this integer n to indicate that, let me say that ω is put a subscript ω_n and $\omega_n L/c$ would be equal to $n\pi$. And this gives me an expression for ω_n which will be $\frac{n\pi c}{L}$.

So, these are the normal mode frequencies for this problem. So, one thing is very clear that arbitrary values of normal mode frequencies are not possible. So, you have some basic quantity that is given by $\pi c/L$ and integer multiples of that are the possible normal mode frequency for this standing waves.

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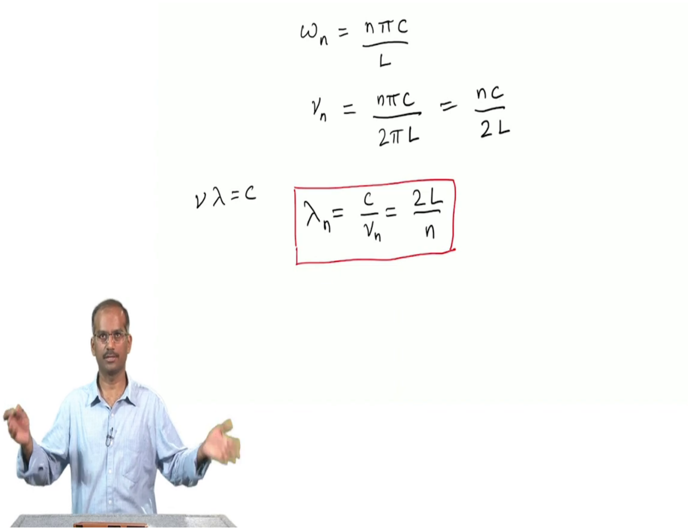

Since I have expression for ω_n which is $n\pi c/L$, where c is the phase velocity of the wave I can also write it in terms of the frequency; not the angular frequency just the frequency. So, that would be ν_n which is the frequency would be equal $\frac{n\pi c}{2\pi L}$ that would be $\frac{nC}{2L}$.

And using this I can also write an expression for wavelength, because we know that ν which is frequency multiplied by the wavelength is equal to c; the velocity of the wave phase velocity, wavelength we will simply be equal to c/ν. So, I can extract that parameter here. So,c/ν_n would be equal to $\frac{2L}{n}$ and that is equal to λ_n. So, this quantity is again very important and useful for our purpose.

So, we know the angular frequency, we know the frequency and we also know what the wavelength is. It is related to just the length of the total length of the string tied between let us say two rigid walls. Let us now visualize the pattern of oscillations given that wave assembled all these results together.

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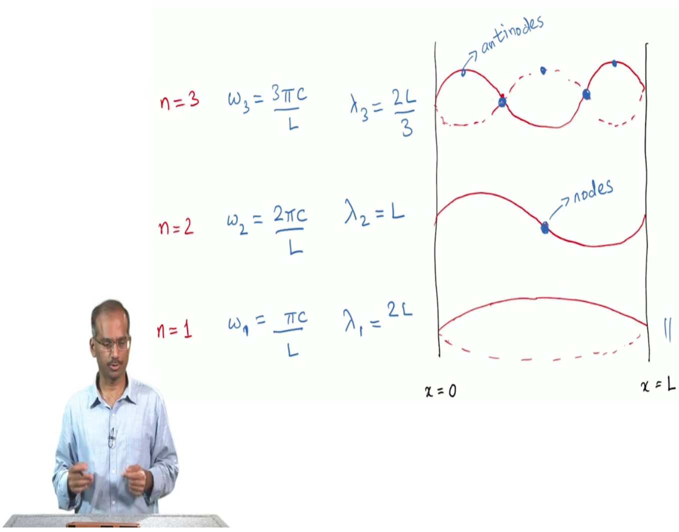

To do that let me draw for you this picture of two walls, and this is $x = 0$ and this is $x = L$. ω_n is $n\pi c/L$ and $\lambda_n = 2L/n$. And since n is equal to one I can erase this n here, so λ_n is equal to $2L$. And $2L$ is twice the length of our total string length that we are considering.

So, which means that between zero and L it should correspond to half the wavelength. In other words, length L is simply equal to half the wavelength when you are looking at the fundamental mode. So, then that is a clue to draw the picture, that would correspond to an oscillation of this type. So, this is one possible pattern for the fundamental mode or $n = 1$ mode. In principle you could have another possibility which is a mirror image of this.

Now let us go to the next mode: first excited mode or next highest frequency. So, in this case I should write it as ω_2 maybe I should have written this as ω_1, let me do that. This is ω_1 and this is of course, λ_1. Now ω_2 will be equal to $2\pi c/L$ and λ_2 will be equal to L itself. So, the entire length of the string that lies between the two walls is equal to one wavelength in this mode. So, I should be able to visualize it as let say this is the midpoint.

Similarly I can do for $n = 3$ as well. Here ω_3 will be equal to $3\pi c/L$ and λ_3 will be equal to $2L/3$. So, if λ_3 is equal to $2L/3$ its equivalent to saying that the entire length between the two walls would be occupied by three by two times the wavelength or one and half times the wavelength. So, that gives us the clue as to how to plot this.

So, let us divide into three parts roughly. So, you see that there is no zero crossing in this, on the other hand you see that there is one zero crossing here which is this point. And here there are two zero crossings in the case of n equal to 3. So, these points where there are zero crossings they are call the nodes node or nodes and these points where the amplitude is maximum right, this ones this ones. So, these would be call the antinodes ok.

So, you can see that there is a progression there is a very clear relationship between n and the number of nodes in your in a particular mode. And you will also notice that for whatever I have drawn n equal to one, two and three cases in all the cases at $x = 0$ and at x = L, y = 0; which is the boundary condition that we demanded and the solutions that we are getting precisely respect those boundary conditions.

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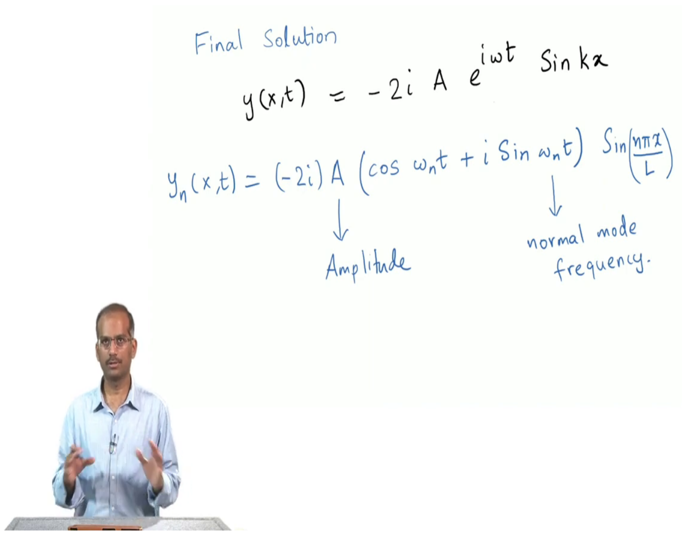

Now, we can write the final solution. So, I have written down what we had obtained earlier that $y(x,t)$ is equal to $-2iAe^{i\omega t}\sin kx$. Now let us write this in a more explicit form.

So,. So now, my solution would tell me that I am writing down the solution for the n-th normal mode; so $y_n(x,t)$. So, n here is an integer that would index the normal mood. So, in this A is the amplitude which can only be determined from the initial condition, ω_n is then normal mode frequency. This provides the complete solution for the standing wave problem.

So, if you are careful about it you would notice that the amplitude has a i, and seems to convey that there is some imaginary part to it, but that can be absorbed as a phase in the $e^{i\omega t}$ term. So, it is not really a problem. So, all these would be settled if you actually look a specific case of a standing wave where you would put in some initial conditions.

And, I would like to leave you with this picture of progression of the patterns that standing waves create as you change the mode number.