Waves and Oscillations
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Lecture - 25
Wave Equation and its Solution

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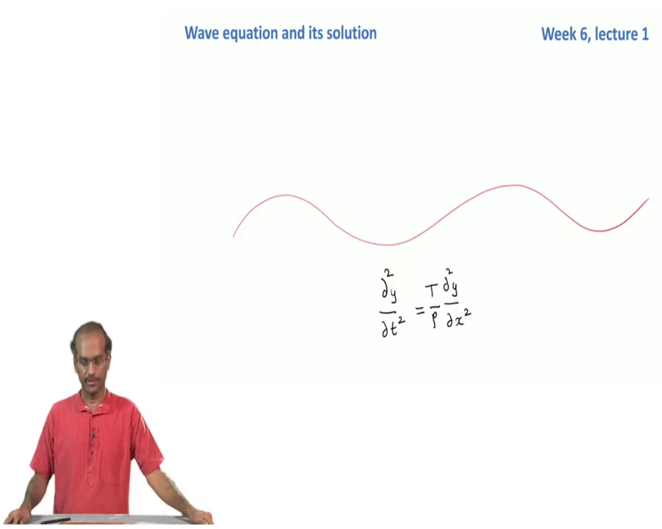

Welcome to the first lecture of 6th week, we were looking at the problem of particles which are coupled together and for us coupling would mean that if I disturbed one of them the others would also get disturbed. So, if I oscillate let us say one of the particles the other particles would also start oscillating soon enough. We could see a progression of Oscillations in the sense that if I oscillate one particular particle its going to convey the disturbance to the next and then to the next and so on.

And when it does so; it does so with some small phase lag and this way the disturbance propagates and this propagation of disturbance is what we call a wave. So, every time we talk of wave say sound wave its propagation of pressure disturbances. So, similarly every wave is in some sense propagation of a certain kind of a disturbance. So, here we are talking of disturbance that we created in a string and finally, we decided to bring the particles closer and closer such that they would form a continuum. So, in the limit of small distance between these particles we derived what are the normal frequencies we derived what are the normal modes and so on and so forth.

So, that is starting from considering single particle, then two particle, then we went to n particle and then we took the continuum limit. Now, we will straight away work with string and ask for what is the kind of oscillations that it would display or pattern of oscillations that it would display. So, the question again is about normal mode oscillations of a string.

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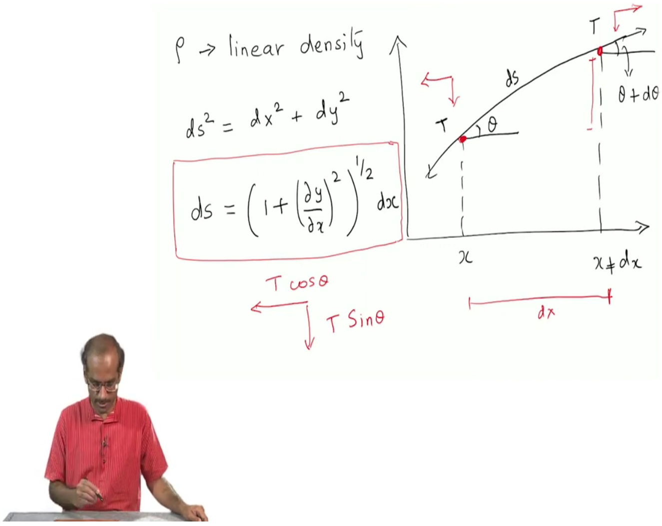

So, the starting point is the consideration of a small portion of a string let us say that let me consider a small portion like this ok. So, the entire string is oscillating I am not worried about what is happening to rest of the part. So, I am just looking at a small segment let me call it a segment that lies between x here and $x + dx$ and this segment at x it makes an angle θ with the horizontal and at $x + dx$ it makes an angle $\theta + d\theta$ with the horizontal and we will assume that the string is of course, oscillating and the string has mass that is uniformly distributed throughout.

So, in other words the density of the string is uniform throughout the string and we shall denote density are more correctly the linear density of the string by ρ and we shall also assume that there is uniform tension T in the string. Clearly from the physics of the problem we can sort of figure out that there is no net force in the horizontal direction. So, the entire string does not move horizontally there are only forces in the vertical direction which leads to oscillations.

So, I am going to denote this length of this segment by ds. So, let me write an expression for ds. So, ds^2 would be. So, clearly dx^2 is this horizontal component now this distance is dx and similarly dy is this distance its easy for me to write the following relation for ds, I will let you do it yourself all you need to do is to divide throughout by dx^2 then you will get the following relation.

So, this is the relation that I need which gives me the length of the segment of string that I am considering. Since there is uniform tension T in the string, so you could imagine that there would be horizontal component of tension which would be $T\cos\theta$ and vertical component that would be $T\sin\theta$. So, this is this would be the case at this point and similarly there would also be another component of tension which would be acting at this point as well ok.

So, in this case the horizontal component would act in this direction and vertical component of course, would act downward whereas, here at this point the horizontal component would act in this direction and vertical component here. So, clearly we see that the horizontal component are acting in opposite directions whereas, the vertical component together are acting downwards.

And we are going to put in an important piece of assumption namely that the amplitude of oscillation is small enough which means that $d\theta$ here would be sufficiently small. So, if I have to anywhere write $\sin(\theta + d\theta)$ I could make the assumption that $d\theta$ is sufficiently small.

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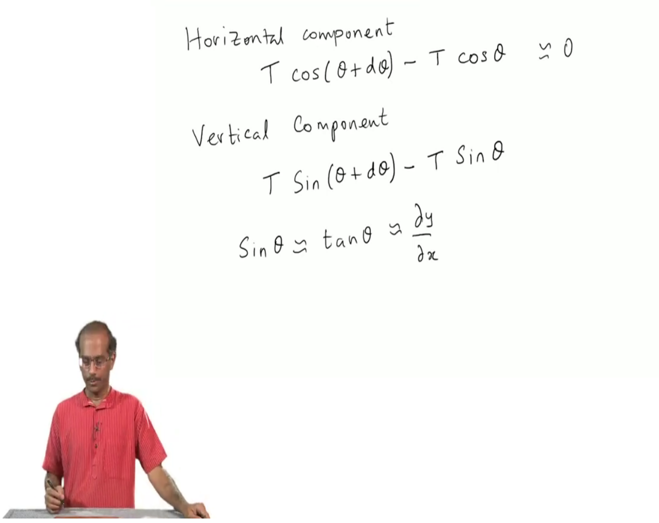

And in the limit when $d\theta$ is small this is approximately zero. So, the two horizontal components balance each other. What about the vertical component of force? So, this vertical component of the force or vertical component due to tension is what provides the restoring force for the string to oscillate. So, in this case I would get $T\sin(\theta + d\theta) - T\sin\theta$ this vertical component could be written as follows.

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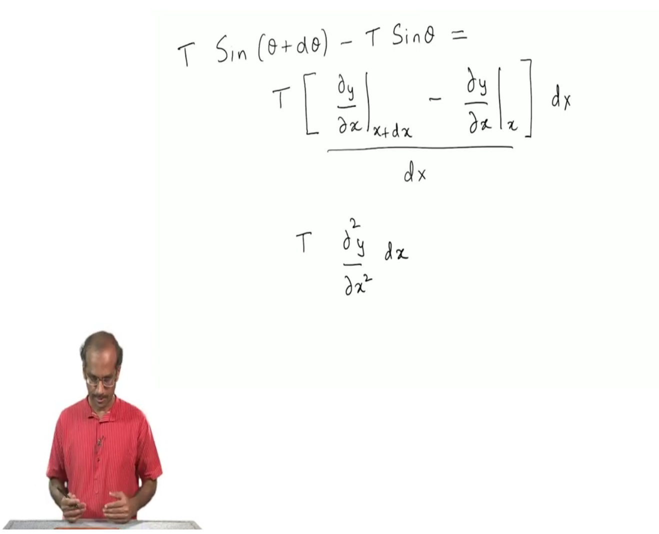

So, that is the expression for the vertical component of tension as you can see I used the fact that $\sin\theta$ is approximately equal to $\frac{\partial y}{\partial x}$ and evaluated at the right positions. I can multiply and divide by dx and with this you will notice that the quantity within the square brackets can be written in partial differential form. So, this would simply be equal to $T\frac{\partial^2 y}{\partial x^2}dx$. Now, of course, I will equate this to mass times acceleration.

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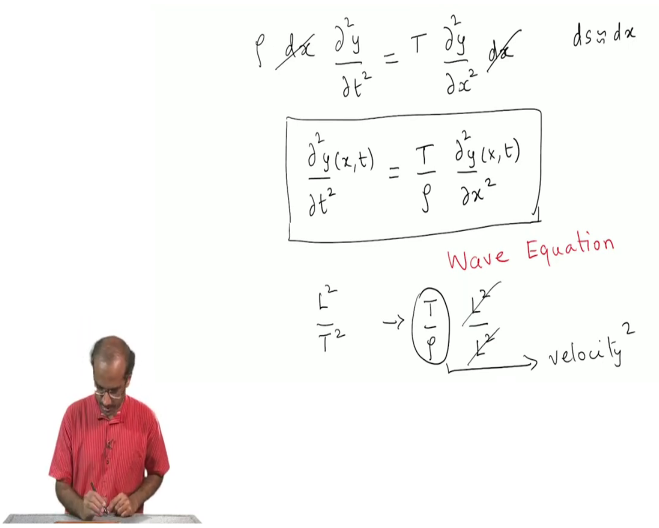

So, mass of the string would be linear density multiplied by the segment of the string that is being considered which is dx that is the length of the segment that we are considering and of course, again in the limit when the amplitude of the oscillations are small we assume that ds is approximately equal to dx. So, if you remember we actually wrote an expression for ds which is the actual length of the segment of the string and now in the limit of small oscillation we assume that ds is very close to dx. So, I use the fact that ρdx is mass of the string and its multiplied to $\frac{\partial^2 y}{\partial t^2}$ and this is equal to Tdx.

So; obviously, now in this expression dx and dx will cancel out I am left with the equation which we have already seen. So, let me for once write the dependence of y on x and t explicitly. So, that would be $\frac{T}{\rho}\frac{\partial^2 y(x,t)}{\partial x^2}$. Next let us look at what this term $\frac{T}{\rho}$

would tell us and to see what it is its more easier to do a quick dimensional analysis. So, if you look at the left hand side of this equation this has dimensions of length square by time square because its $\frac{\partial^2 y}{\partial t^2}$.

So, its length square by time square and here on the right hand side I have of course, $\frac{T}{\rho}$, I will not worry about what it is right now and here I have length square by length square because its $\frac{\partial^2 y}{\partial x^2}$. So, this dimension of course, will cancel one another which means that it tells me that this $\frac{T}{\rho}$ corresponds to dimensions of velocity clearly l by t is of course, the displacement divided by time and l square by t square is square of velocity. So, this quantity $\frac{T}{\rho}$ should have dimensions of velocity square. In fact, it turns out that this simply represent the velocity of the wave. So, typically its written in the following way.

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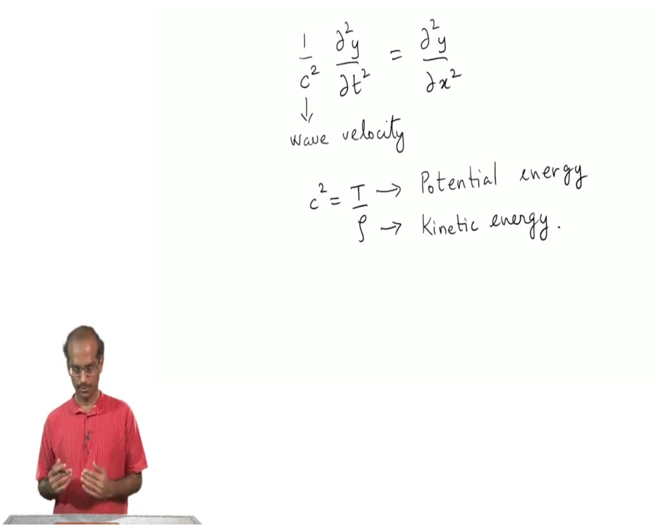

This equation $c^2 = \frac{T}{\rho}$ it depends on tension in the string and the other part is of course, the linear density string is the medium in which the wave motion is happening tension corresponds to the property of the medium. And of course, the potential energy comes from the fact that you have tension in the string, ρ which is linear density is responsible for basically its a property that is related to kinetic energy.

So, its a ratio between something that is responsible for potential energy to something that is responsible for kinetic energy. So, here again I have the wave equation and to again remind you we have not worried about the boundary conditions; boundary conditions will be put in when we actually solved for a specific problem. Now, what about the solutions?

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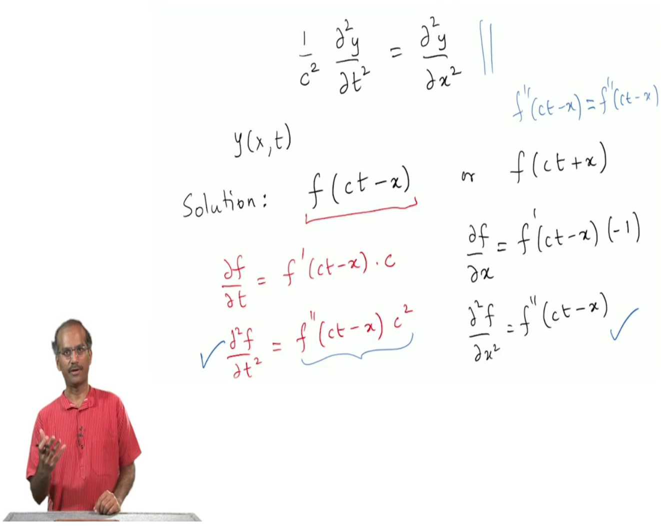

So, when I mean solution I mean the following that I want to know what is y as a function of x and t. In other words I want to know what is the displacement at a particular position on the string at a particular time, if I know this then of course, I can claim that I have completely solved the problem. Its easy to see that one possible solution could be of the form $f(ct-x)$ ok. So, c of course, is our velocity t is time and x is of course, the position along the string. So, either one could take some function of $ct-x$ or some function of $ct+x$ either of these would work as solutions.

Now, I just need to plug in these two quantities this and this in this equation. So, its very easy to see if I plug in the time derivative then of course, this will go in here and the c^2 and c^2 will cancel. So, I will just have $f''(ct-x)$ equal to and of course, on the right hand side I have $f''(ct-x)$ and clearly they are equal to one another. So, we are sort of convinced that our ansatz for solution which is some function of $ct-x$ is indeed a solution to the wave equation and in a similar way you could also plug in $f(ct+x)$ and convince yourself that any function of $ct+x$ would also be a solution of wave equation. I would like to now specify some function.

So, we know that we are dealing with oscillatory system string is oscillating. So, we should expect to have sine and cosine functions around. So, if you take any small segment of a string that is like maybe one particle and its basically going up and down. So, if I focus myself on one infinitesimal segment of a string its doing nothing, but just going up and down oscillations for which we know the solution already right from our very first week of lectures. So, the solution should be either some sine function or cosine function.

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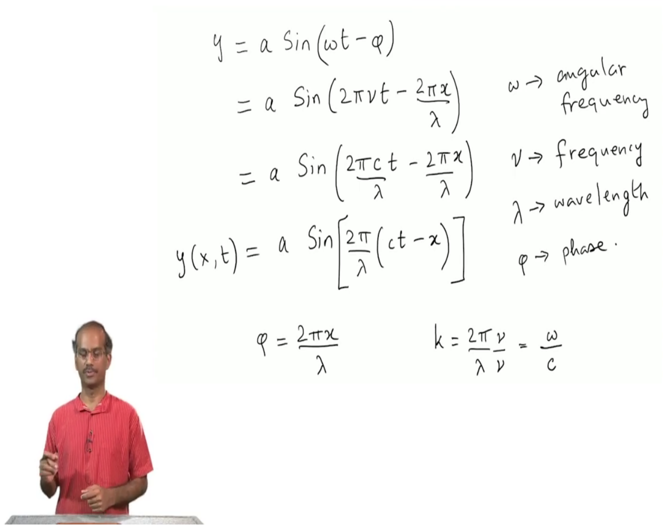

So, based on this I can say that for a small segment y displacement should be $a\sin(\omega t-\phi)$. So, that should be a possible solution after all that segment is oscillating. So, somehow I want to bring in $ct-x$ in which case to do that I can rewrite it as $2\pi\nu t$ minus this ϕ which is essentially in radians let me write it as $\frac{2\pi x}{\lambda}$ and now to get $ct-x$ I could replace this ν by $\frac{c}{\lambda}$. Now, you will see that in this I can take $\frac{2\pi}{\lambda}$ outside. So, I will have $ct - x$. So, clearly what I have obtained is a solution for the wave equation. So, y is the displacement which is a function of position and time and as we wanted we will written this solution in terms of some function of $ct - x$.

And in doing this we were motivated by the fact that a small segment of the string does nothing, but oscillate about the mean position. And to remind you ν of course, is the frequency, ω is the angular frequency and λ is the wavelength and ϕ is of course, the phase we can also introduce another quantity called the wave vector which is k which will be $\frac{2\pi}{\lambda}$.

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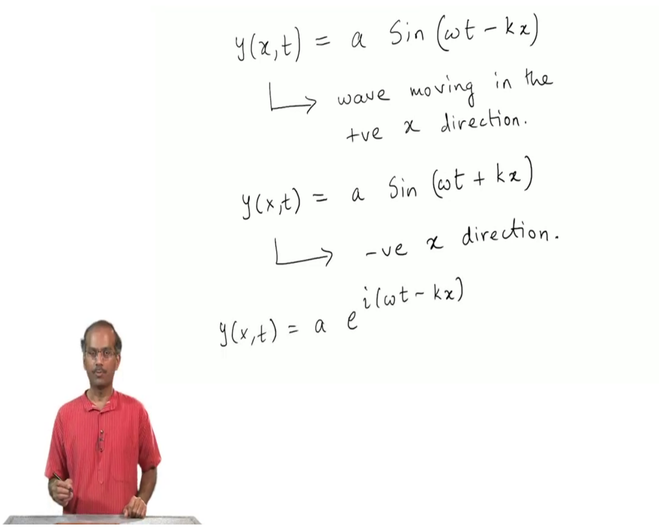

So, k is the wave vector and if I plug in the k, I can write the solution in terms of k and that would become $y(x,t) = a\sin(\omega t - kx)$ and this you could easily figure out that this corresponds to wave that is traveling in the positive x direction. Similarly, I can write out another solution which is $y(x,t)$ is equal to $a\sin(\omega t + kx)$. So, this would correspond to a wave that is traveling in the negative x direction.

And as usual whenever sine and cosine functions are involved these are not the only possible solution that we can think of there are ways of writing it in terms of exponential functions we had already seen that earlier on. Finally, let me also point out that we were written k as $\frac{2\pi}{\lambda}$, but if you multiply it and divide by ν you could also write it as $\frac{\omega}{c}$.

The next module we will continue by looking at more properties of these solutions we will try and solve some real problems. You will note that in obtaining these solutions these are general solutions in the sense that we have not any boundary conditions, but in the next few lectures we will put boundary conditions and obtain specific solutions to specific problems.