Waves and Oscillations
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Lecture - 20
Coupled Oscillators:
Problems

Welcome to the last module of this week. So, in the problems that we did earlier we had skipped that small portion about finding the normal modes we found the normal frequencies. So, in this module we will try and spend some time obtaining the normal modes for the coupled oscillations. So, just to make a very brief and quick recap of what we had done. So, we started with the coupled system of equation.

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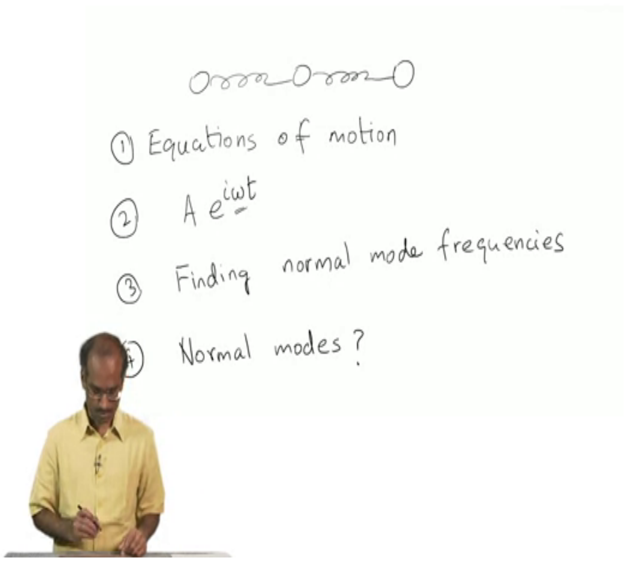

So, it could be for instance something like 2 or 3 oscillators or particles which are coupled by a spring. So, I have in front of me one coupled system. So, the way to do it is, you write the equations of the motion first and then see if by a simple trick of adding subtracting whatever you can uncouple the equations. If you have managed to do that I have quite luckily solved the problem if I have not done that, then start by assuming solutions which are of the form $A e^{i\omega t}$.

For each of the particles you assume that the displacements are of this form and crucially the important point is that, this ω that I have here has to be same for all the 3. So, if I set it to oscillation all of them together are going to show me one possible frequency of oscillation. So, again I stress the different part of couple system cannot in general operate under different frequencies at least for the kind of problems we are doing.

So, once we determine these different possible values of ω, angular frequencies. So, we solved one part of the problem which is finding normal frequencies or normal mode frequencies. One part that is left is to find the normal modes themselves. So, in simple terms normal modes are simply the collective pattern of oscillation that are shown by your coupled system. So, when we have a coupled system it could be made of two particles, 3 particles, a large number may be in general n particles.

So, when you see something like this we are not really worried about what is one particle doing out there. So, you are asking question about what is the kind of dynamics that is exhibited by all the particles collectively together. So, we are interested in collective oscillations ok. So, these normal modes are different possible collective oscillations collective dynamics that can be exhibited by a coupled system.

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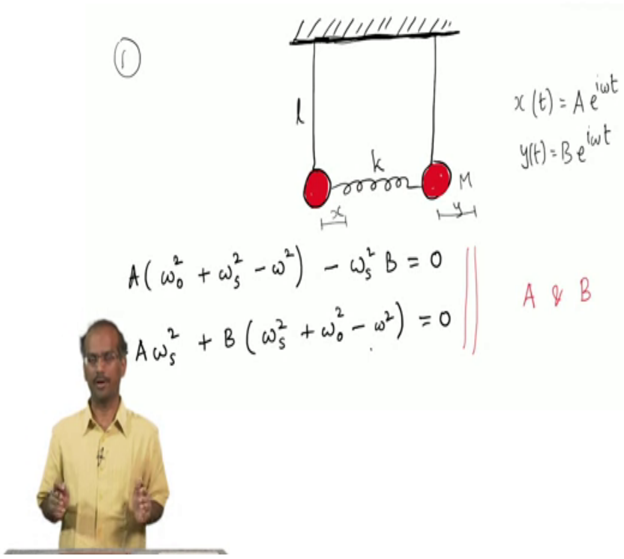

So, now let us go to the a part about finding the normal modes. So, the first problem that we did was this problem of two pendula and their independent pendula, but once you coupled them with the spring whose spring constant is k, then they become a coupled system. So, you assume that a mass of the two bobs are same and it is equal to m and l is the length of the string of each pendulum and it is equal for both the pendulum ok. And, if you remember so, we wrote down the equations of motion for this system, and then we assumed solutions of the following form. So, we said that so, we called one of them as the x pendulum with some displacement x and the second one we said is a y pendulum we called it y pendulum and it has displacement y.

I assumed solutions of the form $x(t)$ is $Ae^{i\omega t}$ and $y(t)$ is $Be^{i\omega t}$. So, you substitute this in the equations of motion and finally, write everything in terms of coefficients of A and B and you will end up with this sort of equation that I have here ok. And, we wanted to solve for the two unknowns which are there in these equations, the two unknowns are A A and B and as you can see a very trivial solution for this cases when A is equal to zero and B is equal to zero.

But we do not want that trivial solution simply because it would simply mean that there is no oscillations because the amplitudes are zero all the time, there is no oscillation we do not want such a solution. And this solution will always exist, but we should ignore that solution. Other values of A and B which is non-zero for which this two sets of equation are satisfied. So, you rewrite this equation here the one that is written here in matrix vector form very simple to do that and let us call this matrix M and χ as the vector for our vector made up of A and B.

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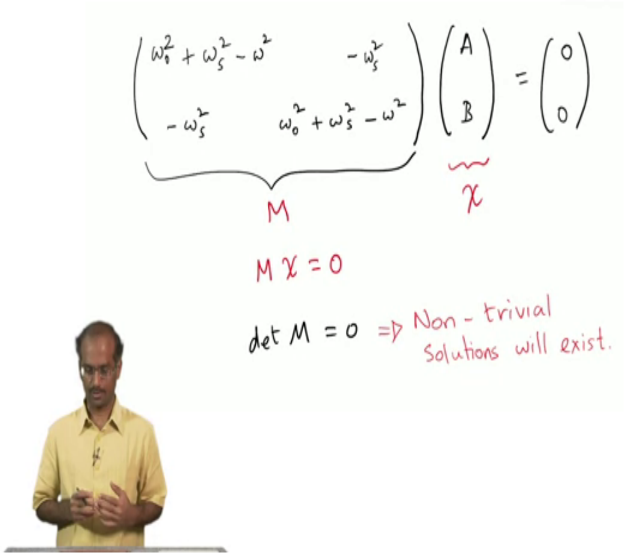

Then this matrix vector equation can simply be written as

$$M\chi = 0$$

and for non trivial solutions to exist determinant of M should be zero. we were able to obtain the values of normal mode frequencies.

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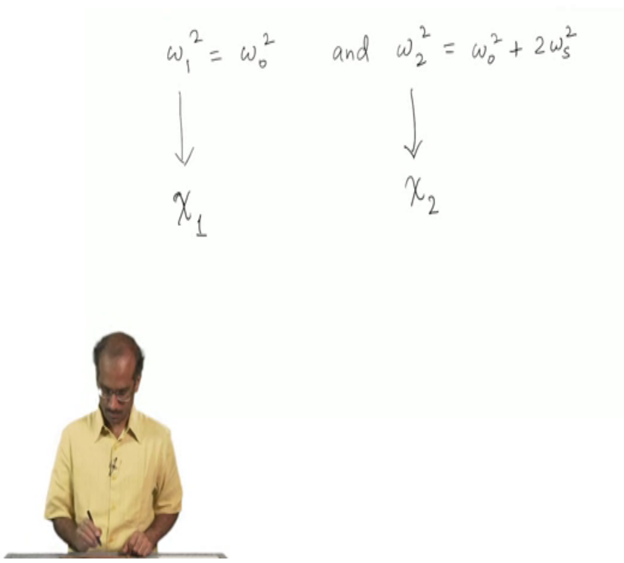

And, for this problem if you remember clearly we obtained two possible frequencies one is ω_1^2 which is equal to ω_0^2 and the second frequency was ω_2^2 which is equal to $\omega_0^2 + 2\omega_s^2$. So, these are the possible normal mode frequency that we had obtained. So, we had already solved this part of the problem.

Now, I will find the normal modes themselves and for each of these normal mode frequencies there will be a normal mode. So, corresponding to this there will be normal mode, let me call it χ_1 and corresponding to ω_2 there will be a second normal mode let me call it χ_2.

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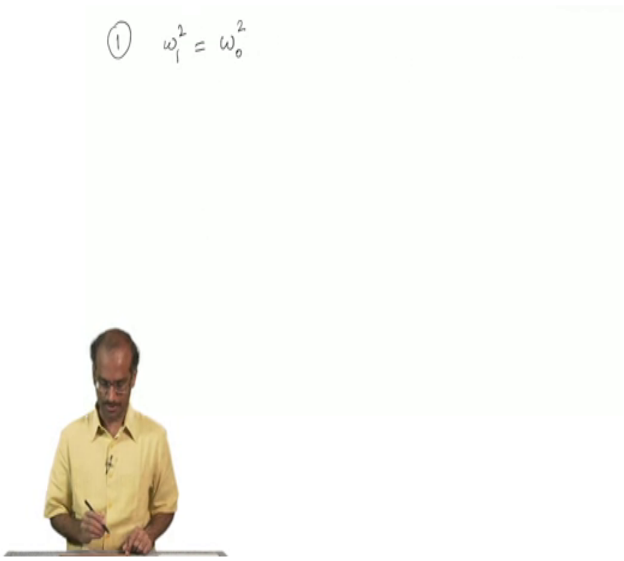

So, let us start with ω_1 we will substitute this in our matrix vector equation.

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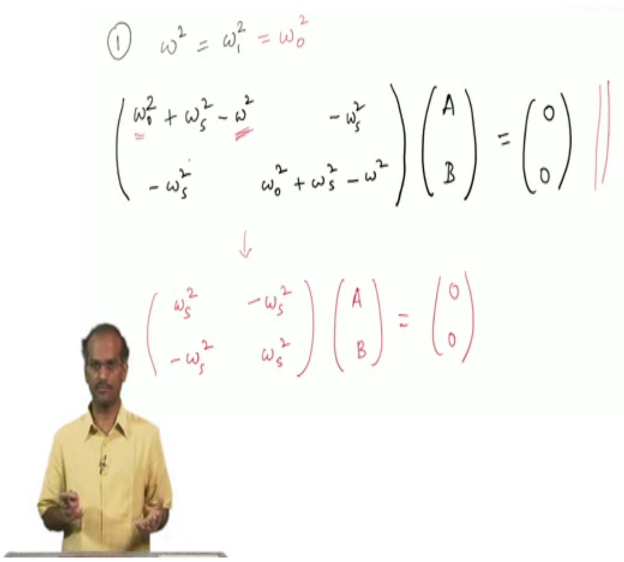

So, I have just copied the equation that we had before. Now what I need to do is to substitute for this value of ω^2 by ω_1^2. So, I would get the following matrix vector equation now I need to solve this for A and B.

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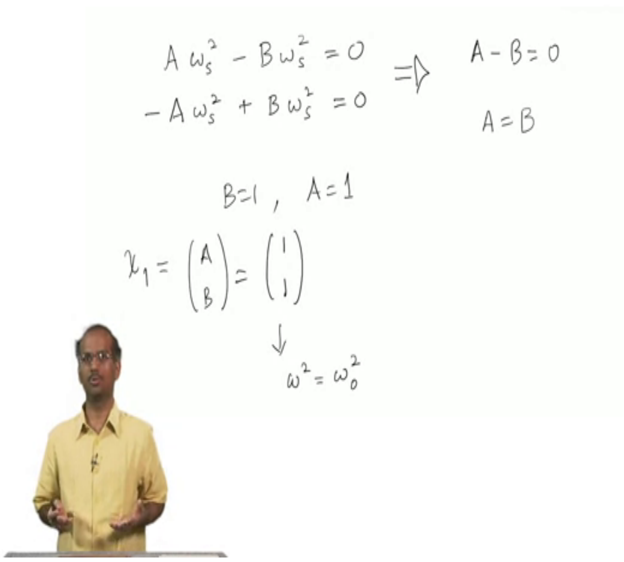

So, if you do that overall change of sign in second equation that is exactly the first equation. So, this equation can be rewritten in a easier form $A - B$ is equal to zero. So, I can cancel of ω_s^2 throughout it is $A - B$ equal to zero and this implies that A is equal to B

and since you have freedom to fix one of them I can say that B is equal to 1 in which case A will also be equal to 1 and the solution that I get χ_1 will be $\begin{bmatrix} A \\ B \end{bmatrix}$ which will be equal to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So, this is the normal mode corresponding to the frequency, which is equal to ω_0^2. Now, similarly I need to go through a similar procedure to find out the second normal mode frequency let us do that.

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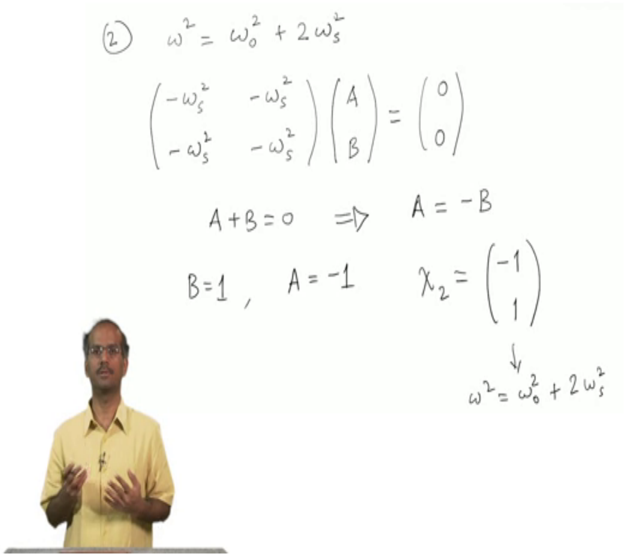

So, going back to the equation, now for the second normal mode the frequency is ω^2 will be $\omega_0^2 + 2\omega_s^2$. Now substitute this value of ω^2 in this equation as we did before and if you do that and cancel off certain ωs here is what you will get. So, this is the equation that we have and now if you rewrite it as a standard linear equation this would simply be $A + B$ equal to 0.

And this would imply that A is equal to $-B$ as usual this case we have 1 degree of freedom. So, if I set B is equal to 1, A would be equal to -1. So, now, I have my second normal mode χ_2 which will be $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. I have now both the normal modes in I can now write down the general solution.

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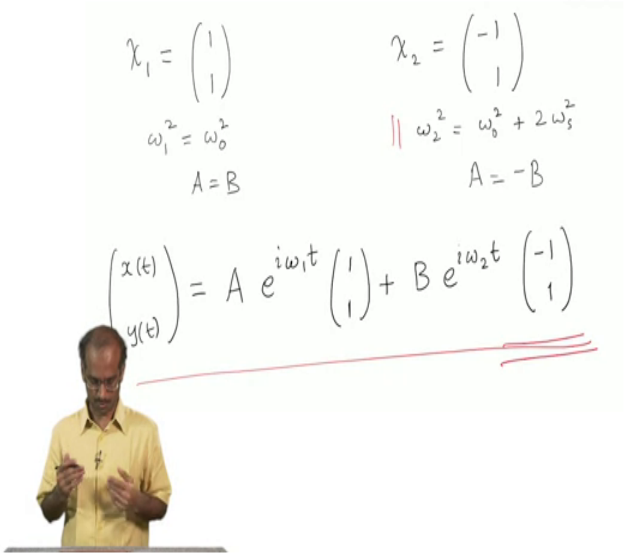

Before we write down the general solution let us look at the physical content of the results that we have in the first normal mode is this $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and if you remember where it came from, it came from the result that A is equal to B, A and B are amplitudes. So, here this is telling me that the two amplitudes of the two pendula are exactly equal at all times. So, the physical picture is their displacements are exactly like this. They are in phase ok.

So, this is one of the possible collective mode of oscillation for this system. So, they go together. So, that is this normal mode. On the other hand if you look at the second normal mode χ_2 corresponding to the value of second frequency ω_2^2, there it tells me that A is equal to $-B$ or B is equal to $-A$ either way it is the same thing.

So, here it tells me that one displacement is negative of the other and equal in magnitude. So, it is like this they do this. So, two of those pendula would be oscillating in this mode. So, that corresponds to the second modes. So, this entire system of two pendula coupled by a spring they have only two normal modes one where they are in phase like this and the other one where they are out of phase and like this.

Now, in general any possible oscillation of this system can be written as a linear super position of these two modes $x(t)$ and $y(t)$. So, x and y are the original coordinate system in which we wrote down the equations of motion and we have the solution in precisely the same coordinate system. So, this here this whole thing constitutes a general system a general solution where the any possible dynamics of this couple system can be written as a superposition of these two normal modes, which is what we have done here and in particular you could adjust these amplitudes and in general phases as well to obtain a particular kind of oscillation, that this system might exhibit.

So, that is the sort of significance of normal modes. Any solution that is exhibited by system can always be resolved as linear super position of it is corresponding normal modes.

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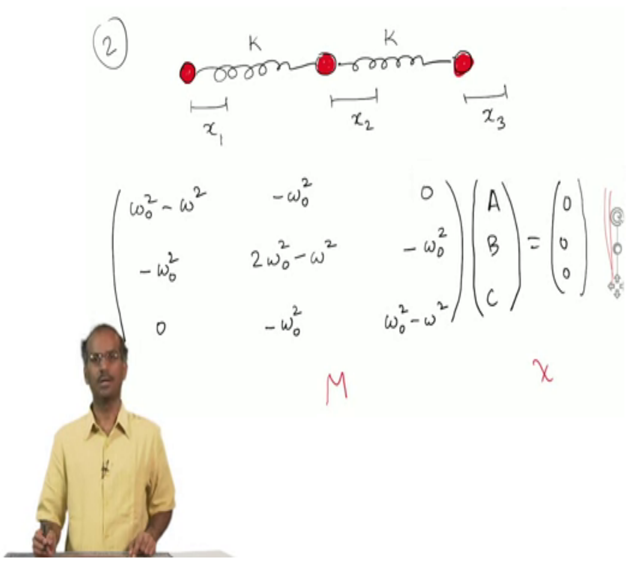

Let us now go to the second problem that we had done earlier and let us look at the solution and obtain the normal modes. If this is the system of 3 blocks coupled by springs in this case we have assumed that the two springs have exactly the same spring constant. We also assumed that to solve the problem originally we have to write down the equations of motion, we assume that these 3 blocks or balls could be called x_1, x_2, x_3 blocks and the displacement of each one of them is x_1, x_2 and x_3 as it as it is shown in this figure.

So, then we went ahead wrote down the equations of motion and as usual our recipe says that assumed solutions of the form $Ae^{i\omega t}$ may be $Be^{i\omega t}$, $Ce^{i\omega t}$ for x_1, x_2 and x_3 substitute that in the equations of the motion finally, you get this matrix vector equation $M\chi = 0$

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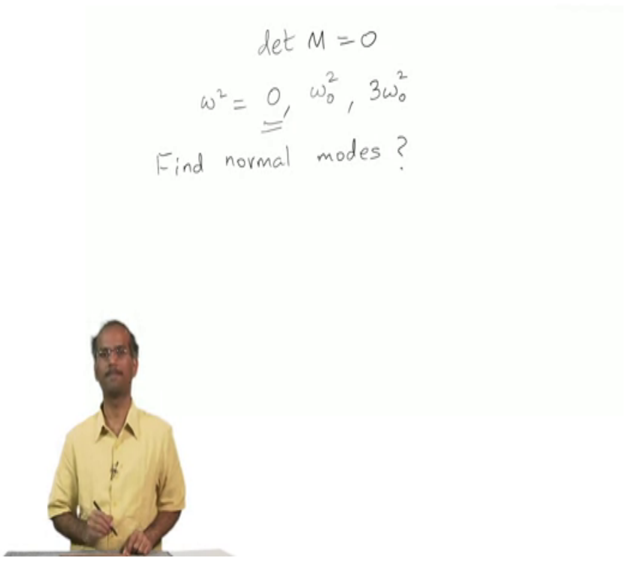

We put in the condition that determinant of M is equal to zero and that gave us three possible normal mode frequencies now the question is find normal modes.

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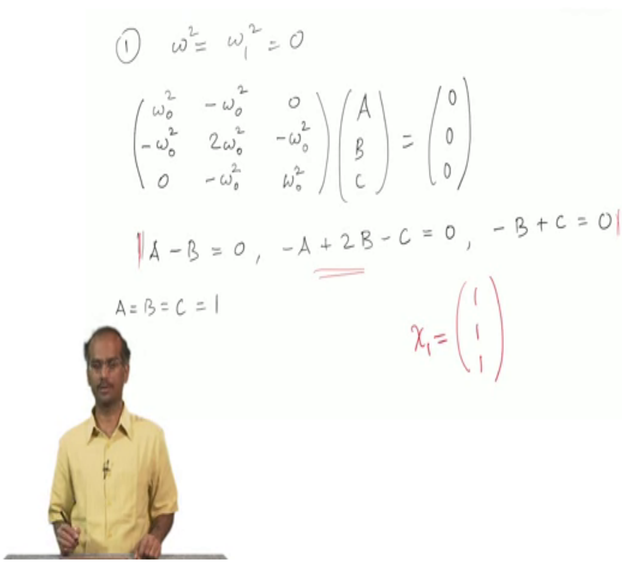

So, as usual let us treat the first case when or corresponding to the first case of ω which is equal to ω^2 is ω_1^2 and that is equal to zero that corresponds to this. And, the way we do the problem is substitute that value of ω here. So, put zero for wherever ω^2 occurs in this equation in this matrix vector equation and when I write down the resulting equation this is what I will get. Now I need to find suitable values of A B and C which are non-trivial solutions.

Now, if you look at collectively these equations you will see that if I take A and C to be 1 and also B to be 1 everything would be satisfied. So, let us say that I take A is equal to B equal to C equal to 1. So, if I do that then the first equation is satisfied which is this equation is satisfied and this equation will also be satisfied and this would also be satisfied. So, my normal mode is $\begin{bmatrix}1\\1\\1\end{bmatrix}$. So, that is for the first case.

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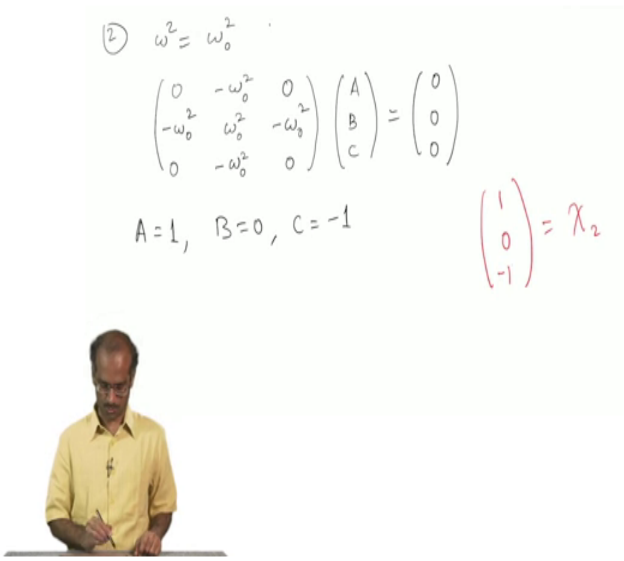

The second case corresponds to the frequency ω_0^2 ok. So, here again the usual way is substitute ω^2 to be ω_0^2 in this equation that you see here right now this one and you will get a simplified matrix vector equation from which you determine A B and C.

So, let us do that if I now write the substitute ω^2 to be ω_0^2 this is what I will get and once again looking at this equation in entirety it is very easy to see that, if I take A to be 1 and

B to be 0 and C to be -1 these set of equations will be satisfied hence my normal mode will be $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ that is χ_2 ok.

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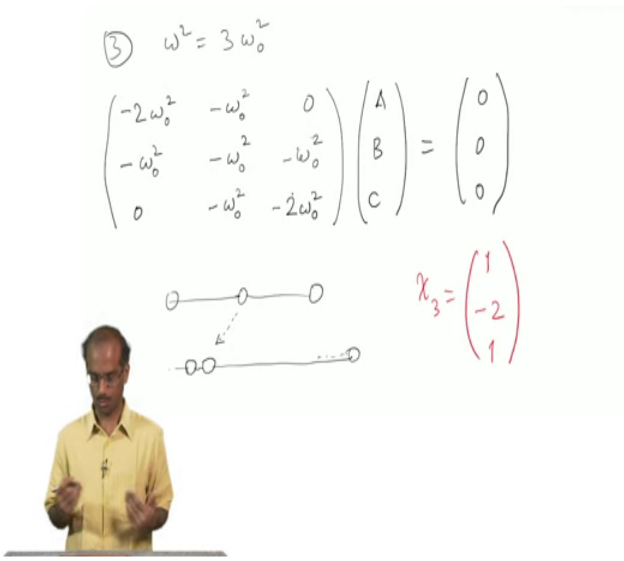

And, similarly we can go ahead and determine for the third case which is ω^2 is equal to $3\omega_0^2$ again go through the usual procedure. So, this time I will directly write down what you should be getting and again I have 3 unknowns A B and C to be determined in a sort of consistent way. And, if you look at the equation, if you are more comfortable with it write it out in normal equation form and in this case we should be able to get the following result, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

So, now I have all the 3 normal modes. So, χ_1 is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, χ_2 is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and χ_3 is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and of course, we can write the general solution. So, if you think about the physical picture that this conveys let us look at the two cases for which the frequencies the normal mode frequencies are not 0 say the second case. So, in this case it tells me that the A and C which are the amplitudes of the first and the third particle they are off by a phase of π.

So, which means that and the middle one is zero. So, the middle one is going to be static and the ones which are on the either side of the middle one they would be doing like this where as the middle particle is not going to do anything. So, that is one possible collective motion of this system and this corresponds to frequency ω_0^2. It is easy to see why it should be so, because the middle one is static is like not moving at all you can imagine like it is a hard wall and the other two are simply simple harmonic motions which depend only on the spring constant of the corresponding springs, which is why the frequency is ω_0^2 on the other hand you go to the third case

In this case the frequency is $3\omega_0^2$, here it corresponds to saying that the middle one let us say that these are the normal equilibrium positions of the 3 cases the middle one would be shifted here like this and this would be shifted here and this would be shifted here. So, that is going to be the kind of oscillation the amplitudes are going to maintain this ratio throughout ok. So, that is the third case and the first case think about it, what is ω equal to zero basically meaning no frequency.

So, it is not a periodic motion simply because when would you get angular frequency is zero. So, you would get that when time period is infinity it does not quite correspond to any kind of oscillatory motion. Now, it is possible to write a general solution as a linear superposition of all these 3 cases and it is exactly the way we did for the previous problem.

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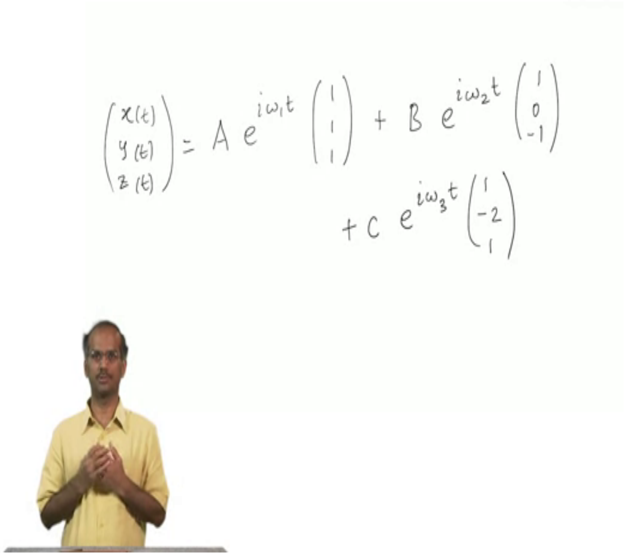

So, here I have the general solution in front of me. So, again to reiterate the point that I have been saying in general if you have n particles there can be *n* normal modes. But, in general any possible dynamics that is shown by a coupled system can be written as a superposition of these normal modes which is why it is easier to think of the system as showing these normal canonical modes. From this we can manufacture pretty much any other dynamics that the system would show by manipulating the phases and the amplitudes. There are more problems of similar type in the assignments I urge you to do it.