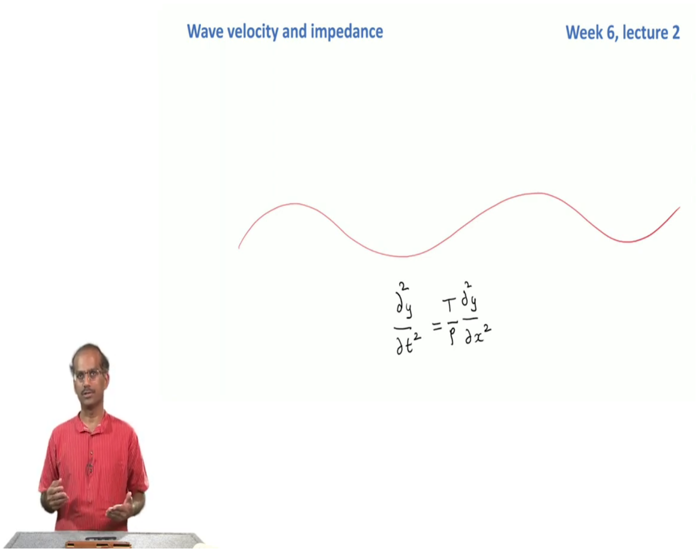
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture – 26
Wave Velocity and Impedence

(Refer Slide Time: 00:15)



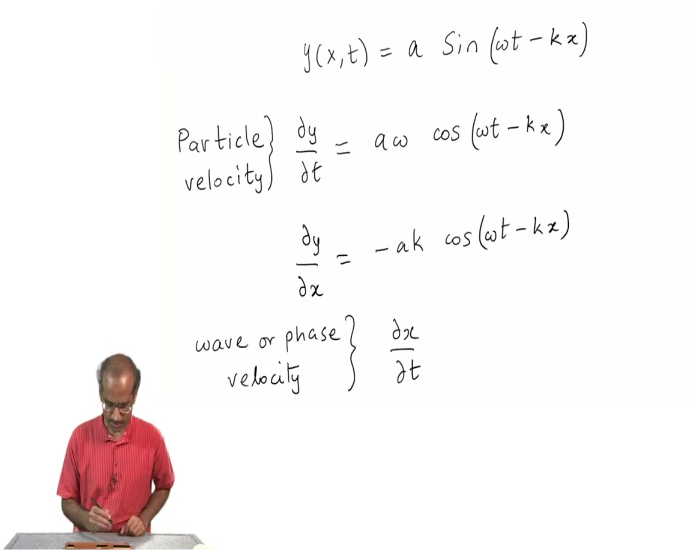
So, welcome to the second lecture of sixth week. In the last module, we derived the wave equation by looking at a small segment of a string. So, it is called wave equation because it describes how a wave propagates in a string and we also saw that it also tells us a way to obtain velocity of the wave that is generated in a string.

So, in this particular case we were actually looking at what would be called transverse wave, simply because each small segment of the string is simply executing simple harmonic oscillation; up and down oscillation whereas, the disturbance as a whole travels in a direction that is perpendicular to this oscillation. So, it is called the transverse wave. So, in today's lecture, let us look at this idea of wave velocity a little more closely.

Simply because there are 2 actually 3 possible velocities that you can identify, at least for the present purposes, let us look at 2 different possible velocities that we can identify. So, one as I said if I look at a specific small segment of a string, it is simply executing up and down motion, simple harmonic motion. So, there is some velocity that is associated with this up and down motion. So, that is one possible velocity and we also were arguing that a disturbance that we create at one place propagates through the string.

So, in this string is a medium or you could imagine that string itself is the medium through which the disturbance propagates and there is a velocity associated with the speed with which the disturbance propagates. So, that is another velocity and there is no reason that these 2 velocities should be the same. To begin with let us write down one possible solution of the wave equation.

(Refer Slide Time: 02:49)

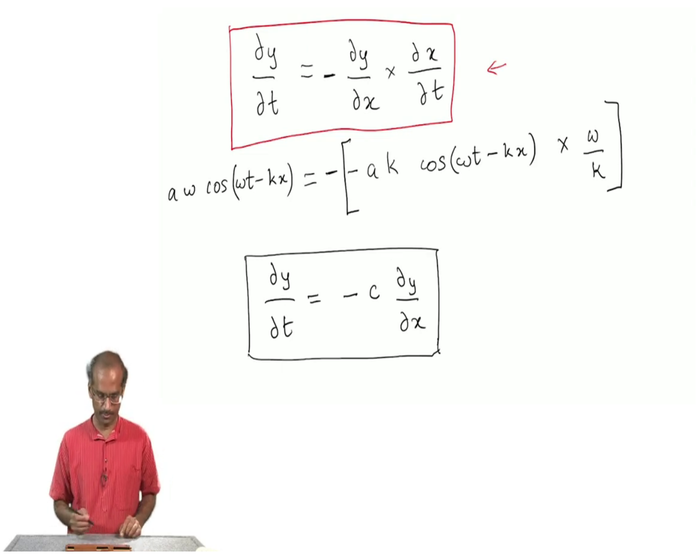


So, this solution that I have chosen corresponds to a wave that is travelling in the positive x direction.

So, one possible velocity that I can define, which would be called the particle velocity is $\frac{\partial y}{\partial t}$, if I differentiate this with respect to time I would get, $a\omega \cos(\omega t - kx)$. This as you can see is defined as the displacement or rate of change of displacement as a function of time. So, this is essentially telling me the velocity of a small segment of string, that simply executing up and down simple harmonic oscillation. And, let me know in anticipation of what is going to come also define $\frac{\partial y}{\partial x}$, which is the gradient, this context.

So, this is telling me how the displacement is changing as a function of position along the string. The other velocity is of course the wave velocity or phase velocity that would of course be defined as $\frac{\partial x}{\partial t}$. So, that is how fast the disturbances travelling. Now, we can relate these 3 quantities.

(Refer Slide Time: 04:39)



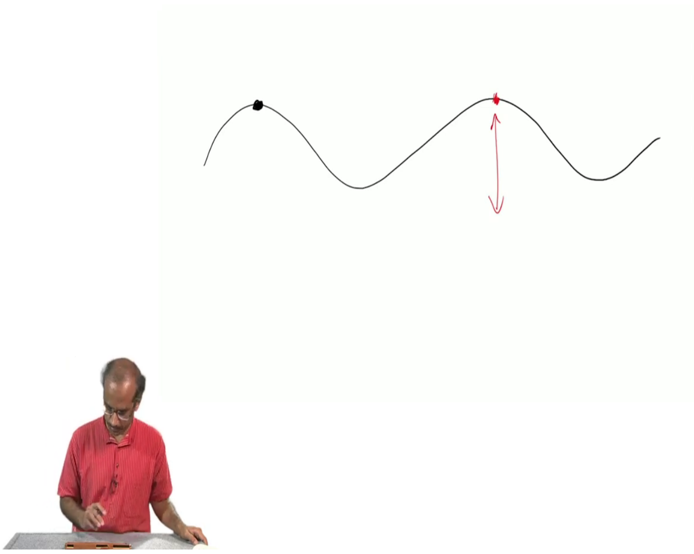
So, I can write $\frac{\partial y}{\partial t}$ as $\frac{\partial y}{\partial x}\frac{\partial x}{\partial t}$ and I can now substitute the known values.

So, $\frac{\partial y}{\partial t}$ is of course my particle velocity and this phase velocity which we shall call as c, we had used the notation c in the last lecture that would be $\frac{\omega}{k}$. So, now, let me put all those together. So, $\frac{\partial y}{\partial x}$ be equal to $-k\cos(\omega t - kx)$ multiplied to $\frac{\partial x}{\partial t}$ which is $\frac{\omega}{k}$ and $\frac{\partial y}{\partial t}$ we had already calculated that earlier on that is $a\omega\cos(\omega t - kx)$.

So, you will notice that there is a problem of sign here, there is an overall an additional minus sign that has crept into this equation. So to balance that, we need to put in a minus sign here. So, then that would take care of the additional minus sign. Now, you will see that both sides are equal. So, minus and minus would give me plus k and k would cancel so, it would be $a\omega\cos(\omega t - kx)$.

So, this for us is a very useful relation. So, just to repeat what I have been saying. So, $\frac{\partial y}{\partial t}$ is simply the particle velocity, it is related to the phase velocity or the wave velocity and the constant and the term that relates these two is the gradient of y with respect to x and that is given by this equation. And, phase velocity is the speed with which the phase of a wave travels for instance; let me draw for you a particular wave form something like this.

(Refer Slide Time: 07:47)



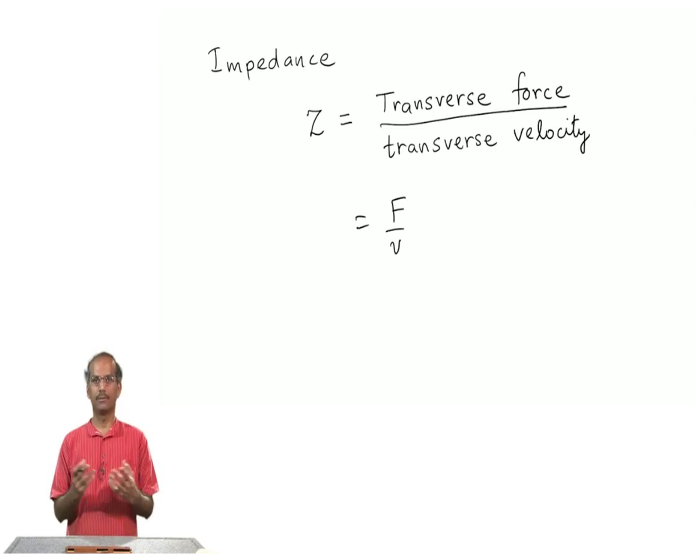
So, let us say that this corresponds to some phase difference with respect to some reference point; the phase velocity basically tells me how fast this particular point moves. Since, every point is moving at the same speed in this case, it is also the speed with which the entire wave is moving. On the other hand, your particle velocity corresponds to this small segment, this small segment, which is making up and down oscillations.

So, the speed with which this up and down oscillation is executed is given by the particle velocity. So, whenever you want the wave to carry say momentum or energy then, physically the important velocity which is relevant in all such cases, is the phase velocity of the wave. And, just to complete this section, let me quickly rewrite this relation as $\frac{\partial y}{\partial t}$, which is my particle velocity to be $-\frac{dx}{dt}$.

Let me indicate by c, the notation that we have been using to indicate the phase velocity multiplied by $\frac{\partial y}{\partial x}$. So, this is another way of writing the same information and, if you remember what we learnt in the previous module and we looked at the transverse force or the vertical component of the force which was acting downwards, there we said that the vertical component of the force is related to $\frac{\partial y}{\partial x}$ and here it turns out that $\frac{\partial y}{\partial t}$ the particle velocity is related to the vertical component of the force, which is $\frac{\partial y}{\partial x}$.

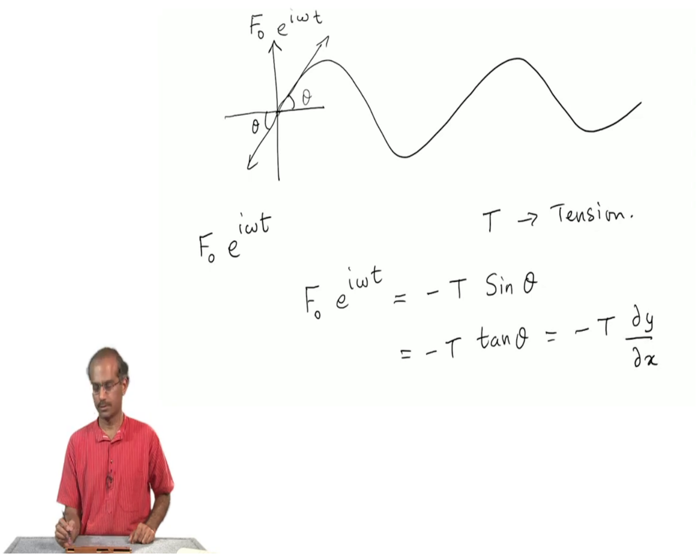
So, clearly the vertical component of the force on a segment of a string is proportional to particle velocity. So, that is another lesson we can draw from this equation, the quantity which we need to introduce here, to quantify how much resistance the medium offers to passage of a wave, is what is called the impedance.

(Refer Slide Time: 10:39)



And, impedance denote by this quantity Z, which would be the ratio of transverse force to transverse velocity. So, when I say transverse it is transverse to the direction of motion of the wave. So, if you wave is let us say travelling in this direction positive x direction, transverse force is the one that is perpendicular to it. So, it is transverse velocity, velocity in this direction. So, we are going to now use this quantity impedance and this impedance will clearly depend on properties of the medium.

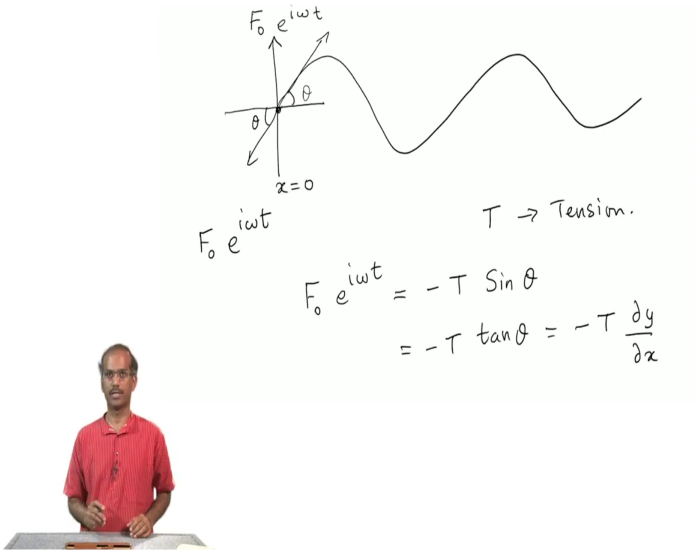
(Refer Slide Time: 11:31)



To illustrate some of these ideas, let us look at this simple problem, of a string which is being forced. So, in other words, string is the first oscillator. So, I would have something like this, that is my string and since this is a forced oscillator, there is a force that is applied in this direction vertically, in oscillating the string up and down. So, the transverse force is $F_0 e^{i\omega t}$. So, that is the force that I apply continuously.

And, of course, upon application of such a force the string begins to oscillate and I also assume that there is tension T in the string. Now, when we do the balance of forces turns out that this $F_0 e^{i\omega t}$ would be equal to $-T\sin\theta$ where θ would be this angle here. Now, in the spirit of small oscillations, I am going to write this as $-T\tan\theta$ and as usual we will substitute it by $\frac{\partial y}{\partial x}$.

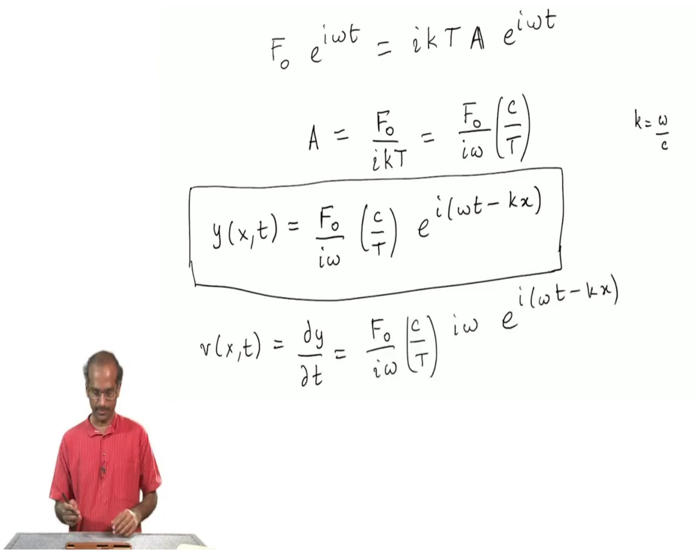
(Refer Slide Time: 13:07)



So, this is my equation for balance of forces and since it is basically a wave form that I expect, I am oscillating one end of the string and its going to generate wave forms, I can assume that the solutions will be the solutions of the wave equation. So, $y(x,t)$ is $Ae^{i(\omega t-kx)}$. So, this represents a wave that is propagating in the positive x direction.

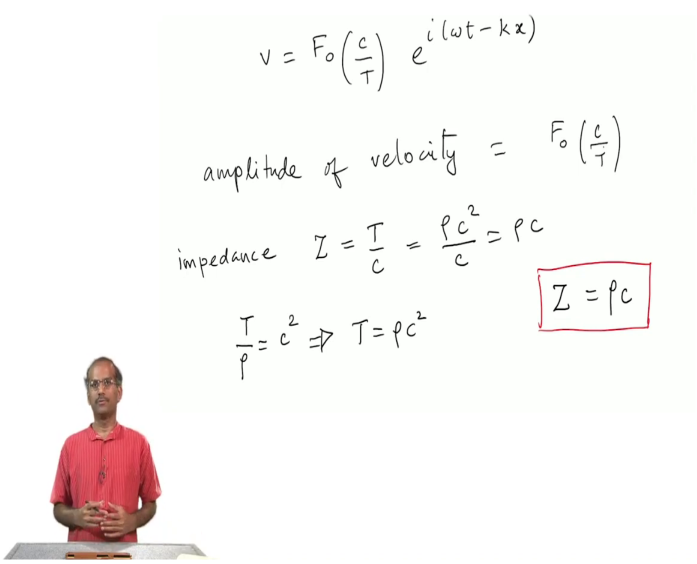
And, I should remind you that this amplitude A could be a complex number, $\frac{\partial y}{\partial x}$ is equal to $-A$. So, remember that A is complex therefore, that is $F_0e^{\imath\omega t} = (-T)(-Ake^{\imath(\omega t-kx)})$ Now, I would like to balance these forces at $x = 0$ simply because, this point here corresponds to $x = 0$, where the forces are balanced. So, if I do that, I will get the following relation.

(Refer Slide Time: 14:33)



So, A would be now, I am ready to write down the full solution. So, $y(x,t)$ would be $\frac{F_0}{\iota\omega}\left(\frac{c}{T}\right) e^{\iota(\omega t - kx)}$. So, this represents my solution to the problem where I assume that at position x, a forcing is applied to the string and so it creates a wave. I can also calculate the velocity, which will be a function of course, position and time. So, that would be $\frac{\partial y}{\partial t}$. So, I have this expression for the velocity, this can be simplified as follows.

(Refer Slide Time: 15:29)



So, v would simply be equal to F_0 into c by T, $\iota\omega$ and $\iota\omega$ will cancel. So, $e^{\iota(\omega t-kx)}$ and if you remember the definition of impedance F_0 is of course, the transverse force, this implies that T is equal to ρc^2 and if I substitute that here, it would give me $\frac{\rho c^2}{c}$, which is equal to ρc.

So, there are these different forms for impedance. So, impedance is Z is ρc. So, ρ is the linear density and c is the wave velocity or the phase velocity. So, so we have now simpler relations that relate impedance to properties of the medium and clearly impedance itself is a property of medium. So, it is not surprising that it is related to ρ and c. So, now, at this point what we have is we have seen the wave equation, we will solve the wave equation, we obtained different possible solutions for the wave equations. And, we also defined at least 2 different possible velocities that are associated with the wave and we also defined the impedance. To remind you again impedance is simply the sort of resistance that the medium offers to the passage of the wave.

Now, in the next few modules, we will use all these machinery that we have learnt in the last, in this module and the previous one, to look at how waves behave when they go from one medium to another medium. So, we look at all these problems in the subsequent modules.