**Waves and Oscillations**
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**Lecture - 22**
**Solutions for Loaded String**

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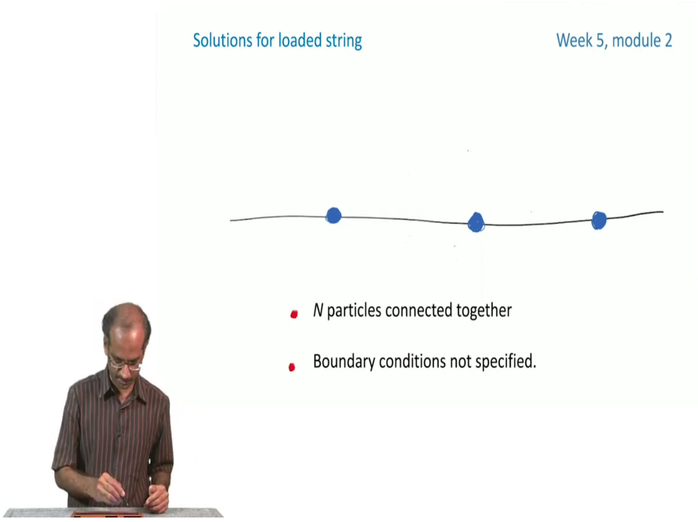


Welcome to the second module of the 5th week. We will started with loaded string. So, in this you have a collection of particles which are connected by a string. So, you should really imagine it as a collection of coupled oscillators; coupled in this case through a string, but it could be coupled by any other means. So, this is a sort of abstract system that we are studying to understand what happens when you put together large number of particles and couple them.

So, the central idea is that you have $N$ of these particles capital $N$ number of particles and we have not really said anything about what the boundary conditions are, maybe the string is tied at the two ends, but that will be left to each problem. So, when we attempt the problems, we will specify the boundary conditions and take the solutions as it comes, ok. So, right now at this stage we are not specifying what happens to the boundaries of this loaded string.

So, let us quickly recap what we did with this loaded string.

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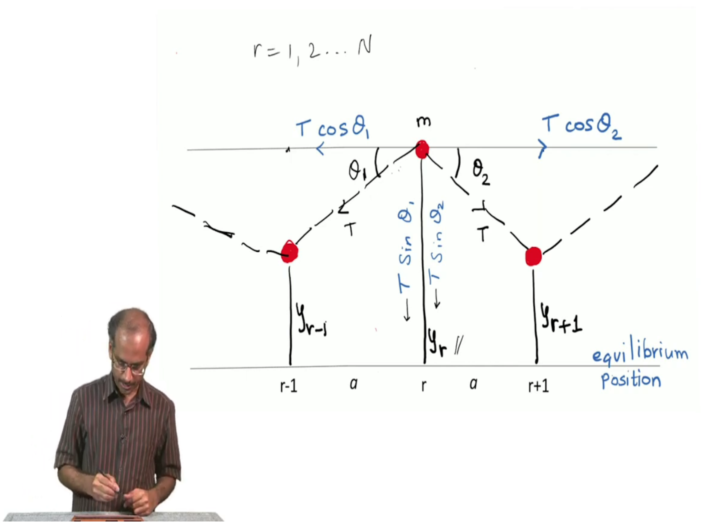


So, what I have here is the figure that was shown to you in the last module. So, there are these three beads or three particles which are in red in color, and $r$ is the index that numbers the particle. So, it goes from 1 to $N$. And what I have is one snapshot of a possible oscillation that three of these particles are executing. So, one of them makes an angle $\theta_1$ with respect to horizontal and the other one makes an angle $\theta_2$, and I have an equilibrium position, all the oscillations are up and down, above and below this equilibrium position.

So, if you do not do anything to this string, it will simply settle at the equilibrium position. Once you give it a tap, it will start oscillating about the equilibrium position. In particular we are assuming that all the oscillations take place in one vertical plane. So, there are no components in perpendicular directions, and we have denoted three possible displacements here of the three particles. So, index by $r$ which also incidentally specifies the position of the particle in our assembly.

So, $y_{r-1}$ is the displacement of the $r-1$ th particle, $y_r$ is the displacement of the $r$ th particle, and $y_{r+1}$ is the displacement of the $(r+1)$th particle. And we have assume that there is uniform tension $T$ in the string. Without tension of course there is not going to be any oscillations. So, you need tension, it is this tension which provides the restoring force for oscillations to take place.

So, what we did is to look at a particular configuration like this, and we realize that in the horizontal directions the tension components which are $T\cos\theta_1$ and $T\cos\theta_2$ in the limit, this is very important. In the limit that $\theta_1$ and $\theta_2$ are very small, they are equal and opposite in direction, and so they cancel out. So, there is no horizontal movement of these particles, there is only vertical movement.

There is a net vertical component which is directed downwards, which is $T\sin\theta_1$ and $T\sin\theta_2$, they add up together. And both $\sin\theta_1$ and $\sin\theta_2$ can be replaced in terms of $y_{r-1}$ $y_r$ and $y_{r+1}$ purely from the trigonometry and geometry of this configuration.

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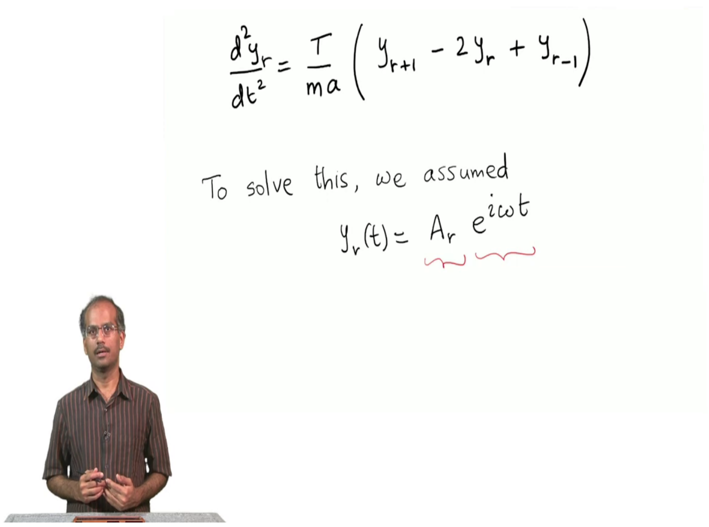


So, if you do all that and rearrange the equation, finally you get an equation of motion for this coupled system. So, this is something that we derived in the last module. So, we will note do that at again. But you will notice that what we have written down is an equation for the displacement of the $r$ th particle. So, it says $\frac{\partial^2 y_r}{\partial t^2}$. But you will notice that this displacement of the $r$ th particle depends on the displacements of its neighbors, the displacement at $r + 1$ th position and $r - 1$th position, so that is a sign that it is a coupled system.

So, what happens at one point or displacement at one point, it is related to the displacement at neighbouring points within the approximations that we are considering. So, it is indeed a couple system and to solve this we assume that $y_r(t)$ which is a solution that we are looking for a displacement as a function of time for the $r$ th particle or a particle at position index by $r$ is given by $A_r$ multiplied by $e^{i\omega t}$. So, $\omega$ is the frequency of oscillation.

So, there are two things to note here in this solution, one is that this $A_r$ is the amplitude. And the amplitude depends on the position of the particle which is why it says $A_r$, $r$ is the index of the position. So, the amplitude is not constant for all the particles in your system, it depends on where the particle is. And then most importantly this $e^{i\omega t}$. So, omega is the frequency of oscillation and the frequency of all the particles is the same.

So, whenever we look at a collection of particles like this, coupled together, executing oscillations, we are interested in what is this normal frequency, what is the frequency with which the entire collection oscillates. So, we are not really interested in one particular particle what is it doing ok. So, we are interested in this collection. So, our solution; the assumed solution is $A_r e^{i\omega t}$. So, we are going to start from this point onwards and obtain general solution for this equation.

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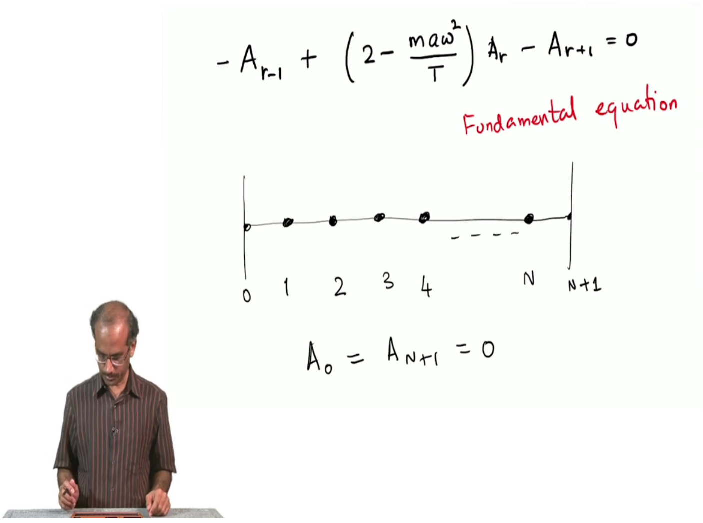


We had obtained this so called fundamental equation. So, it relates the amplitude at $r$ th point to the amplitudes at $r-1$ and $r+1$ position. For doing this part of the problem, I am going to assume that the string is tied at two ends so, it is going to look something like this. So, I have a string and so it has these particles which are equally spaced. So, I will number it as 1, 2, 3, so that is the index $r$ which takes numbers 1, 2, 3, 4 up to $n$. So, if you want you could imagine that there is a particle here and here, but they do not oscillate. So, I will call it 0th particle and $n+1$th particle.

So, if I am going to assume a configuration of this type where the string is tied at both ends, I can specify the boundary conditions as $A_0$ which is the amplitude of the 0th particle is equal to amplitude of the $N+1$th particle, and it is identically equal to 0 simply because these two particles at the extreme ends do not oscillate, so everything else oscillates. So, this is the conditions under which we are going to derive our result.

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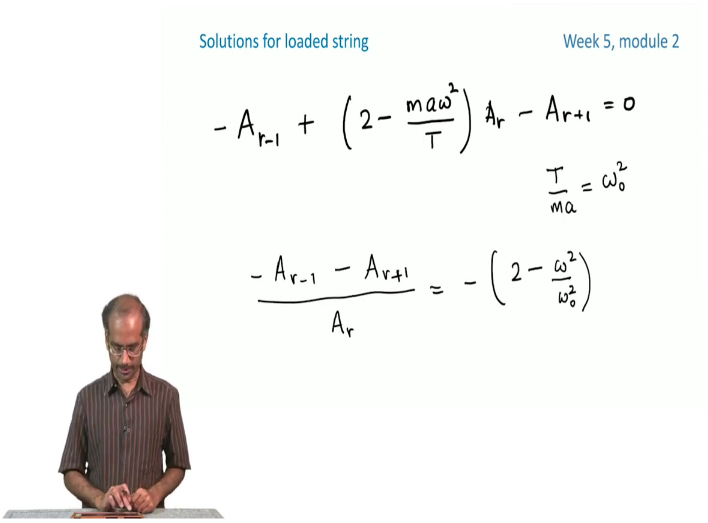


In this equation you will recognize that this quantity $\dfrac{T}{ma}$ is equal to $\omega_0^2$, something that we designate as natural frequency to put it that way. So, if I do this, I can rewrite this equation differently. So, I am going to keep all the terms involving $A$; $A_r$, $A_{r-1}$ and $A_{r+1}$ on one side and move everything else to the other side of the equation. So, I am going to have something like this. Now, let me change signs overall and I will be able to get the following equation.

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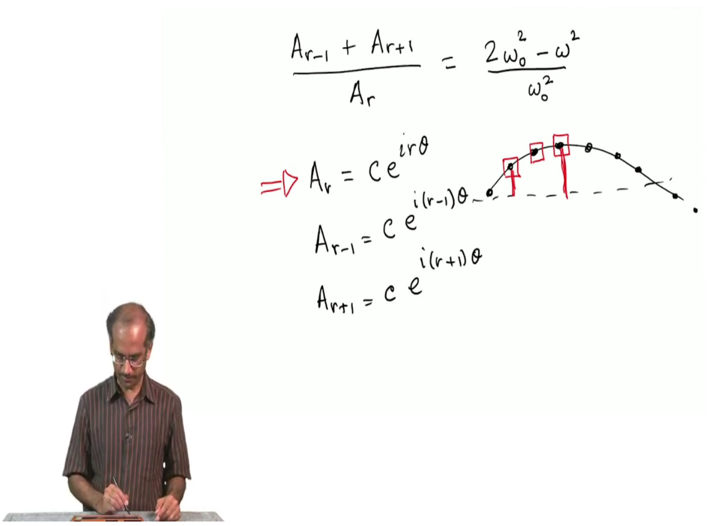

So, I am going to assume that this amplitude $A_r$ amplitude of the $r$ th particle is $C$ which is a constant times $e^{ir\theta}$. If you actually visualize string with beads which are oscillating up and down, you would see a pattern maybe something that might look like this. So, the successive amplitudes of the particles maintain a small constant phase difference with respect to their predecessors. So, if you say look at one particular particle here, and look at what is happening to this, they are separated by some they are off by phase so is the one which is proceeding it and so on.

So, what we have captured is essentially this information that at $r$th position the amplitude is some constant value, but it is multiplied to something which has a phase to it. So, at some certain values of phase, this quantity here could be one in which case it has the maximum amplitude something like may be this one here. It has the maximum amplitude, but at some other points maybe let us say for instance in this case, the amplitude is smaller than any other particles at least these two particles. So, it is a reasonable assumption to make. Now, our next step is to simply substitute this assumed form for $A_r$ into this equation that we have.

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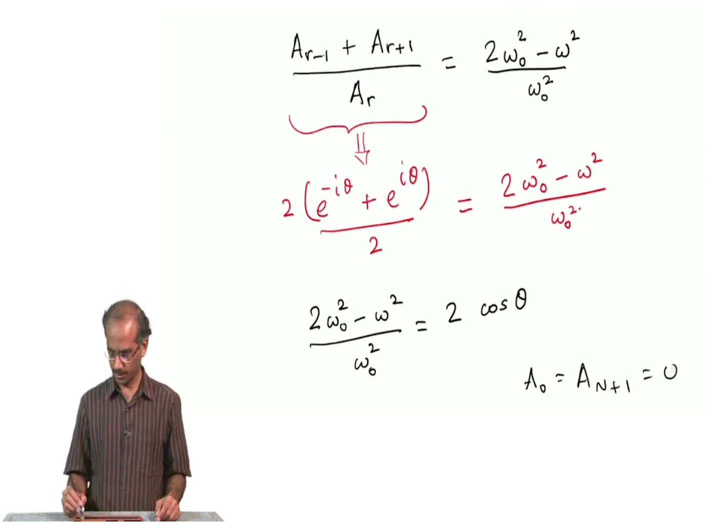


When I substitute all that in this part of the equation, it is a simple exercise I argue to do it yourself. You should be able to show that, this quantity reduces to $e^{-i\theta} + e^{i\theta}$. And clearly it is in very suggestive form; I can multiply this by 2 and divide by 2 and of course, on the other side, this is equal to $\frac{2\omega_0^2 - \omega^2}{\omega_0^2}$. So, the quantity here is $\cos\theta$. So, my final result is; if you notice this quantity $A_r$ which is $Ce^{ir\theta}$ that we introduced $\theta$ is unknown, it is a phase, ok. And we need to determine that phase and we will do that using the boundary conditions. And notice that till now we did not make use of the boundary condition. So, this is the point when we will use the boundary conditions which is that $A_0 = A_{N+1}$ both are equal to 0.

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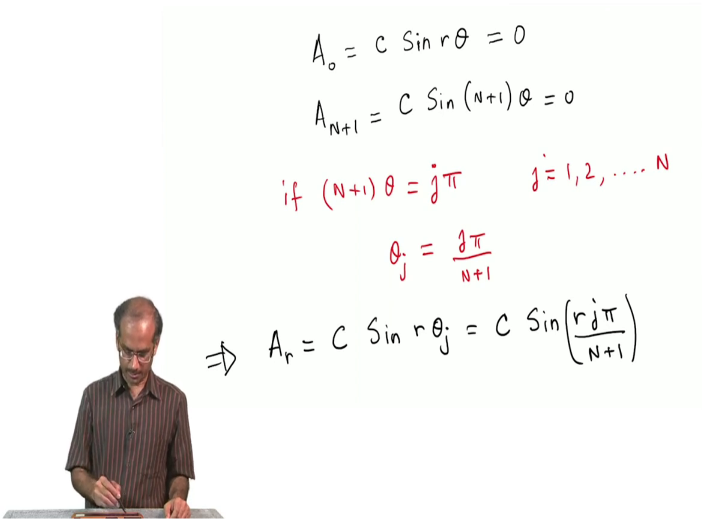


So, if I need $A_0 = 0$, I should have $A_0$ to be $C \sin r\theta$, because if $r$ is 0, $\sin r\theta$ is 0 and it will automatically satisfy the boundary condition that $A_0$ is equal to 0, so that is one useful thing that we can get from putting in the boundary condition. And for $A_{N+1}$, I could write it as $C \sin(N+1)\theta$ and that is equal to 0. It is equal to 0, because boundary condition dictates that $A_{N+1}$ should be equal to 0.

Now, under what condition would this be equal to 0; so, this will be equal to 0 provided $(N+1)\theta$ is equal to some integer times $\pi$, so that is the condition when this will be equal to 0, and of course, $j$ will be equal to and this so on. From this, we can write expression for $\theta$.

So, now, I will call it $\theta_j$ to indicate that there is an integer $j$ involved here, which will index the possible values of $\theta$. So, $\theta_j$ is $\dfrac{j\pi}{N+1}$. And in general now it is possible to write an expression for $A_r$. So, now, that I have an expression for the amplitude of the $r$th particle. The next step simply is to substitute all that in this equation. So, we know what is $A_r$, we know what is $A_{r-1}$ and $A_{r+1}$, substitute it here and extract $\omega^2$ from that, so that is the next step.

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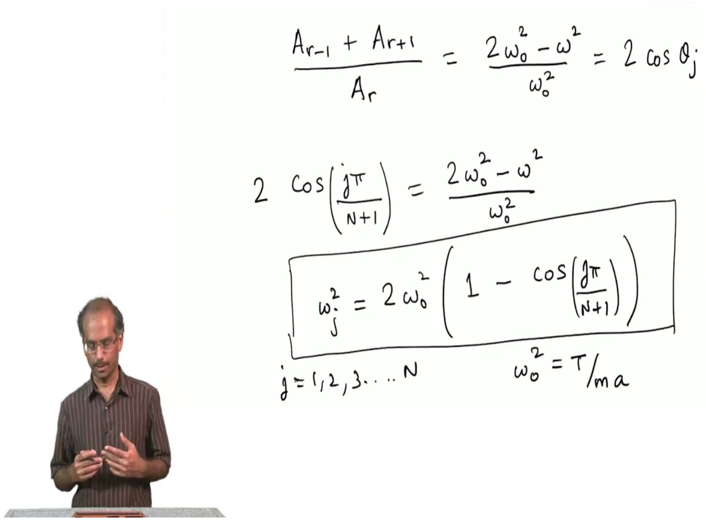


To, we know that this is equal to $2\cos\theta_j$, and we know the value of $\theta_j$. So, we have everything that we can use here. So, my answer would be $2\cos\theta_j$ is $\dfrac{j\pi}{N+1}$ from here. Now, we simply rearrange this equation to extract the value for $\omega^2$. So, let me do that, where small $m$ is the mass of each of those beads and $a$ is the distance between any two beads in our system. So, this is the required result for $\omega_j^2$. Now, we can do few simple manipulations on it and make it a little more elegant, ok.

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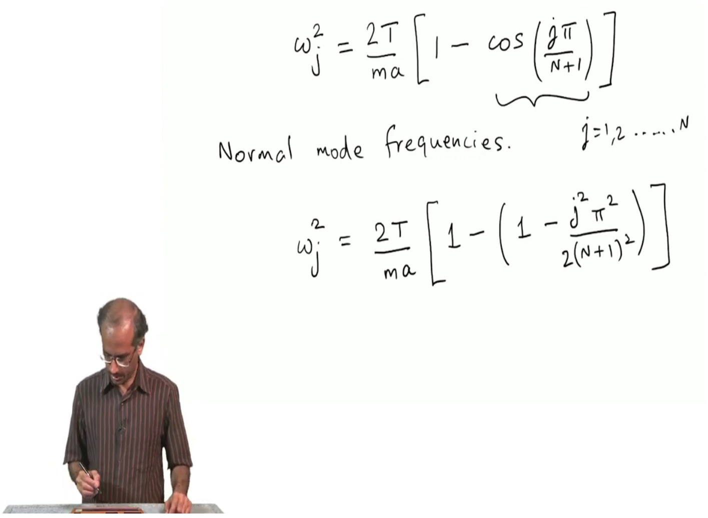


So, I have just read it in this expression for $\omega_j^2$ and I have substituted for $\omega_0^2$. So, this is the expression the general expression that we need to use and these are the usual normal mode frequencies for our problem. And here as you will notice we had $n$ oscillating particles in our system, and we have $n$ normal mode frequencies. So, $\omega_j$ where $j$ runs from 1 to $N$ basically tells you that there are $n$ possible normal mode frequencies.

So, we already saw a bit of this in the simpler version of the problems that we did in the previous module. Now, I can make it a little more attractive provided I assume that I want to look at small values of $j$ and may be typically $N$ is very large. In that case, this quantity here at least $\frac{j\pi}{N+1}$ will be small, so I can expand this in a Taylor series. So, all I have done is to expand $\frac{\cos j\pi}{N+1}$. And since I am assuming that $j$ is small, and $N$ is much larger basically we have large number of these particles. So, I am truncating the expansion with the first term up to the second term. Of course, here one and one would cancel, so I should be getting.

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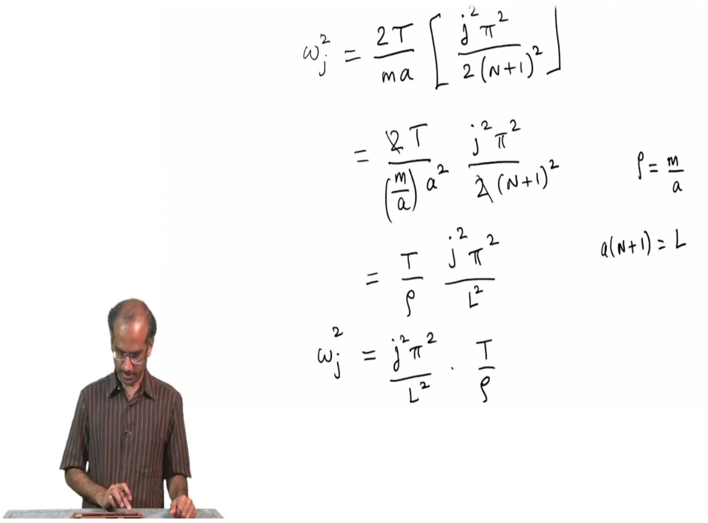


Now, here let me do small rearrangements. So, I am going to multiply and divide by $a$ in the denominator. So, I will divide by $a$ and multiply by $a$. So, that will make it a square. So, there will be of course, $\frac{j^2\pi^2}{2(N+1)^2}$. Of course, this 2 and 2 will cancel. Let me designate this $\frac{m}{a}$ which is mass divided by length as linear density, let me replace it by $\rho$.

In that case, I am going to have $T$ divided by $\rho$ into $j^2\pi^2$ divided by. So, I have a square $(N+1)^2$ in the denominator, so that is $a(N+1)$ is simply the total length of the string. It is easy to see that $a$ is the distance between any two particles, and you have $N$ of those particles. So, there are $N+1$ gaps. So, $a(N+1)$ is the total length of the particle which means that I am going to have $L^2$ here in the denominator. So, now everything looks nice. So, in the limit when $j$ is small and $N$ is larger $\omega_j^2$ is $\frac{j^2\pi^2}{L^2}\frac{T}{\rho}$.

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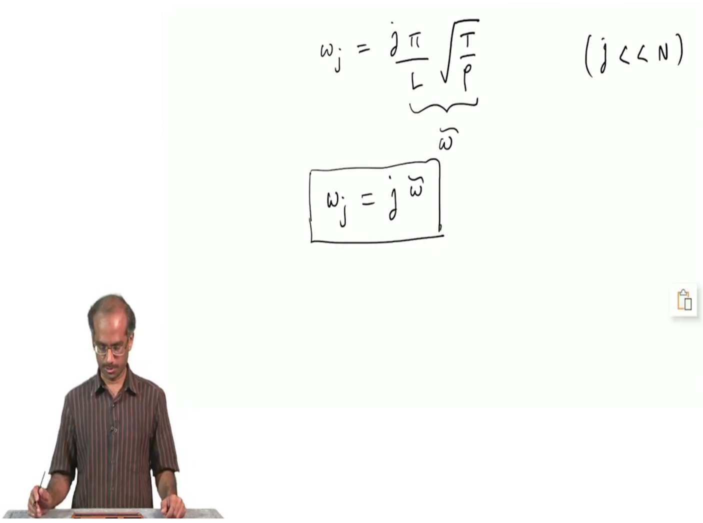

And this can be rewritten as so now I have written a expression for $j$ th normal mode frequency in the limit when $j$ is small compare to $N$. And you could do a little more here. So, you could identify this quantity as some $\widetilde{\omega}$ or what is often called the fundamental frequency. Therefore, $\omega_j$ is equal $j \times \widetilde{\omega}$. So, the successive frequencies are the first overtone as it is called is simply 1 multiplied by $\widetilde{\omega}$, second overtone would be 2 multiplied by $\widetilde{\omega}$ and so on. So, this is a little more elegant, but it is valid in the limit when $j$ is much smaller than $N$.

With this result, I will stop this modules lecture. And we will continue and look at the consequences of the all these results in the next module.