Waves and Oscillations

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Lecture - 23

Oscillations of Loaded String

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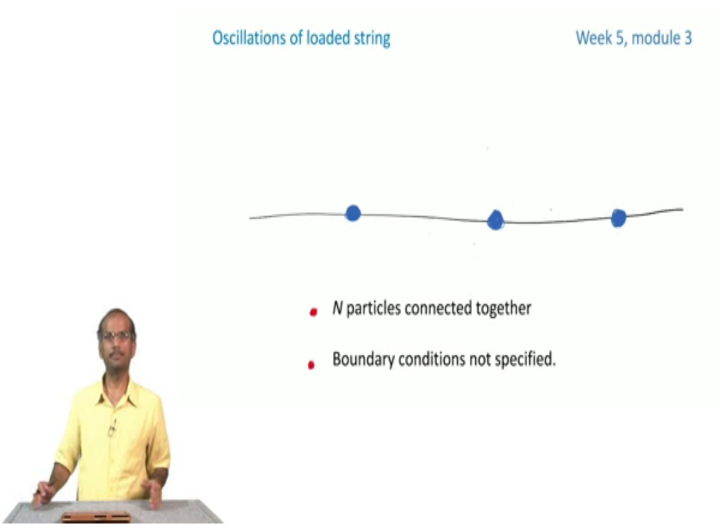

Welcome to the 3rd module, we will continue our explorations with the Oscillations of the Loaded String; a system of connected particles and we also saw that we do not have to be specifically worried about boundary conditions until we get the equations of motion. But, when we want to solve it for a specific case we need to put in some boundary condition to be able to get some suitable results.

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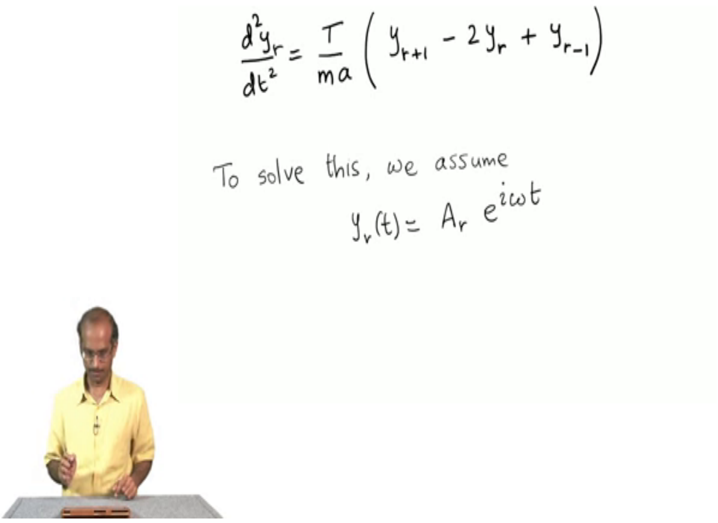

And, here if you look at this equation, this is the equation of motion and we have assumed that there is uniform tension T in the string and r tells you which particle you are looking at, it is the index for the particle. So, you have N particles and r will go from 0 to $N + 1$. So, there are N particles ranging from 1 to N. The zeroth particle at one extreme N and the $(N + 1)$th particle other extreme N will not oscillate, they are tied to the walls, so that in some sense is also are boundary condition.

And to solve this equation of motion we assume solutions of this form $Ae^{i\omega t}$, where is ω is the normal mode frequencies that we wanted to determine the entire system with oscillate with one frequency. You cannot have a situation at least in this kind of cases where one part of the system are, let us say half particles oscillate with one frequency and other half oscillate with another frequency that cannot happen.

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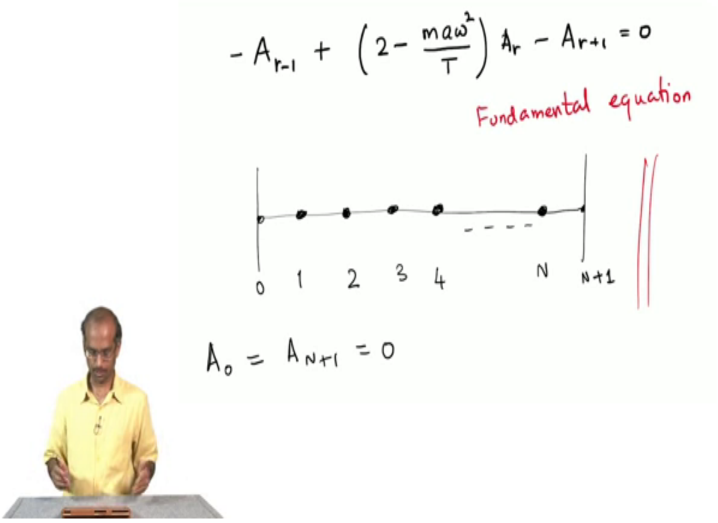

So, here is a picture of a the system that you are looking at. I have two rigid walls and zeroth particle, $(N+1)$th particle is embedded in the wall and rest of the N particles are free to oscillate strictly in the vertical plane, A_0 is the amplitude of the zeroth particle A_{N+1} is amplitude of the $(N+1)$th particle, and with all these assumptions and substituting in this solution we derived the results, the results being one is the normal mode frequencies which in general is given by this one for N particle system N normal mode frequencies, this j is essentially a index for the normal mode frequencies.

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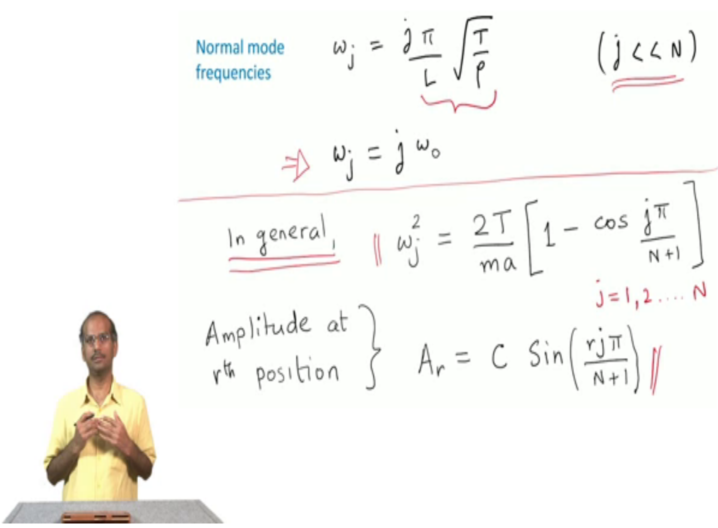

So, j goes from 1 to N and $j = 1$ would be the first normal mode frequency and so on. And r as I said is the index for the position of the particle along your string. So, A_r would tell me what is amplitude of the r th particle. And again the amplitude of the r th particle depends on which normal mode you are looking at. So, the amplitude A_r depends on both the induces r and j as you can see here.

And this general expression for the normal mode frequencies ω_j^2 which is here, can be simplified, if you if you are looking at only the lower frequencies which means that j is much less than the number of particles that you have. In that case we saw that we can write that ω_j is equal to j times ω_0, where ω_0 is this quantity $\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$. ρ is often called the linear density or mass per unit length.

With this background we will address few related questions, questions related to this problem. So, the first thing we should do is to see if all that we did is right. So, we already solved for example, a problem with two beads.

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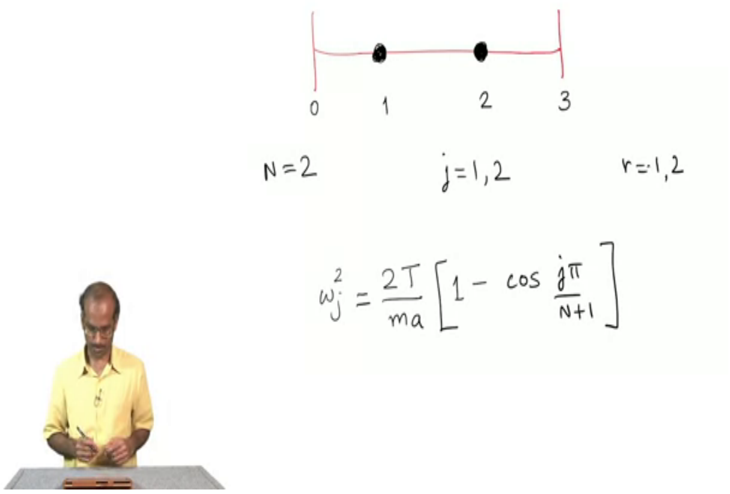

So, suppose I have this simple system, instead of considering N particles, my N is now 2. So this is the zeroth position $(N+1)$th position or maybe I should say third position and this is 1 and 2.

We solve this problem already in one of our earlier module, we know the results. Now we should check, now that we derive a more general case we should check if we are still able to reproduce those known results from these general results. So, if we have this problem in this case N is equal to 2 and I am going to have two normal modes. So, j will run from 1 to 2, and there are two particles, so r will also run from 1 to 2.

So, let us see if we can first get the known results, let us first get the normal mode frequencies. So, I have this expression that we derived earlier. So, I just need to plug in all the values, so if I do that. So, let us first get the lowest frequency.

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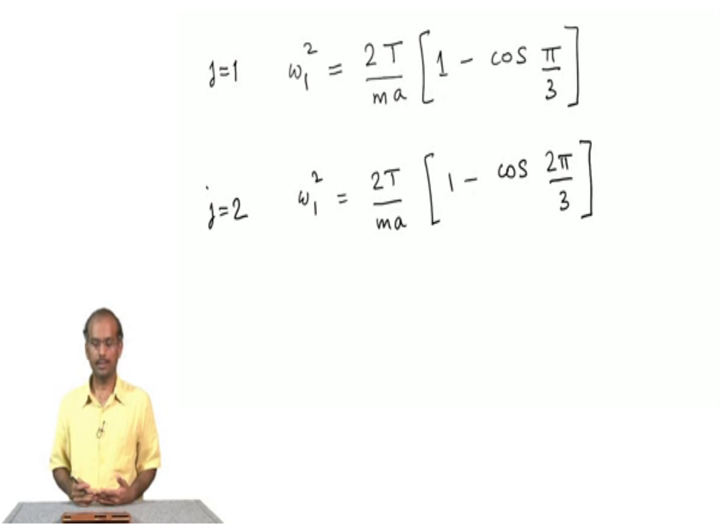

So, I have this expression, $1-\cos\dfrac{\pi}{3}$ and if I set j to be equal to 2 which will be the second normal mode frequency that would correspond to,

$$\omega_1^2 = \frac{2T}{ma}\left[1-\cos\frac{\pi}{3}\right]$$

$$\omega_1^2 = \frac{2T}{ma}\left[1-\cos\frac{2\pi}{3}\right]$$

So, I will live it to you as an exercise to substitute the values for $\cos\frac{\pi}{3}$ and $\cos\frac{2\pi}{3}$ and you should be able to recover the known results for ω_1 and ω_2.

Now, that we have the normal mode frequencies, let us see if we can get the normal modes themselves; the pattern of oscillation. To get the normal mode pattern we need to work with the expression for the amplitude which is given by this expression.

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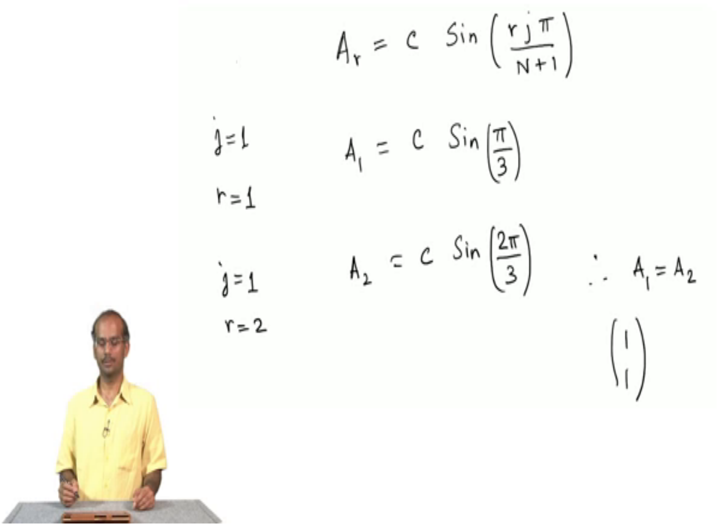

So, now we will repeat the same exercise. So, let us set j equal to 1, which means that I am going to get the normal mode for the lowest frequency. And in the normal mode I want to get the amplitude of the first particle. So, that would mean that r is equal to 1 as well. So, now simply plug in all these things, so I will get A_1 is equal to $C \sin \pi$ by of course, N is 2, so that is $\frac{\pi}{3}$. Now what about A_2?

So, there are two particles and again we are still looking at the lowest frequency corresponding to j equal to 1 and now r equal to 2. So, that would give me $C \sin\frac{2\pi}{3}$; $\sin\frac{\pi}{3}$ is equal to $\sin\frac{2\pi}{3}$; therefore, we can write $A_1 = A_2$. So, this is our normal mode or if you write it in vector form this will be $\begin{bmatrix}1\\1\end{bmatrix}$. So, that is our normal mode. Now, so we look at the second normal mode corresponding to the larger of the frequency.

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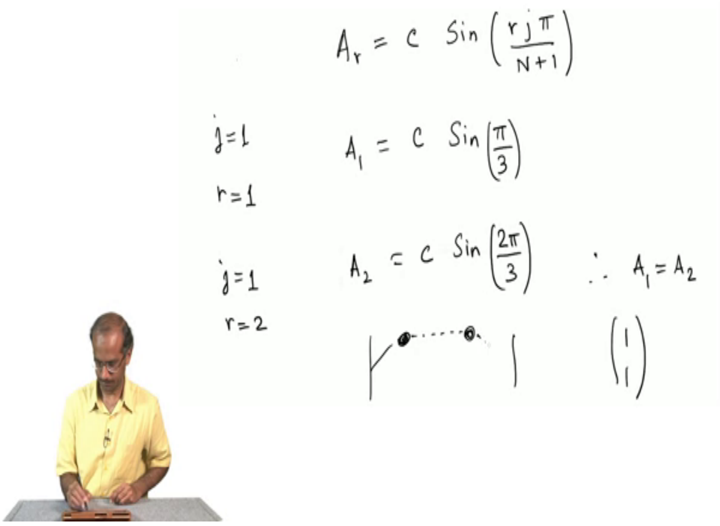

So, that would correspond to $j = 2$ and again I will work with do it for the first particle corresponding to $r = 1$. So, in this case the expression would be, and now I stick to $j = 2$; the same normal mode, but for the second particle. So, now, its easy to verify again we can put in the values for sin $\frac{2\pi}{3}$ and sin $\frac{4\pi}{3}$, you will notice that $A_1 = -A_2$. So, that would correspond to saying that my normal mode is $\begin{bmatrix}1\\-1\end{bmatrix}$ ok.

So, again if you remember what we had done the previous module, this correspond to two distinct patterns of oscillations; one in which the two particles just go up and down like this together, the so called in phase oscillation and the other one where, when one of the particle is above the equilibrium point then second one is below the equilibrium point. So, you have this anti-phase osculation when one goes in the other goes down and so on, so that is this second mode. So, if I have to quickly sketch that it would correspond to something like this whereas the first one would correspond to something like this.

So, in general we kept arguing that if you have N particles you should have N normal mode frequencies. Can we say something about what is the largest frequency that is possible. So, let us address that question.

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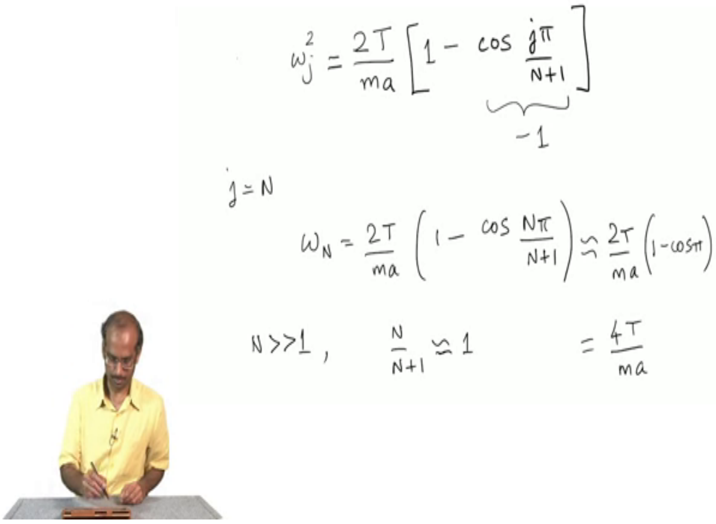

Now, I need to determine what is the maximum normal mode frequency, that is possible. It is easily seen from the formula that if this quantity is equal to -1 that is when ω_j would be maximum and that would happen if $j = N$. In other words we are looking at the largest mode, so the frequency of the first mode is the lowest; call it the zeroth mode and the next one has higher frequency and so on. So, the Nth mode should have the highest frequency.

So, I am going to set $j = N$, in which case ω_N will be equal to,

$$\frac{2T}{ma}\left(1 - \cos\frac{N\pi}{N+1}\right).$$

And of course, if N is sufficiently large enough you could sort of say that in the limit when N is much greater than one, $\frac{N}{N+1} \approx 1$, in which case this whole thing simply boils down to,

$$\frac{2T}{ma}(1 - \cos \pi),$$

and that is simply equal to 2. So, this would just be equal to $\frac{4T}{ma}$.

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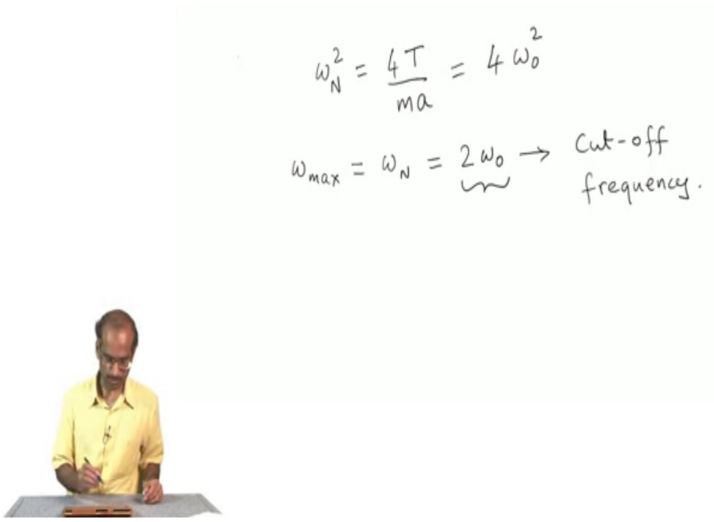

So, my result says that

$$\omega_N^2 = \frac{4T}{ma}$$

and this can be written as $4\omega_0^2$. Hence ω_{max} or the maximum frequency possible is equal to ω_N which is equal to $2\omega_0$; so, this quantity $2\omega_0$. So, if you remember ω_0 is the lowest frequency possible for your system and the largest frequency possible in your system is simply twice the lowest frequency that is possible; $2\omega_0$. So, this is called the cutoff frequency and this is true in general for this class of systems.

So, when I say this class of systems it means systems which are in one dimensions, like the one that we are considering and the particles are periodically arranged. So, if you consider this class of systems in one dimension periodically arranged particles, then this cutoff frequency $2\omega_0$ is a common a result for all such cases. So, we have seen what is the largest frequency, that is supported by our system. What about the normal mode itself, what is the pattern of oscillation corresponding to this largest frequency?

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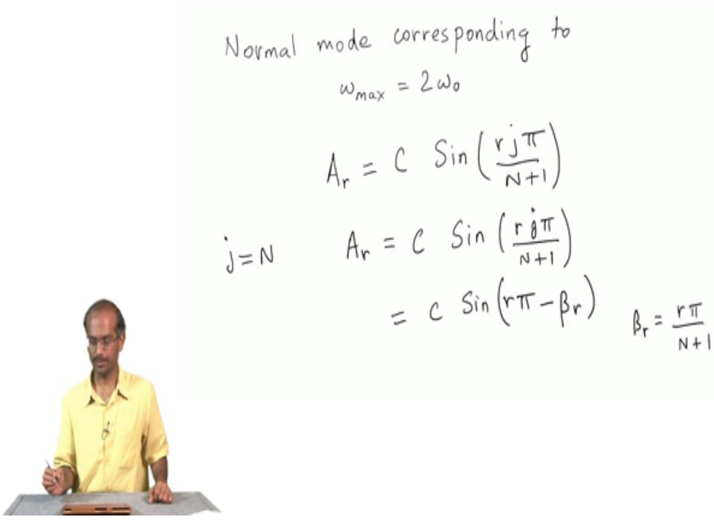

This can be obtained by again going back to the equation for the amplitudes. So, just to remind you once again, the equation for amplitude depends on two things; one the normal mode that you are looking at and the position of the bead along the string. So, just to fix the parameter, so I am going to look at the normal mode corresponding to the largest frequency.

That means, that $j = N$ and of course, r will go from 1 to N. A_0 and A_{N+1} as usual are equal to zero. And this quantity $C \sin \dfrac{rj\pi}{N+1}$ can be rewritten slightly differently. It can be written as $C \sin(r\pi - \beta_r)$, where this quantity β_r is equal to $\dfrac{r\pi}{N+1}$. To get some information about the pattern of oscillation I also need what is A_{r+1}.

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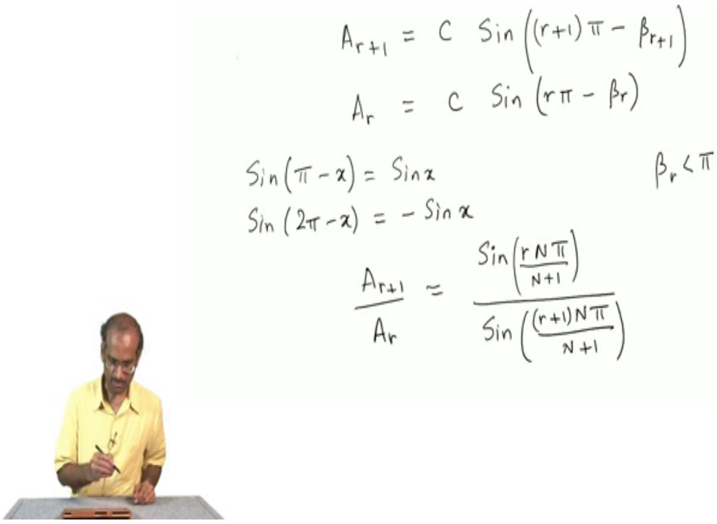

Let me for compression also write A_r which will be $C \sin(r\pi - \beta_r)$. Remember that r is an integer going from 1 to N. So, if A_r is $\sin(\pi r - \beta_r)$, A_{r+1} comes with $\sin(r\pi - \beta_r)$So, in other word if r happens to be a odd number $r + 1$ would be even number, or if r is a even number $r + 1$ would be odd number.

So, under all the circumstances if I take A_{r+1} divided by A_r, the ratio will always be negative, simply because one of them is always going to throw up a negative sign for a given value of r. So, which means that the neighbouring amplitudes are always going to be negative of the other; so, there is always going to be a negative sign in this ratio of A_{r+1} divided by A_r. So, with all these information now we can roughly plot the pattern of oscillation.

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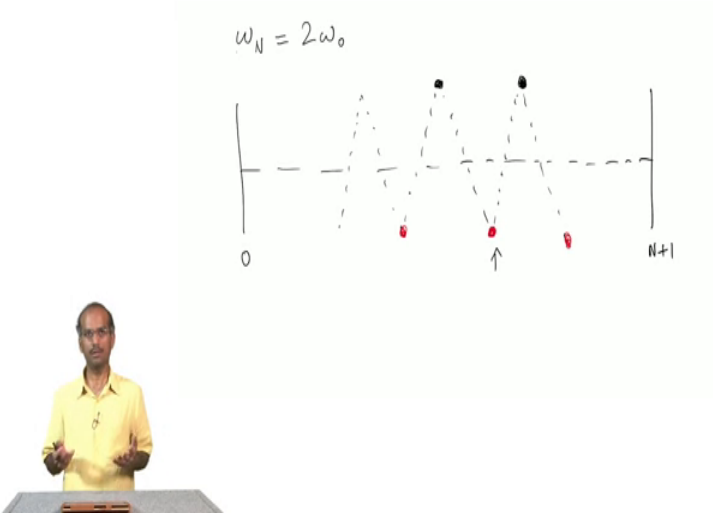

And we can in general also look at what is the overall profile of these oscillating particles. to do that we just need to go back to the formula which is the ratio of the r th and $r+1$th amplitude which is written down here.

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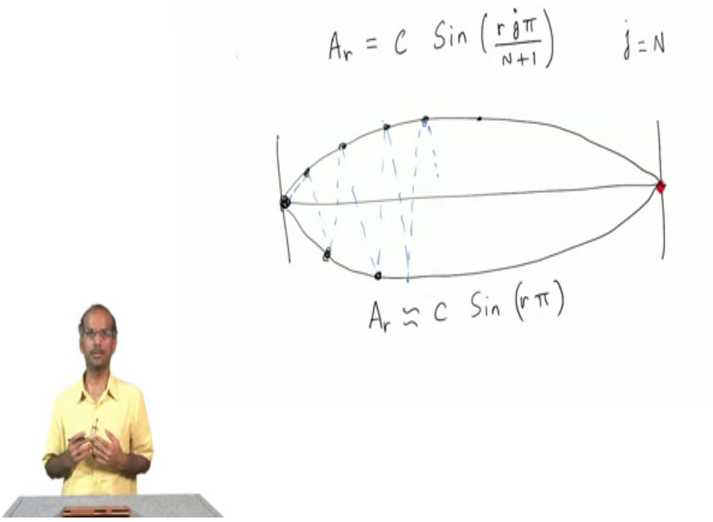

Again in the limit when N is large, so I will take N divided by $N+1$ to be equal to 1 approximately. So, I have $A_r \approx C \sin r\pi$. And when r is zero which is this point and r_{N+1} is this point, so both are points at which the amplitude is zero. So, which means that this should describe for me and ampli overall profile, that is like this. So, this is for the alternative sequence of particles which might probably be something like this.

Now, you can do a similar analysis for A_{r+1} and it will tell you that the equivalent profile might look something like this and that would correspond to particles here. So, now, you can draw the overall picture. So, the alternative particles in this are always out of phase by π with respect to each other. The neighbouring once maintain a phase difference of π. So, this more or less completes the coupled oscillator system. In the next module we will look at how we can take the limit when the distance between the particle goes to zero and go to the continue limit.