Waves and Oscillations
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Lecture – 02
Superposition of Oscillations: Beats

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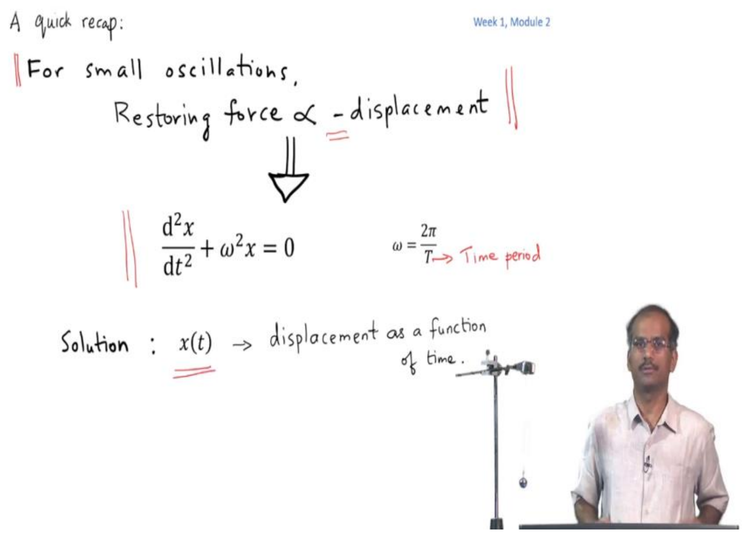

Welcome to the second module of the first week. So, let us begin with the recap of what we did in the first module, we started by motivating ourselves with the physical pendulum like the one that I still have here with me. The central piece of physics was that for oscillatory behavior and in the limit of small oscillations. It is very important that we need to stick to the limit of small oscillations and if you do so, we figured out that restoring force is proportional to displacement with the negative sign, again this negative sign is very crucial, we need it for oscillatory behavior.

So, that this statement that is written here is the central piece of physics which will ensure that finally, when we get to the solutions, the solutions will display oscillatory behavior. Now, starting from this piece of physics we finally, wrote down equation of motion which turned out to be a second order differential equation of this type. So, it is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where this quantity x is the displacement.

So, for instance in the in the case of this physical pendulum that I have here. So, this rest position would be our equilibrium position and I make a small displacement away from the rest position the distance from the equilibrium position to the new position would be my displacement. And this quantity ω we already saw, we already met this quantity and this is simply the angular frequency it is 2π divided by the time period T.

So, we did not solve this equation in the last module, but we know that solution would imply that I should be able to get this quantity displacement x as a function of time. So, that is what I would mean by saying that I have solved this problem ok.

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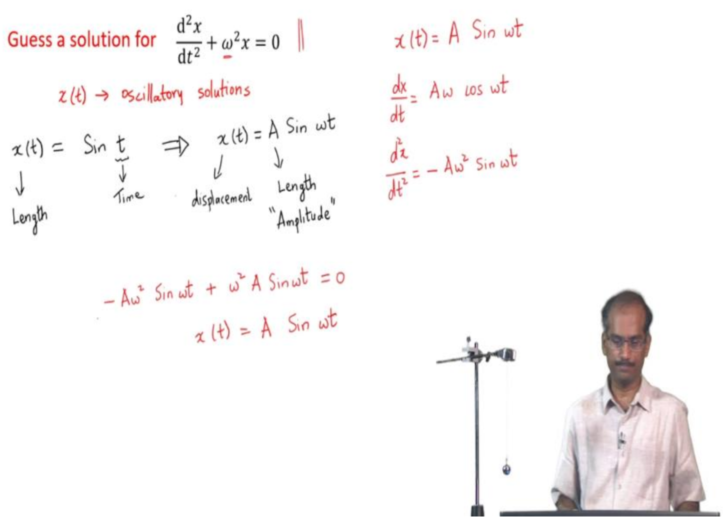

So, in this module we are going to get the solution for this equation so one could take a hard way by actually solving this differential equation, but here in the spirit of what we did in the first module, we will try and guess the solution motivated again by the phenomena that we see.

So, if you look at the phenomena that we are trying to describe and for which it appears that the equation of motion is what I have written down here, you will notice that this is an oscillatory phenomenon. So, any solution for this oscillatory phenomenon should somehow capture these oscillations, in other words the solution mathematically should be a function that oscillates as well.

So, we are looking for functions that would oscillate and they probably could be possible solutions for this problem and again the motivation for such an argument is that because physically we see that the pendulum is oscillating. Or in general whenever you have restoring force that is proportional to displacement with this negative sign, the solutions are expected to be oscillatory. So, I am looking for solutions which are oscillatory in nature. So, let us begin by guessing some solutions.

So, one of the simplest possible oscillatory functions that we know are the sins and cosines the trigonometric function that we studied in schools. For example, I can make an ansatz, I guess of this type that $x(t)$ is $sin\ t$. So, the problem with this guess is that the dimensions do not match, on the left hand side the dimension is that of length because x is displacement and it has dimensions of length. But on the right hand side I have t which is which has dimensions of time and of course, it is embedded inside a more complicated function which is sine function. So, clearly there is mismatch of dimension so we need to correct it.

So, one possible way to correct it is to make a small adjustment here. So, let me say that my $x(t)$ is $sin\ \omega t$ and I know that from what we had seen earlier ω is simply $2\pi/T$ and this has dimensions of inverse time. So, when I put in ω here the time and inverse time they would cancel one another out.

So, this will essentially be a number and in that case again there is a dimension mismatch because on left hand side x has dimensions of length, but on right hand side you have something that is dimensionless, but that can be easily fixed by adding this A here. And A, we can guess from what we have written down should have dimensions of length and for our purposes that is simply the amplitude; amplitude of the oscillation and left hand side of course, you have displacement that is a function of time.

So, amplitude would be something like this. So, this is the rest position and this could be the amplitude the largest displacement that is possible. Now, that we have actually guessed a possible solution next, we need to check if this is indeed the solution for this problem.

So, let me now assume that I am going to work with this solution, let us say this $A sin\ \omega t$ and if it is indeed the correct solution if I plug it back into this equation, it should satisfy the equation and it is very easy to do.

So, $x(t)$ is $A\ sin\ \omega t$ and dx/dt will be $A\omega\ cos\ \omega t$ and d^2x/dt^2 would be $-A\omega^2\ sin\,\omega\,t$. Now let us put the quantities back in this equation in which case I will get, $-A\omega^2\ sin\,\omega\,t + A\omega^2\ sin\,\omega\,t\ = 0$, clearly this cancels one another out so you do get a 0. So, we have discovered one solution for our differential equation $x(t)$ is $A\ sin\,\omega\,t$

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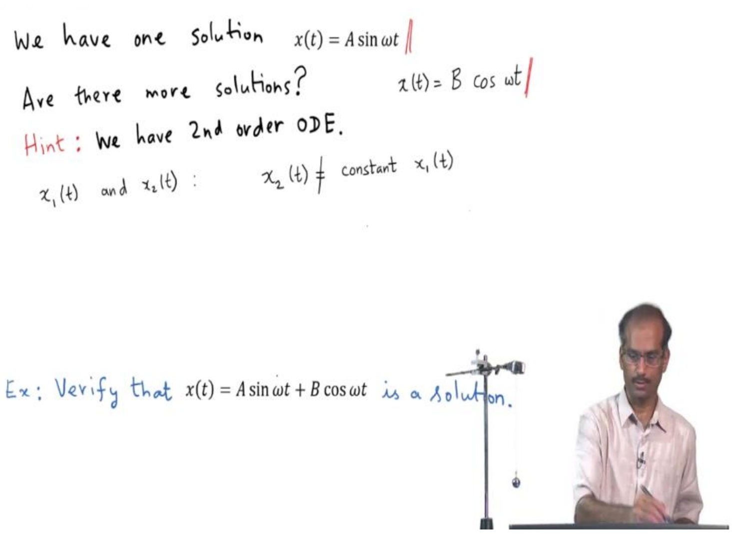

We saw that we found out one solution which is $A\ sin\,\omega\,t$ the question is, are there other possible solutions without even much thinking could say that maybe cos function is also a possible solution. So, I could have by the same argument written down that $x(t)$ equal to let us say some $B\,cos\,\omega\,t$ could have been a possible solution and in fact, you can verify that $B\,cos\,\omega\,t$ is another possible solution as well.

So, the hint for all this comes from the fact that the differential equation that we started with

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is a second order linear differential equations, which means that in principle it should have two independent solutions. Let us say that I have two possible solutions for my differential equation which could be $x_1(t) + x_2(t)$

So, when I say that $x_1(t)$ and $x_2(t)$ are linearly dependent I am making the following statement that $x_2(t)$ is equal to some constant times $x_1(t)$. So, in that case once I know $x_1(t)$, all I need to do is to simply multiply it by a constant I get $x_2(t)$, but in this case what we require is that the two solutions should not be related by this relation. If they are not related by this relation then you would say that the two solutions are indeed linearly independent and again this is something you could verify for yourself that sin omega t and cos omega t are indeed linearly independent functions.

So, at this stage what we have is, we were trying to find solutions for a second order linear differential equation and as dictated by the mathematics, we have found two linearly independent solutions, one of them is $A\, sin\, \omega\, t$ and the other one is $B\, cos\, \omega\, t$. So, in fact, you need not stop with this. In fact, you can also show that if I add $A\, sin\, \omega\, t$ and $B\, cos\, \omega\, t$ that would also be a solution. I am not going to do that it is very easy and just the recipe that we did here in the previous case, if you follow that you should be able to show that $x(t)$ being $A\, sin\, \omega\, t\ +\ B\, cos\, \omega\, t$ is also a solution.

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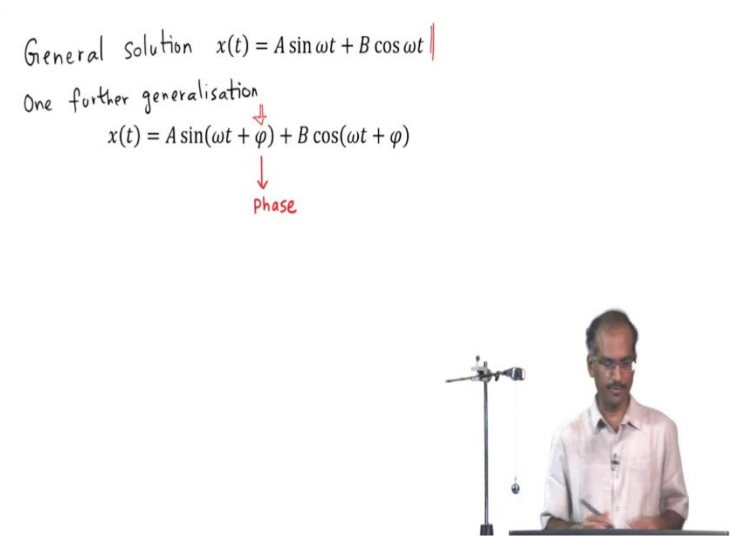

At this stage we have a combination of two linearly independent functions $A\,sin\,\omega\,t$ and $B\,cos\,\omega\,t$ with two arbitrary constants A and B, we will see how to determine them a little later. But this combination is a possible solution to the differential equation that we had obtained for describing the oscillatory function. Now, another question is this the most general function that would be a possible solution? The answer is no you can actually generalize it even further by adding a phase term as shown here. So, this ϕ that is written here is the phase and we will see what this phase is about, to understand phase let us cut down the second part and just stick to this simpler version of our solution.

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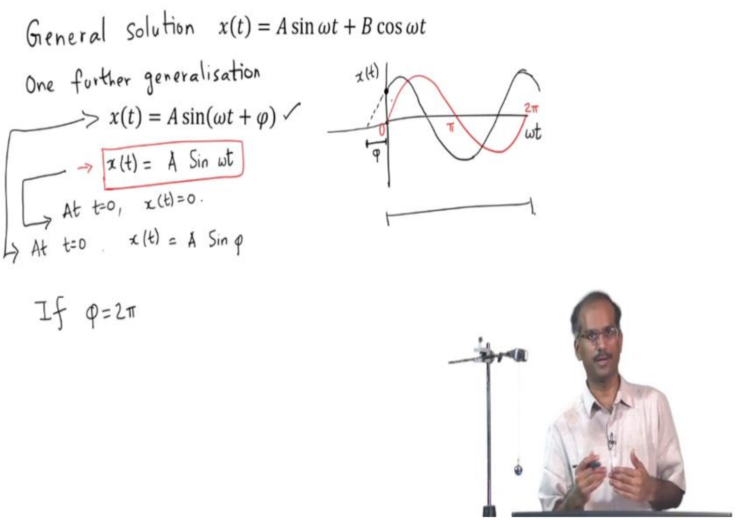

Now, to understand phase here, let me also for comparison write down $x(t) = A\,sin\,\omega\,t$ Now, both these are possible solutions to a differential equation, in one version I have written down $A\,sin\,\omega\,t$ and in the other I have written down $A\,sin\,\omega\,t$ plus this added function ϕ which I am calling the phase.

So, to understand how this works, let us plot the solution. So, I am going to plot displacement $x(t)$ as a function of ωt . So, let me first plot this function $x(t) = A\,sin\,\omega\,t$ might look something like this ok. So, this is your $0, \pi\ \&\ 2\pi$

Now, let me also plot this but before plotting the second one let us say that at $t = 0, x(t)$ corresponding to this case would be ϕ

On the other hand, corresponding to this case at $t = 0$, $x(t)$ would be equal to 0 or $x(0)$ would be equal to 0. So, they differ in their values at time $t = 0$. So, if I have to plot this function now, let me plot it with the black pen at $t = 0$ it is a non-zero value A is not equal to 0 and ϕ is also not equal to 0.

So, let me say that at $\mathrm{t} = 0$, this is my initial point and this is going to be something like this and to complete the picture let me draw it on the negative t direction as well ok. So, you could see that phase is basically this quantity here. So, it is the initial offset that you give with respect to a solution whose phase is 0. So, for the case of this red curve here this corresponding to this solution, the phase was 0.

On the other hand, corresponding to this solution this one your initial phase was ϕ and that corresponds to this quantity here. It can be easily realized that if ϕ where 2π you are moving an entire distance corresponding to 0 to 2π and there would be no difference between a solution that has no phase and the one that has a phase difference of 2π.

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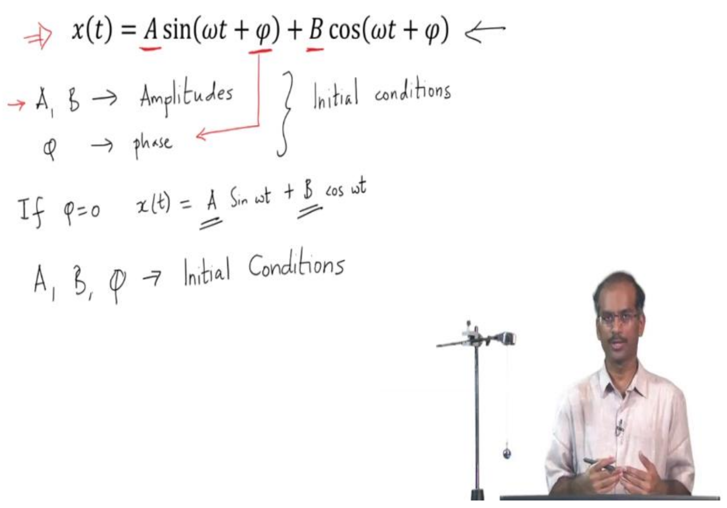

So with this understanding of phase let us put back the complete or the most general possible solution. So, I have written down the most general possible solution here as displacement as a function of time is $\mathrm{A}sin(\omega t + \phi)$ and there is also the second linearly independent function which is $B\cos(\omega t + \phi)$ and as we just saw ϕ is simply the phase and A and B are the amplitudes.

And if $\phi = 0$, the solution would simply reduce to $A\sin\omega t + B\cos\omega t$ and these three numbers A, B and ϕ are arbitrary constants. And the information about these values of A, B and ϕ will have to come from the system that one is studying ok, it is not determined by the equation of motion itself. If you go back to the equation of motion, you will notice that there is no information about initial values in the differential equation that has to be provided as part of the problem ok.

So, these three quantities would be in general called initial conditions. So, if you want to get a specific solution for a particular problem, you need to specify these initial conditions. So, in the tutorial section, we will see how these initial conditions can be provided and used in obtaining the solutions.

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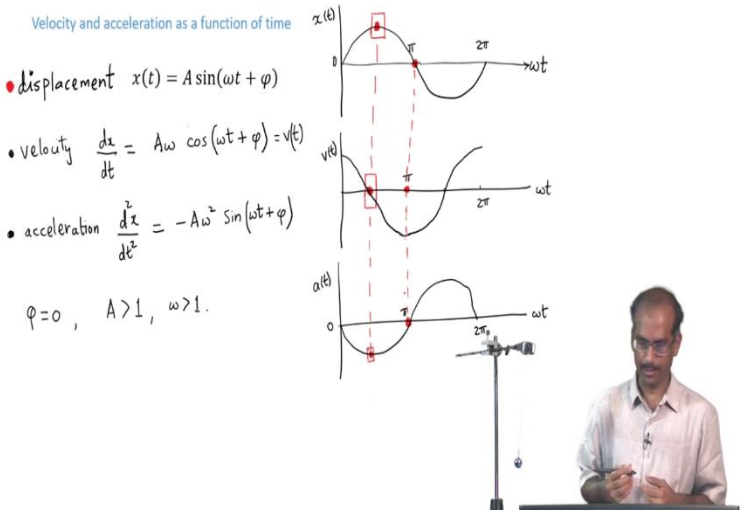

Now, let us get back to our solution once again. So, I will stick to one form of this solution which is $A\sin(\omega t + \phi)$. So, that is my solution for displacement. Now, starting from here, once displacement is given it is easy for me to calculate the velocity. So, I can write an expression for velocity which will be dx/dt and if I differentiate that would be is $A\omega\cos(\omega t + \phi)$. And given the velocity it is again straightforward to calculate the acceleration which will be d^2x/dt^2 or second derivative of displacement with respect to time and that would be $-A\omega^2 sin(\omega t + \phi)$

So, if you notice the relation between the displacement and acceleration you will notice that displacement and acceleration are proportional to one another with the negative sign that should not be surprising given the fact that we actually started from an ansatz which basically stated that restoring force is proportional to displacement with the negative sign. So, this is a sort of consistency check from that ansatz we went to write down a differential equation and from there we obtain the solution and finally, we see that it is indeed consistent with the ansatz that we started with.

So, to have a pictorial understanding of these three quantities again let us plot these functions. So, let me start with displacement as a function of ωt and for the purposes of plotting sketching this function I am going to assume that $\phi = 0$, just a simplifying assumption and I am also going to assume that the amplitude A is greater than 1, I will also assume that ω is greater than 1.

Now, let us sketch the function. so, this is $x(t)$ and the same scale let me see if I can plot velocity which I will call $v(t)$. So, this will be $v(t)$ as a function of ωt. So, this is a cosine function and I have assumed that A and ω are greater than 1. So, it is going to look something like this is of course, your π and 2π.

Let me also sketch, acceleration as a function of time and this as we see here from the figure will be negative of displacement. So, I should have something like this so this is $2\pi, \pi$ & 0. Let us pictorially look at the solution and what they really reveal. So, I have sketched displacement, velocity and acceleration as a function of time.

So, let us see what they physically tell us. So, the first thing to notice is that at the point where say the displacement is a maxima that is precisely the point where the velocity is a minima this point here ok. So, how do we relate it to a real physical system like this? So, this is how say that my pendulum oscillates and if I look at it carefully the displacement is maxima at let us say somewhere here and that is precisely the position where the pendulum comes like this, momentarily stops and then turns back.

So, that is exactly the position where the velocity is 0. So, at the position where the displacement is maxima, the velocity is 0. On the other hand, you can easily figure out that as we pass the equilibrium position here so this is the equilibrium position when it is oscillating it passes through the equilibrium position and when you pass through the equilibrium position its displacement is 0, but nevertheless its acceleration would be a maximum.

So, at the extremes where the velocity is actually changing signs because it is changing directions, velocity would either go from negative to positive or positive to negative. So, the acceleration would be largest. So, at the point where the displacement is maxima, the velocity at the corresponding time would be zero and at precisely the same time, the acceleration is a maxima.

So, that is this position at the extremities either at this end or the corresponding end on the other side. On the other hand, if we analyze what happens let us say at this point here. So, here the position is 0, but the velocity is maximum and that is understandable because really the pendulum is passing through the equilibrium position and its velocity is the largest and here there is no change of direction, there is no change of sign in the velocity. So, in this case acceleration is indeed 0.

So, the velocity would be maxima corresponding to this and acceleration would be 0. So, it would correspond to this. So, essentially what we see is that, at the positions where displacement is maxima your velocity is zero and at values of times when displacement is zero, velocity is maxima and displacement and acceleration have this relation that they are proportional to one another and there is a minus sign between them. So, we see that clearly in this sketch.

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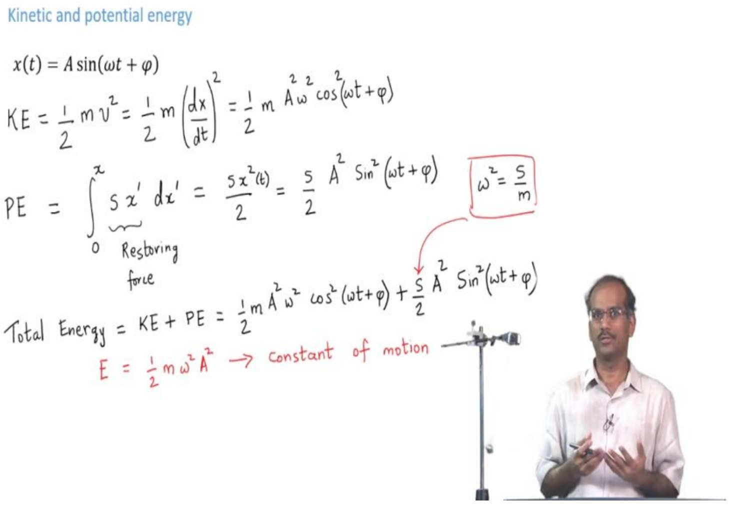

Next, we will try and evaluate the kinetic and potential energy of this of an oscillating system and again the starting point is we will start with one solution and I have taken one of the simplest ones as $A\sin(\omega t + \phi)$ and it is even simpler if I can even set ϕ equal to zero at the it would not change any of results.

The kinetic energy is easy to write down. So, kinetic energy which I will indicate by KE simply half times mass into the velocity square which will be half mass into dx/dt whole square and that is simply $(1/2)m$ into $(\omega\cos(\omega t + \phi))^2$ what about the potential energy?

Potential energy is simply the work done by the system against the restoring force. So, this at this position, we have assumed that the potential energy is zero because we have said the sort of put the scale such that the zero of the position corresponds to the point where the pendulum does not do anything if left to itself. So, that will be the position where the potential energy is zero.

But now when I try and move it on either sides let us say that from here I move it to this side, by act of doing this I have I am putting in some potential energy into the system. And how much potential energy have I put in that would be equal to the work done in moving it from here to this point, that much energy is stored in the system as potential energy so that when I leave it, it will use that energy to oscillate.

So, the physics of the problem is simple. So, this is its rest position, it has no energy no kinetic energy because it is not moving and it has no potential energy because we have said that at this point the potential is zero and when I move it like this a little bit sideways, I am putting in potential energy and the amount of energy is equal to the work done and moving it from here to here and when I leave it, it will start doing its work.

So, the potential energy would simply be equal to the work done against the restoring force. So, that would be something like. So, in general when I write in terms of the stiffness constants, it is s times $x'dx'$. So, I need to do this integral. So, so remember that this was our restoring force. So, all I have done is to multiply the restoring force by the distance that it moves which is dx' and you integrate over the entire distance from 0 to x. So, you add these small products of restoring force multiplied by an infinitesimal distance dx'.

So, when you do this the integral is easy to do, it will be $Sx^2/2$ and if I substitute this value of x here. So, remember that x is a function of time, it will be equal to $(S/2)A^2\, sin^2(\omega t + \phi)$. The total energy would be the sum of kinetic energy and the potential energy, all we need to do is to add these two expressions that we have. So, $(S/2)A^2\, sin^2(\omega t + \phi) + (1/2)m(\omega \cos(\omega t + \phi))^2$

So, if we remember what we did in the first module, there is a relation between $m\omega$ and the stiffness constant S which is that ω^2 is equal to S/m, if we plug in this relation and substitute for S here, this expression can be simplified and it would simply become $(1/2)m\omega^2 A^2$ and there will be this $sin^2(\omega t + \phi)\ +\ \cos^2(\omega t + \phi)$ which will be equal to 1.

So, the total energy is, I am denoting by E will simply be equal to $(1/2)m\omega^2 A^2$ So, you will notice that each of these kinetic and potential energy individually are functions of time because kinetic energy has this $\cos^2(\omega t + \phi)$ and potential energy has the $\sin^2(\omega t + \phi)$. So, individually kinetic energy and potential energy they are time dependent, on the other hand the total energy is independent of time.

So, mass, angular frequency, omega, amplitude all of them are constants. So, the total energy is a constant of motion, it does not change with time. So, what we have is an interesting model, kinetic energy is time dependent, potential energy is time dependent, they change in such a way as to keep the total energy constant. The fact that energy is a constant of motion should also be physically appealing because in an ideal case where there is no dissipation in the system then the energy that you put in when you start off the oscillations like this would not be expended at all.

So, you would expect that the total energy would simply be converting from kinetic energy to potential energy and back to kinetic energy and so on, but otherwise the total sum of the kinetic and potential energy would remain a constant.

And if you go back and look at the equation of motion you would notice that there is no term in the equation of motion which takes care of the dissipation in energy. So, once again all this points out to the same fact that within the scope of the model that we had written down, there is no way the system could lose the energy.

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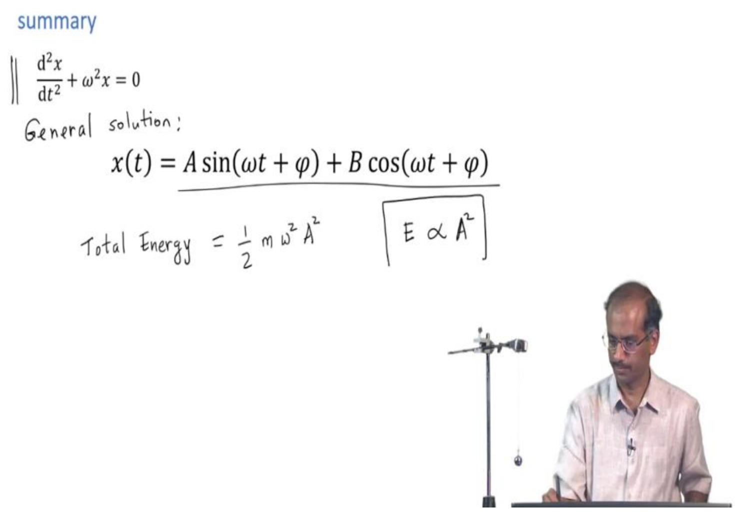

Now, to summarize this module, we started by writing down the equations of motion which is this

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

and motivated by the physical phenomena of say an oscillating particle we guessed a solution, we corrected it for the dimensional problems and the most general solutions; that solution that we could write down $x(t) = A\,sin(\omega\,t + \phi)\; +\; B\,cos(\,\omega\,t + \phi)$ was

And then we also saw that given a solution like this it is straightforward to calculate the velocity as a function of time, acceleration as a function of time. We also saw that there were systemic relations between the values of position, the values of velocity and the values of acceleration that an oscillating system takes.

Finally, we calculated the total energy which turned out to be $(1/2)m\omega^2 A^2$. And the important part of this total energy is that it is indeed a constant of motion and it depends only on the square of the amplitude in other words, a fact that will keep using again and again later on is that energy of an oscillating system is proportional to square of amplitude.

The fact that energy is a constant of motion should also be physically appealing because in an ideal case where there is no dissipation in the system then the energy that you put in when you start off the oscillations like this would not be expended at all. So, you would expect that the total energy would simply be converting from kinetic energy to potential energy and back to kinetic energy and so on, but otherwise the total sum of the kinetic and potential energy would remain a constant. And if you go back and look at the equation of motion, you would notice that there is no term in the equation of motion which takes care of the dissipation and energy.

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