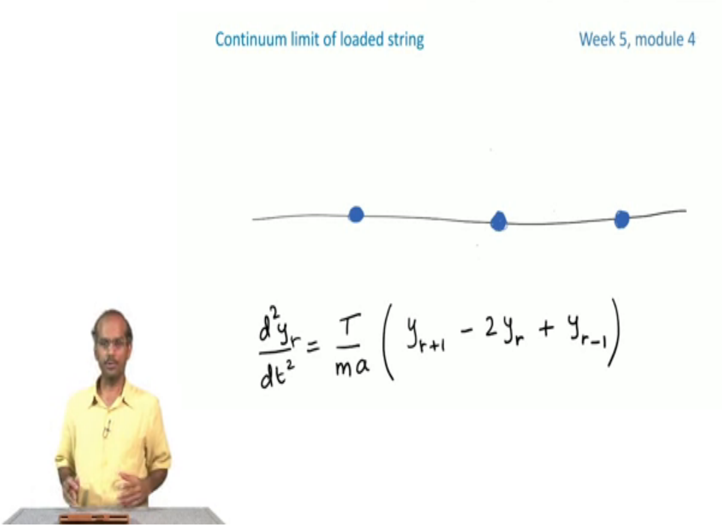
Waves and Oscillations
Prof. M S Santhanam
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 24
Continuum Limit of Loaded String

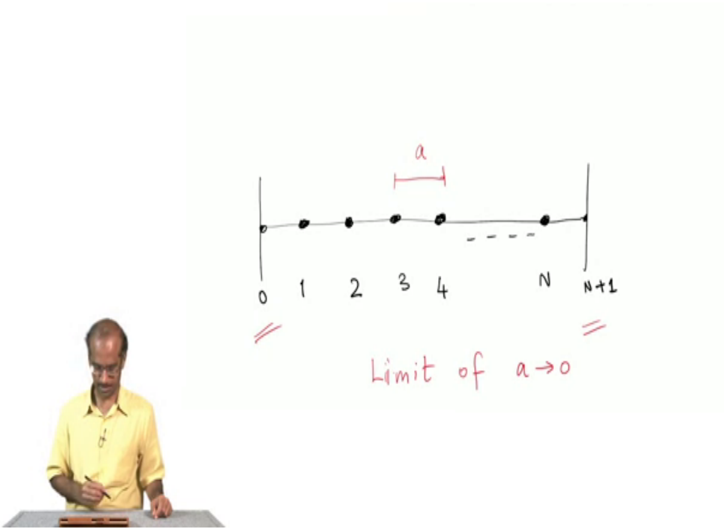
(Refer Slide Time: 00:15)



Welcome to the module 4 of week 5, we are still looking at the problem of loaded string by which I mean particles string together or connected together in some way and here you taken string is a kind of abstract notion that couples particles together. So, we are already seen this problem for the last three year. So, models and we wrote down an equation of motion, we solve the equation of motion, we looked at the case when there are *n* particles tied together by a string and between two rigid walls.

And we saw that if you have n particles there would be *n* normal mode frequencies and we also saw qualitatively to some extent and also quantitatively as to what kind of pattern of oscillation would be exhibited by the normal modes. Especially for the lowest frequency and for the highest frequency. In cases where you have smaller number of particles you can work out all the details and you can basically write down the pattern of oscillation for each and every case. Now, we take the next step towards what would be called the continuum limit.

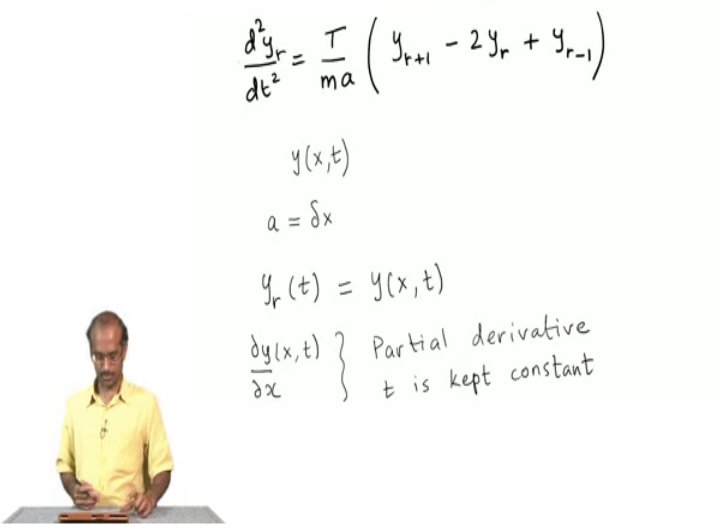
(Refer Slide Time: 01:53)



Here is a canonical example of what we have been seeing all along N particles in a string and there are these boundary conditions that a particle at zeroth position and the one at $(N+1)$th position, the amplitudes are zero at all times which means that they will not oscillate at all. Now when I say that I am going to take the continuum limit what I am going to do is that this distance between two particles which we had called a and all the particles did maintain equal distance from each other which is equal to a, I am going to take the limit of a tending to zero. So, if I take the limit of a tending to zero.

So, the particles are going to come closer and closer to one another and would essentially form a string. So, that is the kind of scenario that we will be looking at. So now, we do not have to anymore worry about particles tied to a string and so on, we will just be looking at oscillations of the string itself in which the particles are so close to each other that you can take the distance between the particles to be equal to approximately zero. To analyze the situation we will again start from the equation of motion that we have already obtained.

(Refer Slide Time: 03:23)

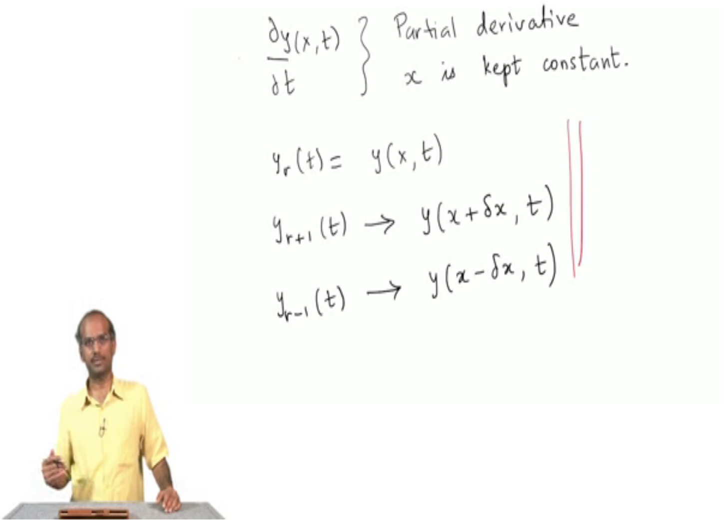


In this equation of motion just to recall what we had done T is the uniform tension in the string m is mass of the beads and a is the of course, the distance between the particles. We have equation for y_r which you would mean that it is the displacement for rth particle ok. So, r to remind you again is the index for the number of the particle along the string. In general as we can see the displacement is actually a function of both position and time. So, it depends on where you are looking at along the string and also the time. So, when we discretize I am going to use new set of notations.

So, I am going to say that a will be denoted by δx and of course we will be taking the limit δx tending to 0. So, let us say r is index for one particle position of one particle and now that is going to be denoted by $y(x,t)$. So now, explicitly it is very clear that why being the displacement depends both on space, that is the position of the particle along your string and time. If this is the case then I can denote few other quantities with respect to y since y is a function of both x and t I can define $\frac{\partial y}{\partial x}$ which is partial derivative of y with respect to x or with respect to position.

So, it is the question about how what is the rate of change of the displacement as a function of the position along this string. So, when you do this you keep T constant. Similarly I can also define another partial derivative with respect to time.

(Refer Slide Time: 05:40)

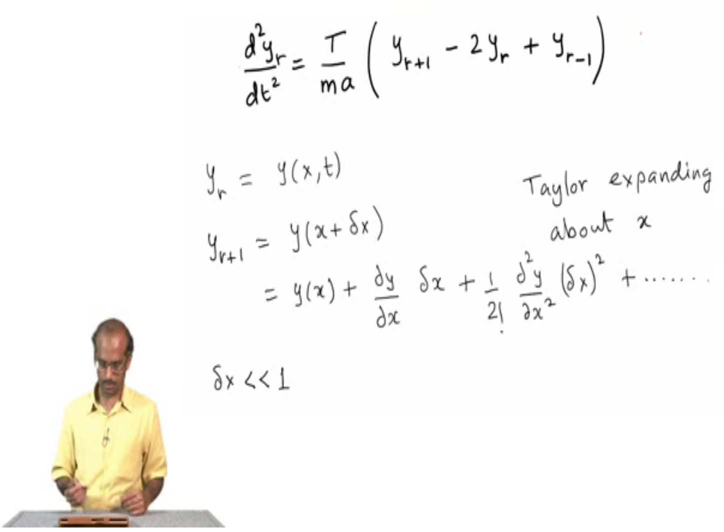


So, that would be $\frac{\partial y}{\partial t}$, so y itself is a function of x and t. So, $\frac{\partial y}{\partial t}$ would imply that it is the rate of change of displacement as a function of time when you keep x constant. So now, let us go back to our notations that we had introduced. So, we said that $y_r(t)$ is $y(x, t)$. So, this is the notational change in the continuum limit and I am going to introduce $y_{r+1}(t)$ which we had used and that will become $y(x + \delta x, t)$.

And similarly $y_{r-1}(t)$ would become $y(x - \delta x, t)$. Now to actually obtain in equation of motion in the continuum limit, all I need to do is to substitute these things in the equation that I have which is this. I should also mention that second derivative is also defined in a very similar way. So, d square or $\frac{\partial^2 y}{\partial t^2}$ square would simply mean second derivative of y with respect to time when x is kept constant and so on ok.

So, it is a standard generalization, if you are already aware of partial derivative and partial differentiation this should be easy. In case you are not familiar with this idea of partial differentiation I urge you to go back and look up any basic book on calculus, where you should be able to easily pick up these ideas.

(Refer Slide Time: 07:47)



Now, let us go ahead and let substitute for all these y_r, y_{r+1} and y_{r-1}. Before I substitute let me make an approximation here, of course y_r is $y(x, t)$ quite standard there is nothing to approximate. But y_{r+1} which is $y(x + \delta x)$ and I am going to suppress this time here for now. But keep in mind that y always depends on time even if I do not mention it explicitly here.

So, δx being small I can now make a Taylor expansion of this quantity about x. So, once again let me also remind you that in case you are not very familiar with the idea of a Taylor expansion, please go back and look up in analysis calculus books from where you can I again pick up the basic ideas easily. So, here I am Taylor expanding about x. So, in other words the question is I know the value of y the displacement at position x, I want to know what is the value of displacement at the point a little bit away from x maybe $x + \delta x$.

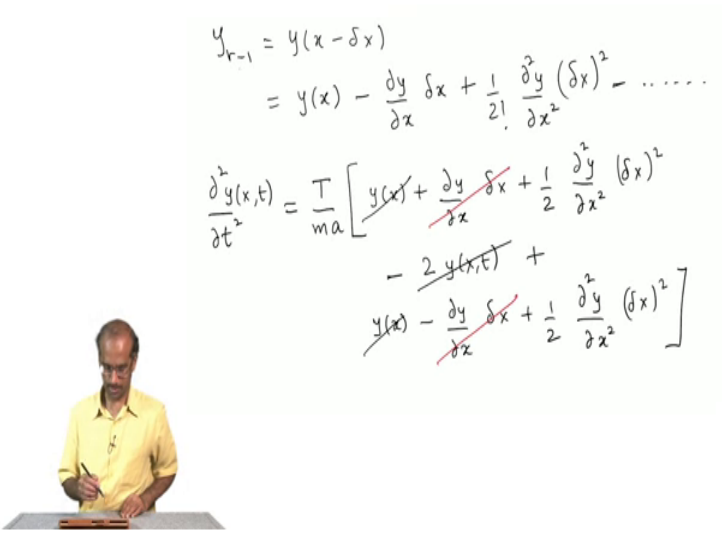
So, then we make a Taylor expansion. So, this will be

$$y(x) + \frac{\partial y}{\partial x}\delta x + \frac{1}{2!}\frac{\partial^2 y}{\partial x^2}(\delta x)^2 + \cdots$$

Of course, there will be higher order terms higher order in δx. So, there will be term which will have some coefficient time's δx^3, some coefficient times δx^4 and so on. But since we have assumed that δx is small.

So, equivalent to saying the $\delta x < < 1$, in which case we can ignore all the terms which are higher order in δx beyond δx^2. So now, before substituting in this equation I am going to also do the same thing for y_{r-1}.

(Refer Slide Time: 10:44)



So, again it is straight forward Taylor expansion it will be $y(x)$. So, it will be $y(x - \delta x)$ it also a function of time. So, it will be

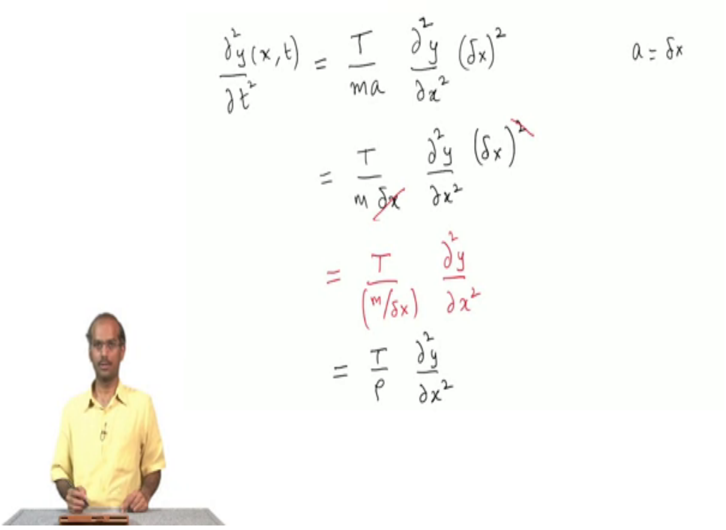
$$y(x) - \frac{\partial y}{\partial x}\delta x + \frac{1}{2!}\frac{\partial^2 y}{\partial x^2}(\delta x)^2 + \cdots$$

and of course there are higher order terms, which I am going to ignore. So, now I have all the ingredients and I substitute all this in this equation. Now if I do that this is what I should get y_r is simply $y(x, t)$. So, I should I have left hand side, $\frac{\partial^2 y}{\partial t^2}$ and it is a function of x and t and then there is $\frac{T}{ma}$.

And now I will substitute for y_{r+1} which is $y(x) + \frac{\partial y}{\partial x}\delta x + \frac{1}{2!}\frac{\partial^2 y}{\partial x^2}(\delta x)^2$. So, I am not going to consider any terms beyond this, $-2y_r$ will be $2y(x,t)$. And then I have $+y_{r-1}$ which should give me $y(x) - \frac{\partial y}{\partial x}\delta x + \frac{1}{2!}\frac{\partial^2 y}{\partial x^2}(\delta x)^2$.

Now, if you look at these equation there will be will be many terms that cancels. So, for example, this $y(x)$, $y(x)$ and $2y(x)$ `that will cancel and so, the first order term in δx; $\frac{\partial y}{\partial x}\delta x$ will also cancel. So, after these cancellations we are only left with two terms which are the second derivative terms, second derivative with respect to position x. So, let us collect all that together and write it.

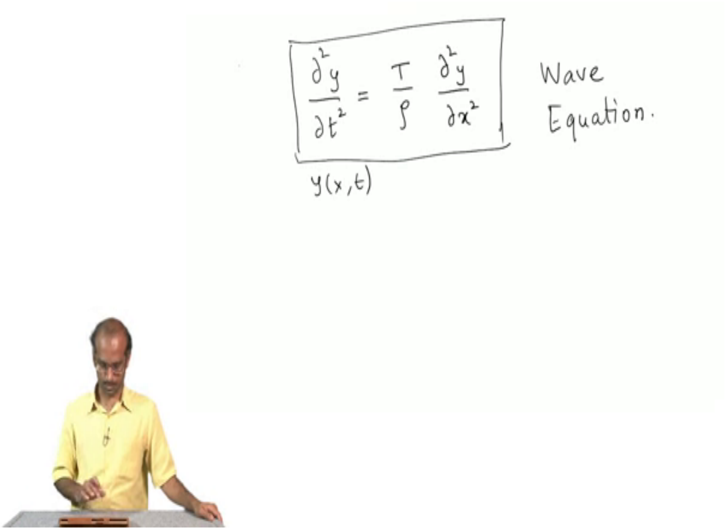
(Refer Slide Time: 13:54)



So, when I do all this the equation is sort of reasonably simple and I get this equation that I have in front of you. Now of course, I should have done this long back. So, we said that $a = \delta x$. So, this would be $\frac{T}{ma}\frac{\partial^2 y}{\partial x^2}(\delta x)^2$.

So, δx this would cancel and there would be only one δx left here and with this I will have $\frac{T}{(m/\delta x)}\frac{\partial^2 y}{\partial x^2}(\delta x)^2$. And $m/\delta x$ of course, is the linear density, so the final result would turn out to be dou by $\frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$. So now we can write the final result. So now, we have obtained an equation of motion in the continuum limit.

(Refer Slide Time: 15:11)



And should keep in mind that y is a function of x and t and when we say that we are going a solve this equation this continuum version of this equation. We mean that we should be able to obtain displacement both as a function of position and as a function of a time. So, this is probably one of the most important equations in physics and is called the wave equation. The problem was to describe the dynamics of collection of particles which are coupled together and there may be a boundary condition there may not be a boundary condition, it is not particularly important as far as the equation of motion itself is concern. So, we are able to obtain an equation of motion whose discrete form was this given right in front of you.

So, physically this corresponds to beads which are tied together by strings ok. Now when you bring those beads closer and closer together you get to the continuum limit, so you take the limit properly starting from that discrete version of your equation of motion. You will end up with partial differential equation and this is called the wave equation. And as again I would like emphasis that here there is no boundary condition imposed in this equation.

So, every time we solve a problem we need to impose a boundary conditions suitable for that problem. So, with this lesson I will stop this module and we will look at how to solve the wave equation and various properties of this and how it relates to real physics of the problem in the next week.