Waves and Oscillations
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Lecture – 19
Coupled Oscillators: More Examples

Welcome to the 4th module, this 4th week. We are still looking at the coupled oscillations. In this module, we will look at two more examples of solving the coupled oscillator problems.

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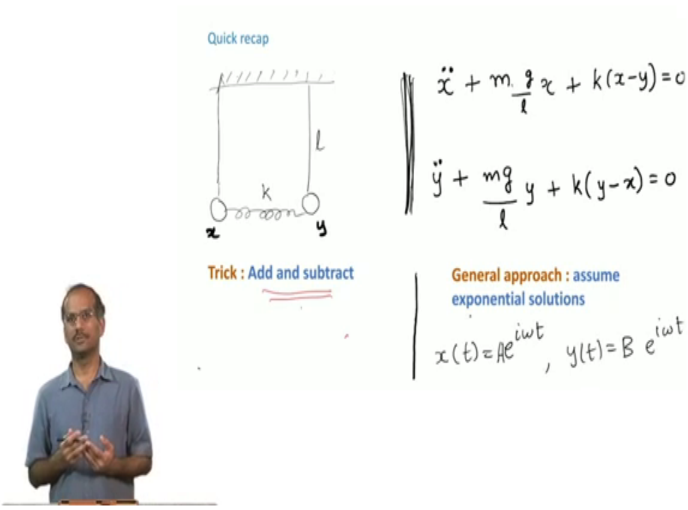

So, we started by looking at the problem of two pendulums connected by a spring. So, what I have in front of me, are the two equations of motion. To solve this particular system the sort of trick that we adopted was to add and subtract these two equations of motion. So, we got two equations of motion each of them corresponded to a single one-dimensional harmonic oscillator. The important point is that we added and subtracted and this is not the way in general coupled problems can be done. Suppose, let us say that we have 3 different pendula connected like this, then we will not be able to do this trick of adding and subtracting equations.

So, there would be many other problems where the simple trick of adding and subtracting the equations will not work. So, the general approach was simply to assume exponential solutions. So, the important point is that this omega is the frequency angular frequency of motion and this is same for the entire system. So, whatever be the pattern of oscillations both the oscillators together display, the frequency of motion is omega. So, there cannot be a scenario where two pendula would oscillate with two different frequencies even when they are coupled together, so that cannot happen. So, that is the important point that we should note. So, from now on we will try and use this recipe for solving.

So, once we have $x(t)$ and $y(t)$ chosen like this you calculate $\dot{x}$, $\dot{y}$ and then $\ddot{x}$ and $\ddot{y}$, substitute it back in this equation that we have here and finally, what we will get is a matrix equation where the two unknowns are simply the amplitudes A and B. So, we will demand that the amplitudes have non-trivial solutions that is they are in general not 0, $A \neq 0$ and $B \neq 0$.

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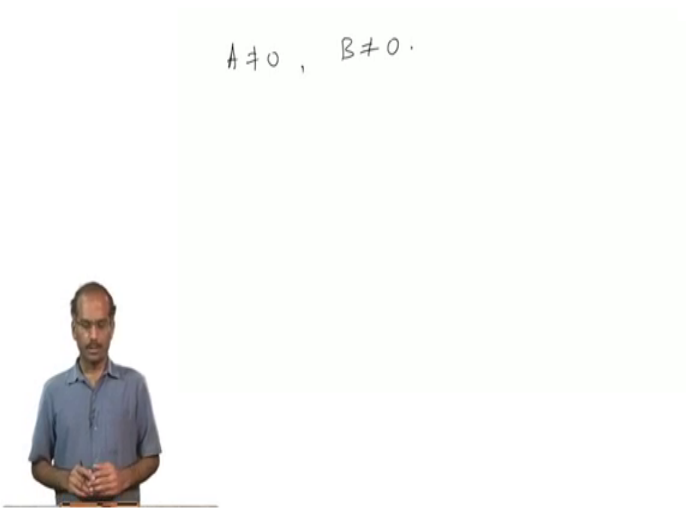

So, that will give us the condition that the determinant should be equal to 0 from which we can determine the two frequencies. So, what you will finally, get are two frequencies which we had called normal mode frequencies. The pattern of oscillations displayed collectively by both the particles in this case two pendulums they would be called the normal modes themselves.

So, today we will do two problems, one where we have two particles and other where we have three particles.

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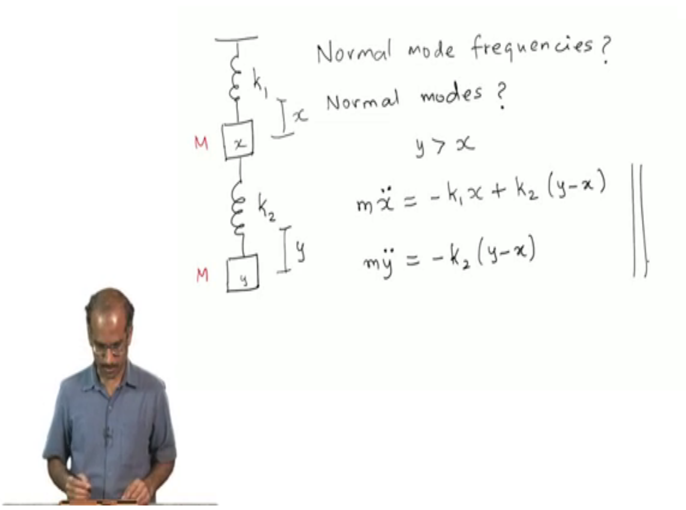

So, here in front of me I have a system with two blocks whose masses are M and they are connected through a system of two springs with spring constant k_1 and another one with spring constant k_2. Now, once of course, you start giving it a little bit of push, they would start oscillating. So, I want to find out what are the normal mode oscillations and normal mode frequencies.

Let us say that the displacement is x and in the case of second pendulum the displacement is y. And I am going to assume in general without loss of generality that $y > x$. The equations of motions themselves will not depend on whether $x > y$ or $y > x$. So, it really does not matter. Here is what could be the equations of motion $m\ddot{x}$ would be equal to. So, as far as the x block is concerned it is being pushed up by an amount x, so the spring k_1 has actually compressed which means that it is going to push down the first block which is opposite to the direction of motion.

So, that would give me $-k_1x$. There would also be a force that arises from the second spring which is the k_2 spring or spring with spring constant k_2. In this case or the net compression or the elongation of the spring would be $y - x$. We have assumed $y > x$, which would mean that the second spring is compressed, so the second spring will favor in which case I should write the second one second term with the plus sign, so that would be the equation of motion for the first block.

Now, let me write the equation of motion for the second block. So, $m\ddot{y}$ that is equal to as far as the second block is concerned, it is it is only in contact with the second spring with spring constant k_2. So, the only force that is going to affect the second block will arise from the second spring, but the second spring itself is in contact with both the blocks, so it would depend on the displacement of both the blocks. So, now we have assumed $y > x$. So, $y - x$ is positive, so which means that second spring has undergone a compression and it is going to push it down hence it will be $-k_2(y - x)$. Now, we will try and solve this system. Again now, there is no point in trying adding and subtracting is not going to work.

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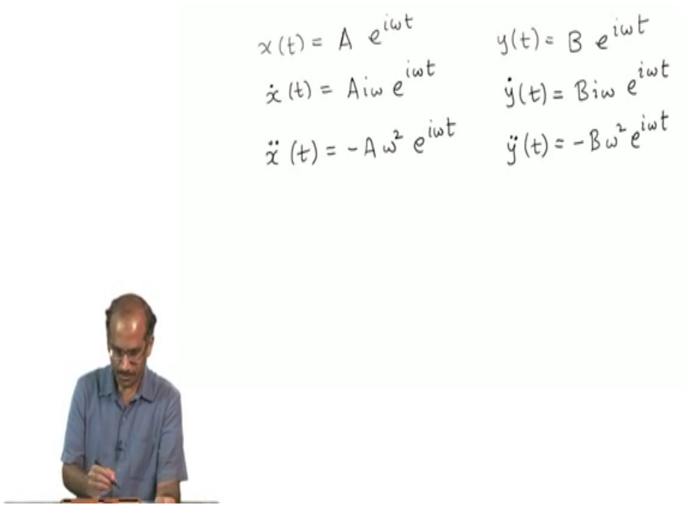

I am going to start by assuming that $x(t) = Ae^{i\omega t}$ and $y(t) = Be^{i\omega t}$. And I would like to stress that when you make a choice like this $Ae^{i\omega t}$ and $Be^{i\omega t}$, implicit in that is some choices already made for some initial conditions. Since, we are not really worried about what kind of initial conditions for which we are solving the problem, we can simply choose initial, any initial conditions which is simple to handle because what we are interested is finding out the normal mode frequencies and the normal modes.

So, I have written everything here. Now, all I need to do is to substitute it back in this equation.

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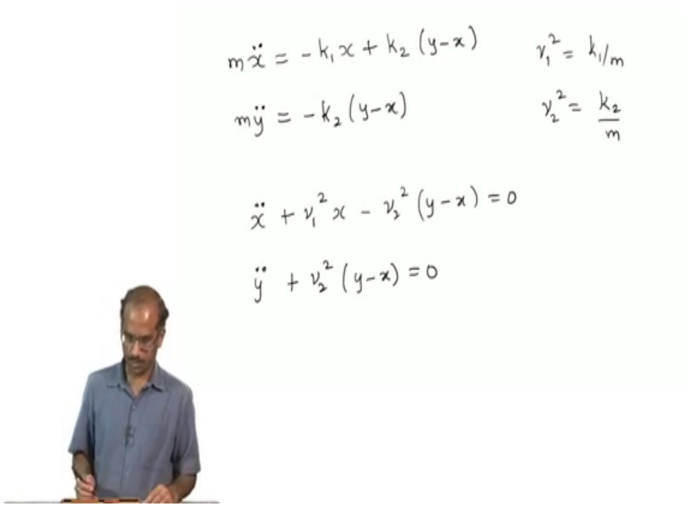

So, if I divide throughout by m, I will have terms like $\frac{k_1}{m}$ and $\frac{k_2}{m}$ on the right hand side, and similarly in the equation for y I will have $\frac{k_2}{m}$. And to simplify that let me say that $\nu_1^2 = \frac{k_1}{m}$ and $\nu_2^2 = \frac{k_2}{m}$. And if I put this in I am going to get the following equation of motion,

$$\ddot{x} + \nu_1^2 x - \nu_2^2 (y - x) = 0$$

Similarly,

$$\ddot{y} + \nu_2^2 (y - x) = 0$$

So, now we can substitute our assumed solutions here.

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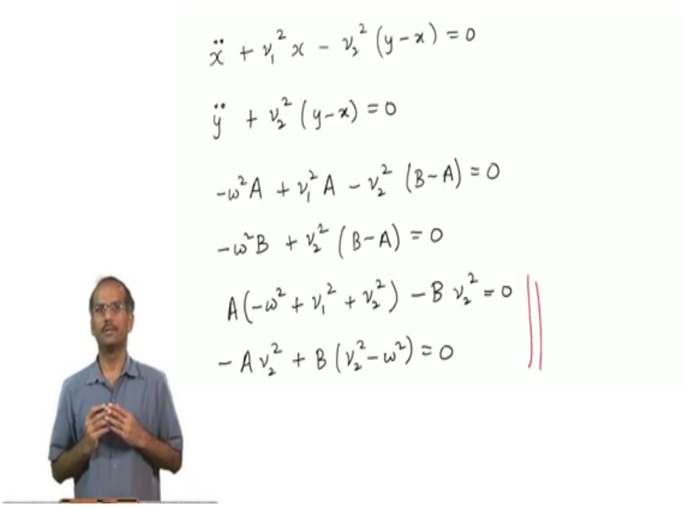

Let us substitute for $\ddot{x}$, x and y and if we do that this is the equation that we will have. So, these are the two sets of equation that I have and this one can now be simplified and I will write it in terms of coefficients of A and B. Basically, I am going to separate out the terms with A and B.

Now, what is relevant for us is these two equations. I am going to write it in matrix vector product form.

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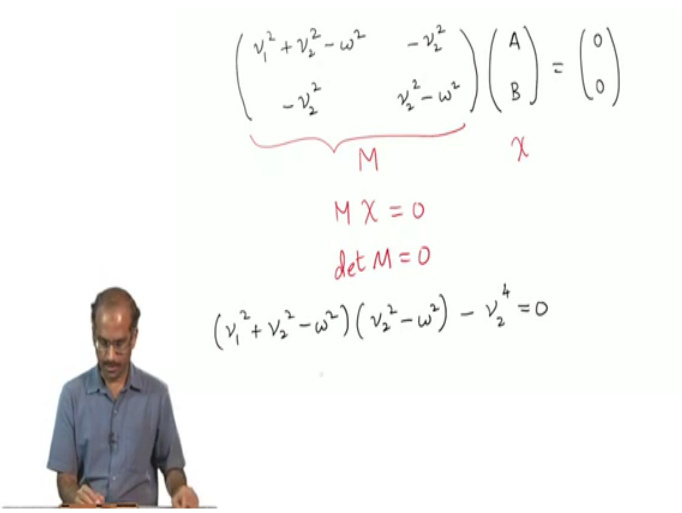

So, now I have written this in matrix vector product form. And as usual let us do a renaming let us call this matrix M and this is χ and of course, this is 0, so now I have a matrix vector product being equal to 0. Both A and B we should remember are the amplitudes, they should not be 0 in general. The condition for A and B not to be 0 in general is that determinant of M should be equal to 0, and determinant of M can be easily calculated. Now, I need to solve this. You will quickly recognize that this is a quadratic equation in the variable ω^2. So, if I multiply all the terms, arrange them in powers of ω it should get the following equation.

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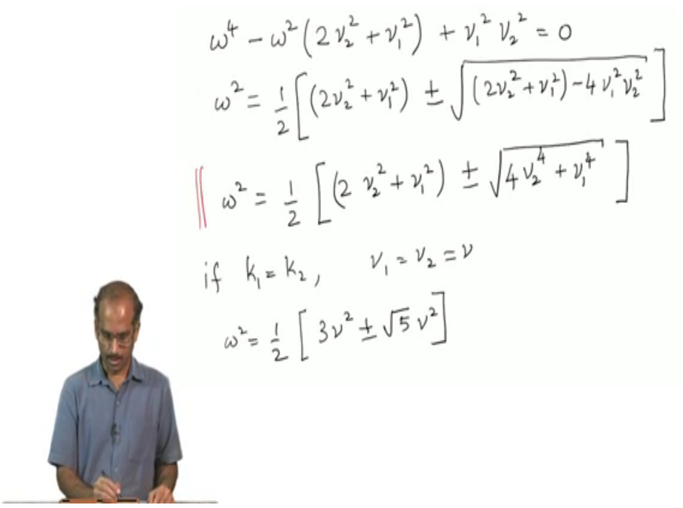

Now, it is very clear that what I have is a quadratic equation in the variable ω^2. The solution would be values of ω^2 that is the one that I want. Now, all we need to do is to simplify this. I urge you to do it yourself and if you do everything correctly you should be able to get the following.

So, here I have two possible frequencies ω_1 would correspond to say the plus sign and ω_2 would correspond to the negative sign. And so, remember that it is a fairly complicated solution that we have obtained, but it can be simplified provided we make some simplifying assumptions. Suppose for example, I assume that k_1 and k_2 are equal, that is both the spring constants are equal, if I make that assumption in such a case ν_1

would be equal to ν_2, in which case this would simplify considerably. So, here I have two possible normal mode frequencies.

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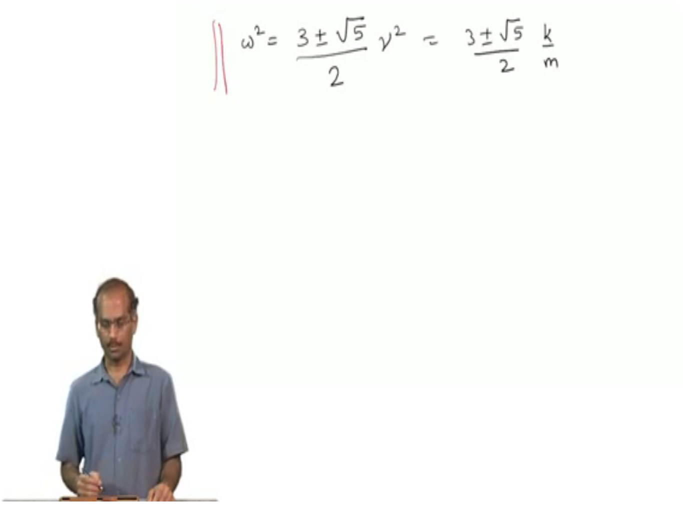

To repeat in general when $k_1 \neq k_2$, the normal mode frequencies are given by this form and when $k_1 = k_2$ it simplifies and these are the normal mode frequencies.

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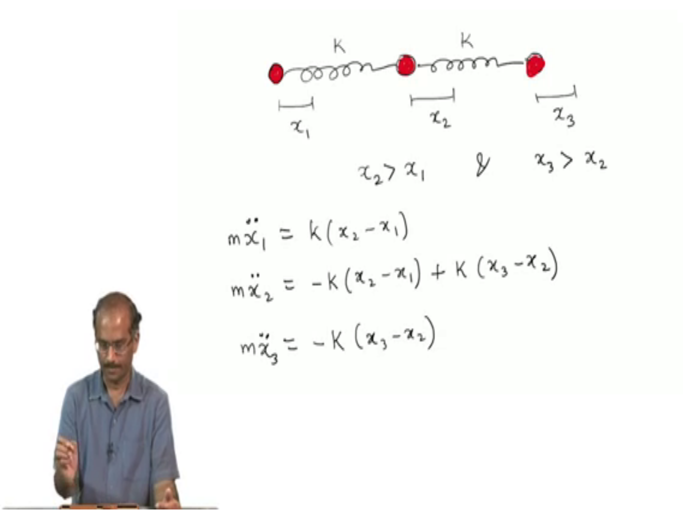

The next problem is 3 blocks again connected by springs. So, in this case we have 3 blocks connected by two springs. So, if we go by the intuition that we gained from the last two problems, we should be able to figure out even without doing any further mathematics that in this case we should expect to see 3 normal mode frequencies. So, when we had a system a coupled system with two particles, we had two normal mode frequencies. And when we were dealing with single particles like the case of a single simple pendulum we had one frequency, so it is reasonable to generalize and anticipate 3 normal mode frequencies in this case.

And I am going to assume that the spring constants are same, both will be given by k. To write the equations of motion I will assume that the displacements are x_1, x_2 and x_3 and I will even call these each of these particles as x_1 particle, x_2 particle, and x_3 particle. And to write the equations of motion let me assume that $x_2 > x_1$, and $x_3 > x_2$.

So, with this assumption let us write down the equations of motion. And once again let me emphasize that this assumption about which displacement is bigger or smaller will not change the equations of motion. So, it does not matter what you assume here.

So, let us start by writing the equations of motion $m\ddot{x}_1$ will be equal to. So, in this case the first particle is connected only to one spring, but this spring is going to depend on the displacements of both the particles in which case I should get $k(x_2 - x_1)$. Now, let us write the equation of motion for the second particle.

In this case, the second particle is affected by both the springs on either side, so it is going to have additional term. Since, we have assumed $x_2 > x_1$, $x_2 - x_1$ is positive which means that the first spring has elongated and is going to pull back the middle particle. It is going to bring it in this direction; hence I will call it $-k(x_2 - x_1)$. And what about the next spring? The second spring that we have here, in this case we have assumed that $x_3 > x_2$ which means that again it is an elongation and by similar arguments we can write for the last particle which is x_3, in this case again it is in contact with only with one of the springs hence I will have.

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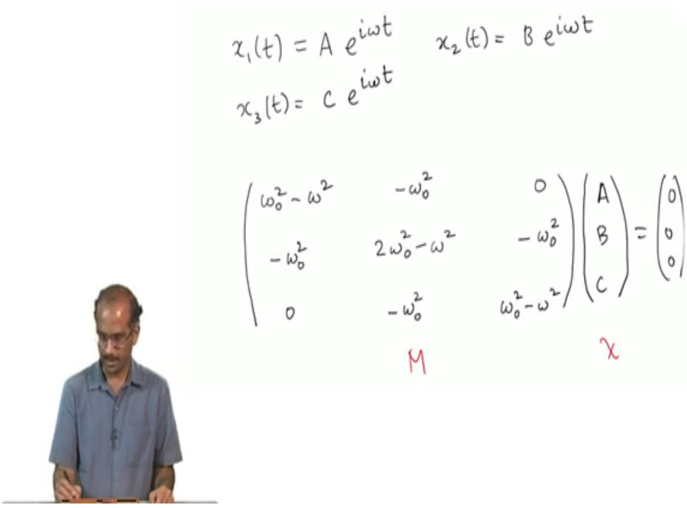

So, let us assume that $x_1(t) = Ae^{i\omega t}$ and $x_2(t) = Be^{i\omega t}$ and $x_3(t) = Ce^{i\omega t}$. So, we do the usual calculation, calculate $\dot{x}$ actually $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and also the second derivative of these displacements. And if you substitute it back in this equation that we have finally, it is going to give us the following equation. Now, this time I will not do all the steps, I will let you try and do it yourself. But if you do it everything if you do everything correctly you should be able to get the following matrix vector equation.

So, I have written down the matrix vector equation. As usual we will call this as M and this is χ and the same argument again goes through that for non-trivial solutions for A, B and C we require that the determinant of $M = 0$.

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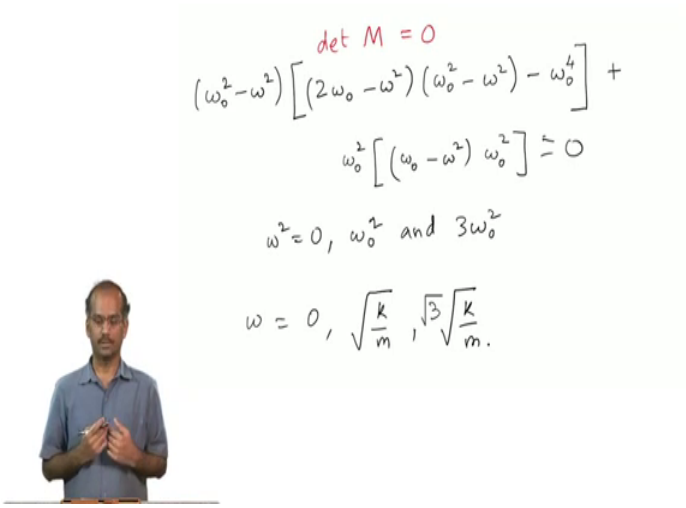

I have written the equation corresponding to the determinant of M and you can easily see that the highest power of ω that appears here would be 6. It would appear as ω^6 and we want to find out ω^2, so it is essentially a cubic equation. So, you should expect to have 3 roots and 3 roots are the 3 normal mode frequencies.

So, again I am going to let you solve for omega square for this, and if you do it correctly you should get the following. These are the 3 roots of this equation, and it is equivalent to saying that the normal mode frequencies ω would correspond to 0,$\sqrt{\frac{k}{m}}$ and $\sqrt{\frac{3k}{m}}$.

So, I have done this problem by skipping many steps and I urge you to complete the steps in between. And as you can see it is a fairly routine exercise. The main ingredient is really to write down the equations of motion. Once that is done rest of it is fairly simple and the straightforward.

In the next module, we will see how to obtain the normal modes for these cases, but as a last part of this module we will see some experimental demonstration of coupled pendulum.

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- In this experiment two identical compound pendula are coupled together by a spring and their dynamics is studied.
- Initially, one pendulum is held fixed while the other is displaced slightly from its equilibrium positon.
- Dissipative effects are neglected and the spring is assumed to be ideal.

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- It is observed that, at different times, when the amplitude of oscillation of one of the pendula is its least, the amplitude of oscillation of the other pendulum is its greatest and vice versa.
- At different times, when the energy of one of the pendula is its least, the energy of the other pendulum is its greatest and vice versa.
- The energies of the pendula are oscillatory functions of time that are out of phase but the total energy of the system is a constant as expected from the theory.