Waves and Oscillations
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Lecture - 21
Coupled Oscillators of Loaded String

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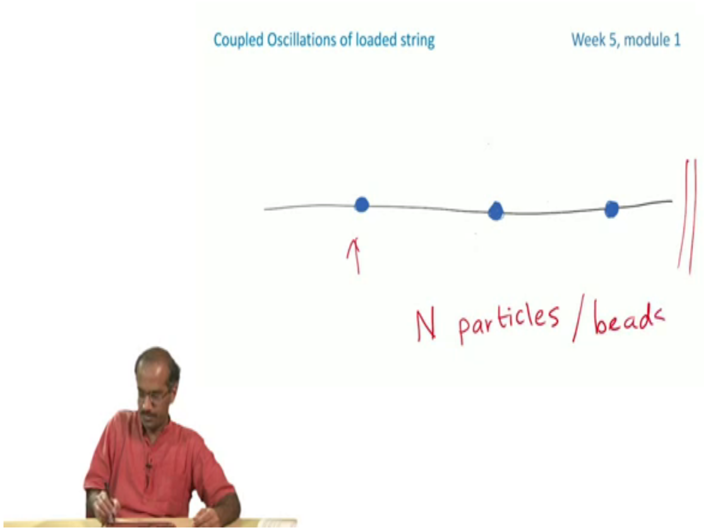

Welcome to the 5th week; this is the first module we are going to look at Coupled Oscillations of what is called the Loaded Spring; it is also a good point to take stock of what we had been doing.

So, we started by looking at simple oscillations and then in order to get more realistic; we added effects due to viscous damping. With oscillations and damping present, we also included the effects due to periodic driving and we saw things like resonances happened; resonances in displacement in velocity and so on. And then we studied coupled oscillations; once you say coupled system you do not worry about what an individual element is doing, you can also talk about that, but largely we are interested in what the entire system is doing as a whole.

So, in that context we introduced the idea of normal frequencies and normal modes which are basically the patterns of oscillations of coupled system. Now, we want to take that idea further; the idea of coupled oscillations. So, we want to go to a limit where a large number of particles are coupled together and in the appropriate limit they would may be form a string for example. So, starting point for that would be something like what I have drawn here.

So, you have what are called beads and they are tied let us say through a spring and if you really make a; make such a system which is very easy to do. And if you try and oscillate one of them or just disturb one of them very soon the one next to it also will get disturbed and a little later the one sitting next to that will also get disturbed.

So, what is happening is the disturbance that you created at let us say at one point here is spreading through the string and that is because there is coupling. So, if this is oscillating the one next to it also is going to oscillate and so on. So, again we have same kind of questions that we had faced in the previous chapter; what are the normal modes. So, we can slowly start progressing with 1, 2 and 3, but ultimately we are interested in N particles or N beads.

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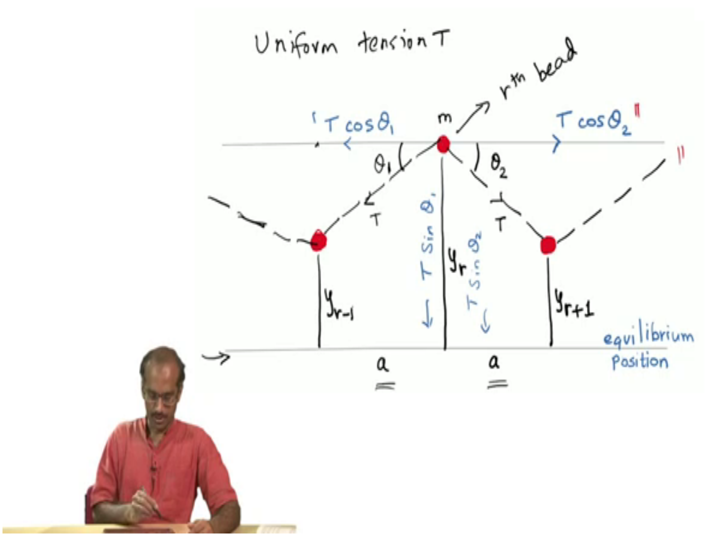

So, we will begin this by considering a small segment of string and beads. So, what I have here in front of me is three beads; which are coupled through a string. Normally, if you have such a system of course, you will probably tie it to walls and so on. So, those are what would be called boundary conditions. So, at this stage let us not worry about the boundary condition; so we are just focusing on this segment of the string. When you do not do anything the string is going to beat out. So, there is some tension T in the string.

I am going to assume that there is uniform tension T in the string. There is an equilibrium position when nothing is disturbed so, that shown here by the lower line. So, when nothing is disturbed that is where the system will rest. Now, if I pull one of the beads and leave it; it is going to start oscillating. And maybe at some point of time this is the configuration of the three beads like the one that I have drawn here.

So, I am assuming the following things that the beads are equally spaced. So, the spacing between the beads is a; the displacement of each bead is denoted by y and there is this index r; that you will see here which, which is the index for the number of the bead. So, for example, the middle bead which is shown in the figure is rth bead and on the left side and right side you have y_{r-1} and y_{r+1}th bead.

So, like this you could number your beads going from 1 some large value of N, but we are focusing on some three of them in between. And in this configuration let me also assume that this angle here is θ_1. So, we can now resolve this tension into two components. You would remember that when we were starting the problem of simple harmonic oscillations, we were always looking at the limit of small oscillations ok.

So, here again we are going to look at the limit of small oscillations which means that tθ_1 and θ_2 are really small. So, in the limit of θ_1 and θ_2 being small; the two horizontal components which is $T\cos\theta_1$ and $T\cos\theta_2$; they would be equal and oppositely directed.

So, there is no net force on the bead in this horizontal direction which means that the only possible dynamics for this bead is to go up and down. So, it is in a plane, it goes up and down; so that is part of our assumptions. So, the net force downward would be sum of these two forces; $T\sin\theta_1$ and $T\sin\theta_2$. And here in this case, the tension provides the restoring force and what we have written down is $T\sin\theta_1$ and $T\sin\theta_2$ is just the restoring force. So, with this we can now write down the equation of motion.

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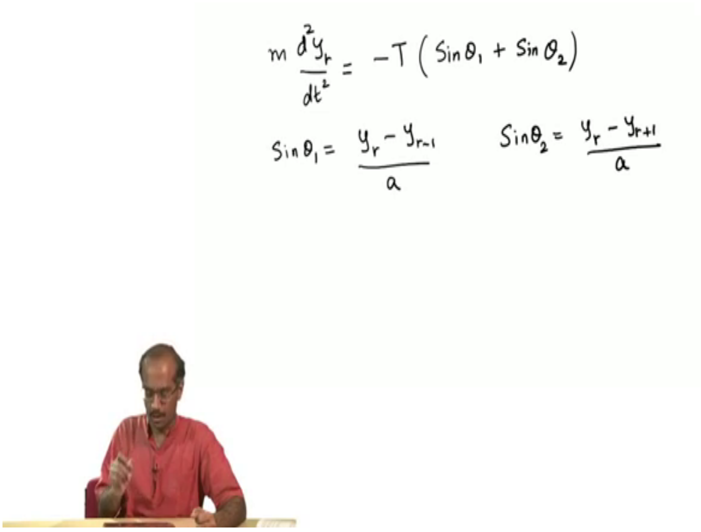

So, here I have written down the equation of motion for the rth bead; I have assumed that all the beads have mass m, T of course, is the uniform tension in the string. Now, if you go back to the figure that we had seen while back; you will notice that we can obtain an expression for $\sin\theta_1$ and $\sin\theta_2$ in terms of these displacements.

So, for example, from this figure it is clear that $\sin\theta_1$ would be equal to $y_r - y_{r-1}$. And in the limit of theta being very small θ_1 and θ_2; we can assume that the length has not changed much; so I will take it as a itself. Similarly, I write an expression for $\sin\theta_2$ that would be $y_r - y_{r+1}$; divided by a. Now, all we need to do is to substitute these two expressions back in our equation of motion and remember that this is the equation of motion for the rth particle.

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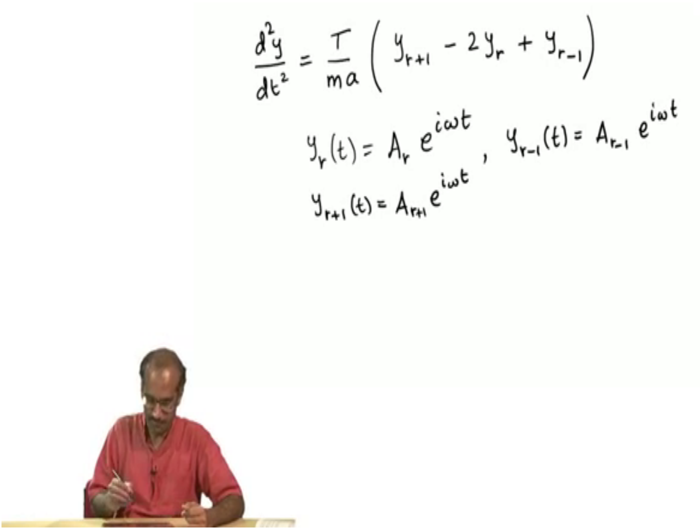

So, now I have my equation of motion and the next part is to actually solve this. We are going to adopt the same kind of general technique that we adopted in the last week. I will write by assuming that y_r which is the displacement at the rth position is going to; of course, be time dependent.

It will be A_r which is amplitude $e^{i\omega t}$ and similarly I can write equation for y_{r-1} and y_{r+1}. Now, we will substitute these three expressions for the displacement back in our equation of motion. If you do that, I will get the following equation it is a simple exercise, so I asked you to check that yourself.

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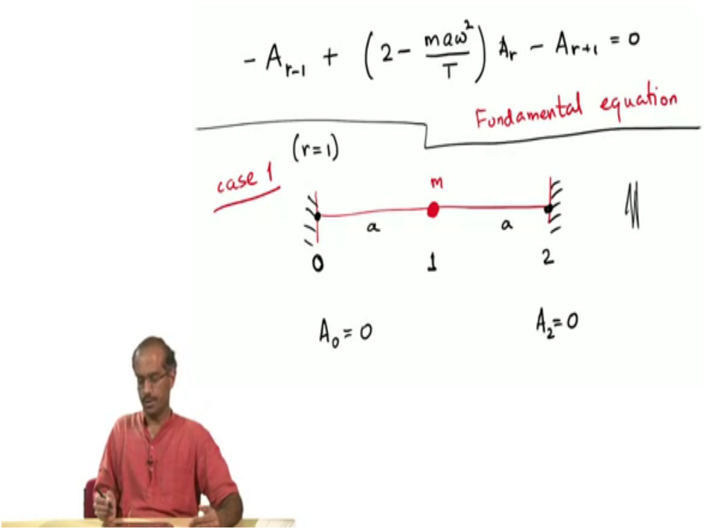

This equation is often called the fundamental equation and you will see why because from this, we could pretty much work out everything that we need. Now, let us a check this for the case of a single bead that is the simplest problem one can think of; of course, it will be an system without any coupling to anything else, but nevertheless it is a simple case to check and I will assume that the distance from these two ends is a.

So, the string is tied to a rigid wall here at this point; there is actually a bead only at position one and if I give it a little bit of disturbance; it is going to oscillate up and down. So, here the boundary conditions are very clear at; at this point where the string is tied to the wall there are no oscillations; it is quite tight there. So, r is the index for position of the particle; so, we have only one particle.

So, we will go from zero to two and there is a particle at position one; zero and two are fixed. So, the boundary condition that we have taken is a reflection of this fact; since A_0 is zero and A_2 is zero; I am left with the following equation.

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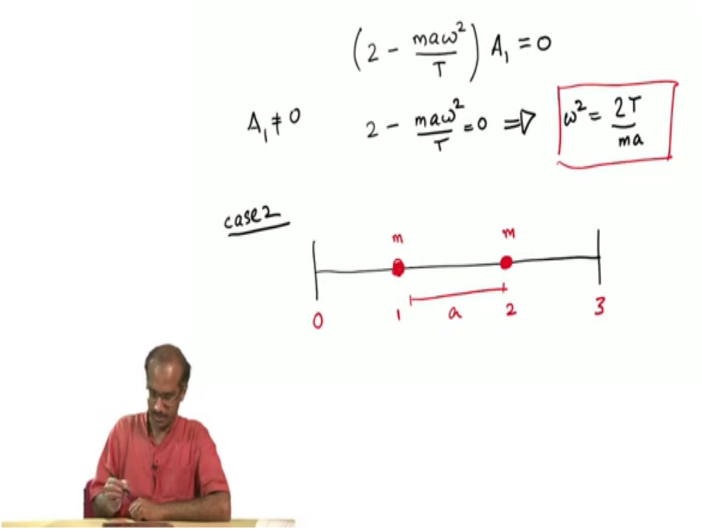

Now, we know that the A_1 is the amplitude of the bead at position 1 and the amplitude in general is not equal to zero. So, A_1 is not equal to 0 in general; hence the rest of the quantities in the bracket should be equal to 0 that is $ma\omega^2/T$ is equal to zero and this will give me an expression for ω^2. So, this will give me $\frac{2T}{ma}$.

So, here I have obtained the normal mode frequency for a single particle; so, it is really incorrect to call it normal mode frequency; it is simply the frequency of a one single oscillating bead. Now, let us go to the next level of complexity. So, we look at the fundamental equation, but again let us write the boundary conditions in this case.

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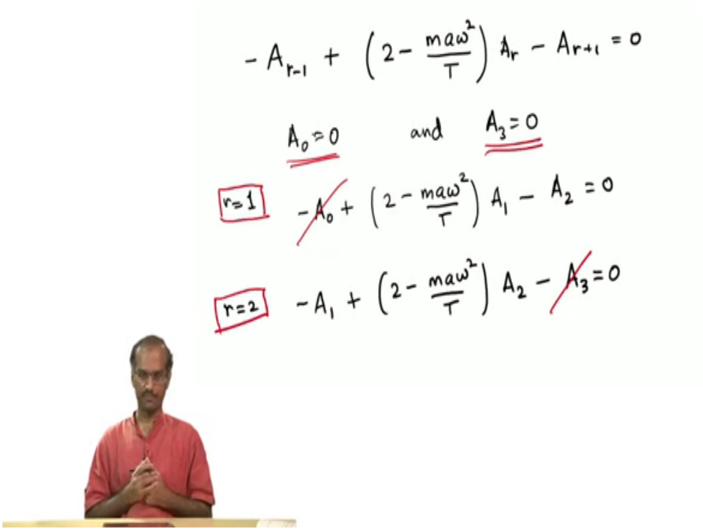

Now, let us right specific equations for the other two positions which is a bead at position 1 and a bead at position 2. So, let us say r is equal to one; if I set r is equal to one and put an r equal to 1 in this equation; I am going to get the following one.

So, all I have done is to simply substitute the value of r from here in this equation and I have got this equation. Now, let us do a similar exercise for r equal to two. So, simply put the value of r equal to two in the fundamental equation. Now, if you remember that our boundary conditions are A_0 equal to zero and A_3 equal to zero and which case this term will go away and this term will go away.

So, now if you look at the rest of the equations that we have; it is two equations and two unknowns the unknowns are A_1 and A_2. So, now we just need to solve for this as usual demand that we need nontrivial solutions which means that A_1 and A_2 is not equal to zero; if we do that we should be able to get two different values of ω^2.

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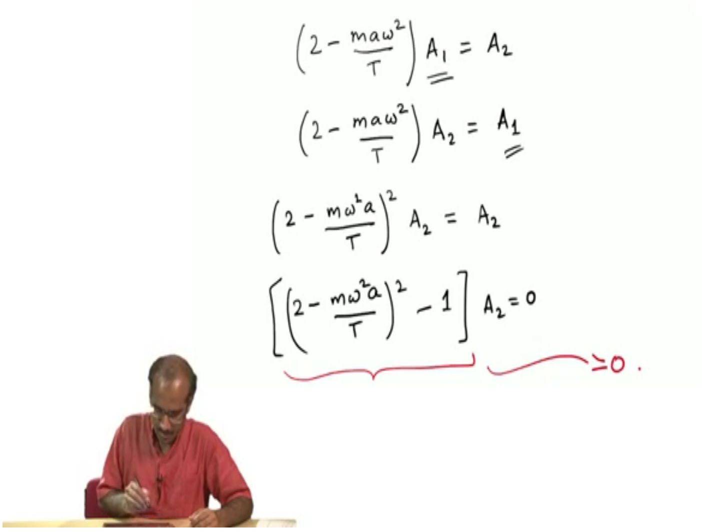

So, I have two equations and two unknowns. So, I can substitute for this A_1 from here; if I do that I am going to get,

$$\left(2-\frac{m\omega^2 a}{T}\right)^2 A_2 = A_2$$

And since A_2 itself is not equal to zero; the quantity here within this square bracket would be equal to zero. Now that the quantity inside the bracket is equal zero; so it is of the form like $A^2 - B^2 = 0$. So, I can split it as $(A+B)(A-B)$ and that would give me the following equation.

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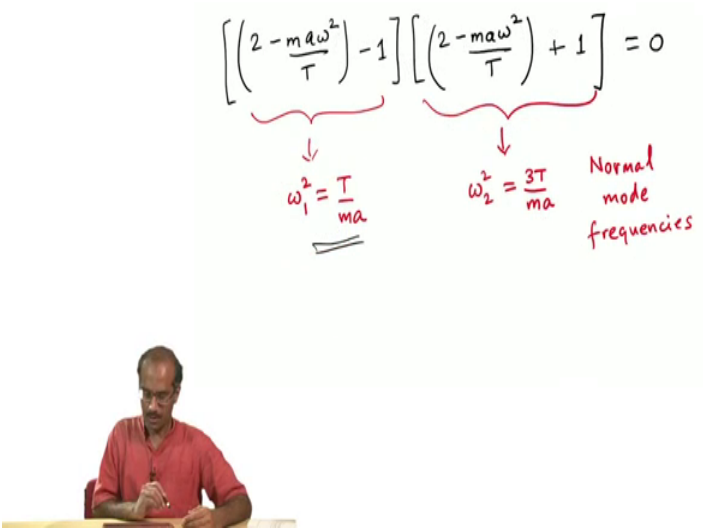

And in this case each of them would individually be equal to zero; in which case I can set each of the term equal to zero and write an expression for ω^2. So, if I do that for let us say this case; it will give me the following expression for ω^2.

So, let me call it ω_1^2 that would be $\frac{T}{ma}$ and if I set this equal to zero; I am going to get the second normal mode frequency which would be ω_2^2 would be equal to $\frac{3T}{ma}$. So, now in the case of two beads, two oscillating beads and I have two normal mode frequencies.

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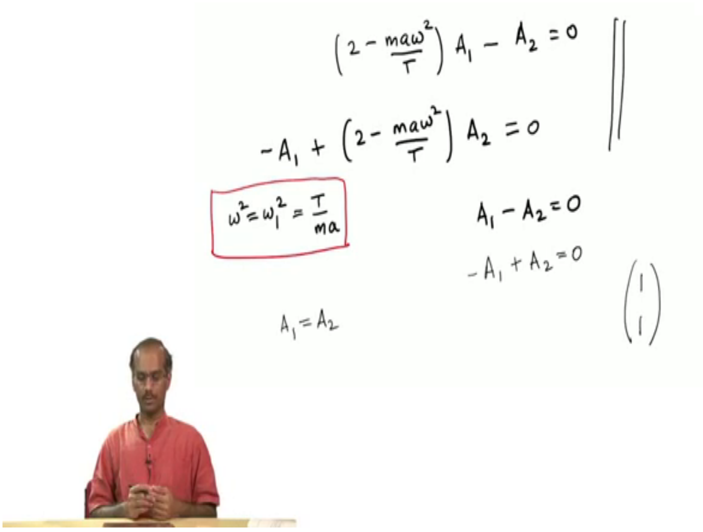

We can also find the normal modes starting from the equations of from the fundamental equations, we substituted the boundary conditions. And after putting in the boundary condition this is the two sets of equation that I have. And now to get the normal mode for each of these cases; I need to substitute for ω^2 here.

So, let us substitute ω^2 to be equal to ω_1^2 and remember that we have already said that ω_1^2 is equal to $\frac{T}{ma}$. So, this is my first normal mode frequency and in this case now if I substitute ω_1^2 equal to $\frac{T}{ma}$ in these two equations; you can see what would happen. We have only one equation; so it can be easily satisfied if we choose A_1 to be equal to A_2; so that would give me a normal mode to be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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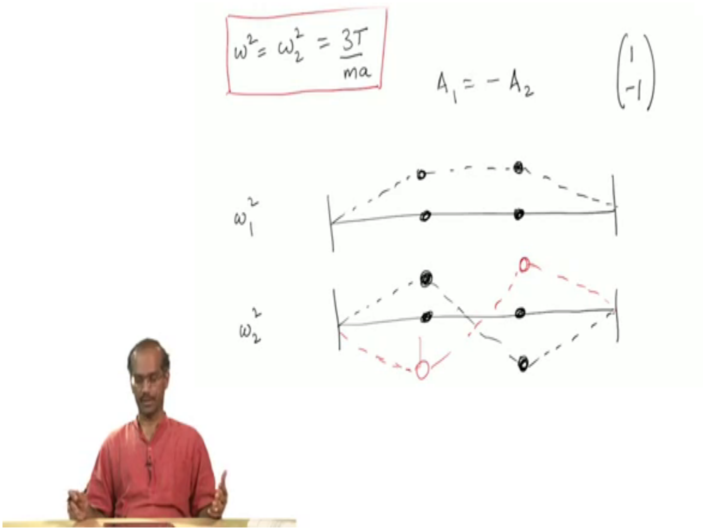

And similarly, if I do the same kind of analysis for the second normal mode; I will have to substitute ω^2 equal to ω_2^2 which is equal to $\frac{3T}{ma}$. In this case, I will leave it to you to try it out yourself; it is exactly the same way that we did for the first normal mode frequency. And if you do it correctly, you should be able to show that A_1 is equal to $-A_2$.

So, this would correspond to the normal mode that is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. And now if you want to transform this result in physical terms; it is equivalent to saying that the first normal mode whose normal mode frequency is given by ω_1^2 will correspond to; both of them will go up together, come down together, go up together and so on; so that is their pattern of oscillation.

On the other hand, in the case of second normal mode frequency their pattern of oscillation is; if one is up the other one would be down. Now, this is not a viable way of solving it if you have a large number of beads. In the next class, we will look at how we can obtain a general solution so that we can scale up obtaining the normal mode frequencies is a normal modes themselves; even for a general case of N beads which are connected by a string.