Waves and Oscillations
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Lecture – 18
General Method of Solving for Normal Modes

Welcome to the third module we are in the 4th week, and we started looking at a coupled Oscillations and we have progressed up to the point where we looked at the solutions of coupled oscillator and we understood that you could do a change of coordinate system in which the solution becomes very simple.

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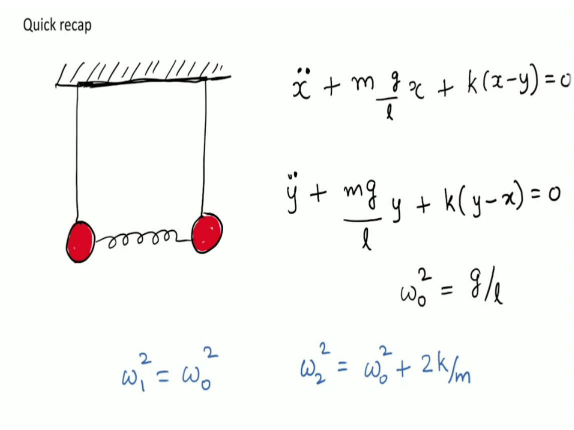

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I $\ddot{u} + \frac{g}{\ell} u = 0$ $\qquad \omega_1^2 = \omega_0^2$ I

II $\ddot{v} + \left(\frac{g}{\ell} + \frac{2k}{m}\right) v = 0$ $\qquad \omega_2^2 = \omega_0^2 + \frac{2k}{m}$ II

$u(t)$, $v(t)$

normal modes

normal mode frequencies

So, just to quickly do a recap of what we have been doing, we wrote down the equation of motion in the u and v coordinate system and once you do this what has happened is now you have two equations of motion which are uncoupled. So, the equation of motion for u does not involve the variable v and vice versa. So, you can immediately write down what are the frequencies ok; so, which is what is shown here. The trick in going from describing the equation of motion in xy coordinate system to the one which is described in terms of u and v was basically a following.

(Refer Slide Time: 01:22)

$$(x, y) \rightarrow (u, v)$$

$$u = x + y$$
$$v = x - y$$

So, you did a coordinate transformation from x, y to u, v and most importantly this is the transformation u was $x + y$ and v was $x - y$. So, in some sense we are lucky to have been able to discover this set of coordinate system in which the equations became uncoupled. And once it became uncoupled it was straightforward to write down the solutions because each of the equation essentially looks like a one-dimensional simple harmonic oscillator equation.

We introduced the term that these two frequencies ω_1 and ω_2 or what would be called normal mode frequencies and this $u(t)$ and $v(t)$ are simply called normal modes or normal mode solutions. And we also saw that since each of the normal mode does not interact with the other normal mode the energies of each of these is a constant and the total energy of the entire system is also a constant. Whereas, if you look at it from the point of view of x y coordinate system you will notice that the equation of motion for x involves y and vice versa; so, the individual energies are not conserved, but miraculously the total energy is a constant of motion. With this background let us today look at a way of solving this in general.

As I said before we are quite lucky to be able to discover this set of coordinate transformation which uncouples the coupled system of equation ok, but there is no guarantee that the same coordinate transformation would work for any other coupled problem. In general, you need to each time find out a different set of coordinate transformation which will uncoupled provided it exists. So, here what we are going to do is not to guess or be lucky in our pursuit of these special kinds of coordinate systems we will write out a systematic way by which we can actually achieve this.

(Refer Slide Time: 03:55)

$$\ddot{x} + m\frac{g}{l}x + k(x-y) = 0$$

$$\ddot{y} + \frac{mg}{l}y + k(y-x) = 0$$

$$x(t) = A\,e^{i\omega t} \qquad y(t) = B\,e^{i\omega t}$$

$$\dot{x}(t) = Ai\omega\,e^{i\omega t} \qquad \dot{y}(t) = Bi\omega\,e^{i\omega t}$$

$$\ddot{x}(t) = -A\omega^2\,e^{i\omega t} \qquad \ddot{y}(t) = -B\omega^2\,e^{i\omega t}$$

Let us continue with the same system that we have been using which is 2 pendula connected by a spring and I have here the two equations. Now, I am going to solve it differently from the way we did, I will not a priori introduce these magical set of coordinates. So, my starting point is this assumption that whatever be the frequency with which the entire system is oscillating it is doing with one particular frequency. So, let me explain it, but by first writing down the solution that I am going to assume.

So, I am going to assume that $x(t)$ is $Ae^{i\omega t}$ where ω is one of the normal mode frequencies and $y(t)$ is $Be^{i\omega t}$. So, what is implicit in this choice of solution is that the entire system is oscillating with one particular frequency. So, physically that should not be very surprising simply because after all its one system coupled system and there is no way that the one part of the system can oscillate with a very different frequency from other part of the system. So, in general the entire system is oscillating with one frequency which is what is implied in this assumed solution.

And also, I have made some assumption about the initial condition basically that the velocity is 0. If the velocity initial velocity is not 0 then I would have had to add additional phases to it but let us simplify the problem and assume that the initial velocity is 0. Now, all that I want to do is to substitute these things back in this equation. So, let us do that as the next step.

(Refer Slide Time: 05:56)

$$-A\omega^2 + \omega_0^2 A + \omega_s^2 (A-B) = 0$$

$$-B\omega^2 + \omega_0^2 B + \omega_s^2 (B-A) = 0$$

$$\omega_0^2 = g/l \quad \text{and} \quad \omega_s^2 = k/m$$

$$A(\omega_0^2 + \omega_s^2 - \omega^2) - \omega_s^2 B = 0$$

$$-A\omega_s^2 + B(\omega_s^2 + \omega_0^2 - \omega^2) = 0$$

So, now what I am going to do is to collect all the terms with A and B together and write out these two equations. So, now I have these two sets of equation is just rearranged from here to here. Now, you can see that I can elegantly write it in matrix form.

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$$\underbrace{\begin{pmatrix} \omega_0^2 + \omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_0^2 + \omega_s^2 - \omega^2 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\chi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$M\chi = 0$$

$$\det M = 0$$

So, this is the matrix multiplies to this vector which has elements A and B and on the right-hand side I have 0 0, this entire matrix I can rename it as M and this vector I could rename it as χ. So, in matrix notation this would simply be M times χ is equal to 0.

Nontrivial solutions for the system of equations needs to exist to say that A and B should have non trivial solution which means that the solution should be something other than 0 because if I put A equal to 0, B equal to 0 you will notice that it trivially satisfies this system of equations. And imagine if A and B are equal to 0 it simply means that amplitude is 0 there is no oscillation.

So, that is not the solution that we want. So, we want some nonzero values for A and B and that will be satisfied if determinant of M is equal to 0. So, there is a standard result in linear algebra in case you are not aware of it I a huge you to go back and look at relevant chapters in any typical linear algebra book. For a system of equation like this a nontrivial solution for χ will exist if determinant is equal to 0. So, I am going to calculate the determinant of M now and that is fairly straightforward to do.

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$$(\omega_0^2 + \omega_s^2 - \omega^2)^2 - \omega_s^4 = \det M = 0$$

$$(\omega_0^2 + \omega_s^2 - \omega^2)^2 = \omega_s^4$$

$$(\omega_0^2 + \omega_s^2 - \omega^2) = \pm \omega_s^2$$

$$\omega_0^2 + \cancel{\omega_s^2} - \omega^2 = \cancel{\omega_s^2}$$

$$\boxed{\omega^2 = \omega_0^2}$$

So, I have written the expression for determinant of M, and it is set equal to 0 and I just need to solve it for the only unknown which is there in this equation which is ω. So, we know what is ω_0 we know what is ω_s. I just need to find out the unknown frequencies which I had started my problem with. So, this is one result again if you recollect one of the frequencies were simply ω_0 itself which is what we have got now.

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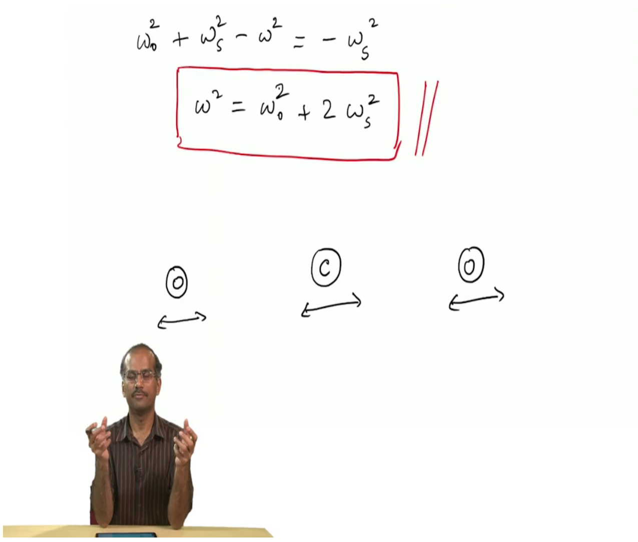

So, I have the second equation, and this can be easily simplified, so this is my second frequency. And now when you compare these two results you will notice that this is exactly the result that we had got earlier around in the last module by just adding and subtracting the two equations from which we started with basically these two equations. But now, purely based on physical motivation that the whole system would oscillate with a single frequency, simply assume that the solutions are of the form $Ae^{i\omega t}$ or some amplitude times $e^{i\omega t}$ for x and y component. And then compute $\dot{x}$ $\ddot{x}$ substituted back and demand that A and B should give you nontrivial solutions.

So, again you have two possible normal mode frequencies which exactly coincides with the result that we had got earlier. And finally, before I close this one might ask like, where are we seeing these kinds of coupled oscillators. In fact, most of the time you do see large number of particles oscillating together whether its sound waves or many other mechanical waves.

In particular for example, one can think of simple examples like say the carbon dioxide molecule say this is a carbon molecule and maybe oxygen molecule here of course, there is no spring, but it's the potential that is responsible for keeping them together and each of them could be oscillating. So, that is an example of a coupled oscillation and in fact, in this case you have 3 particles 1 carbon atom and 2 oxygen atoms.

So, in principle you have 3 normal modes and 3 normal mode frequencies. So, its something that you can sort of generalize we saw that when you had when we had 2 pendulums coupled by a spring because you have 2 particles finally and we ended up with 2 normal modes and 2 normal mode frequencies. And it's also not too difficult to see that when you have things like these carbon dioxide molecules and so on you have 3 particles and you would actually get 3 normal mode frequencies. In general, you should expect to see in N normal modes for an N particle system. And we will see some more examples of the coupled oscillations in the next lecture.