Waves and Oscillations
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Lecture - 17
Coupled Oscillators:
Part 2

Welcome to the 2nd module of the 4th week. We are looking at coupled oscillations and this will be in some sense part-2 of the coupled oscillations. And, as usual before we go ahead let us quickly recap what we have been saying. By coupled oscillations we mean coupling together several particles. So, you can think of it as like several particles coupled by a spring for instance, but in reality you really do not need a spring or any such physical object to couple objects together, it is enough if potentials interact.

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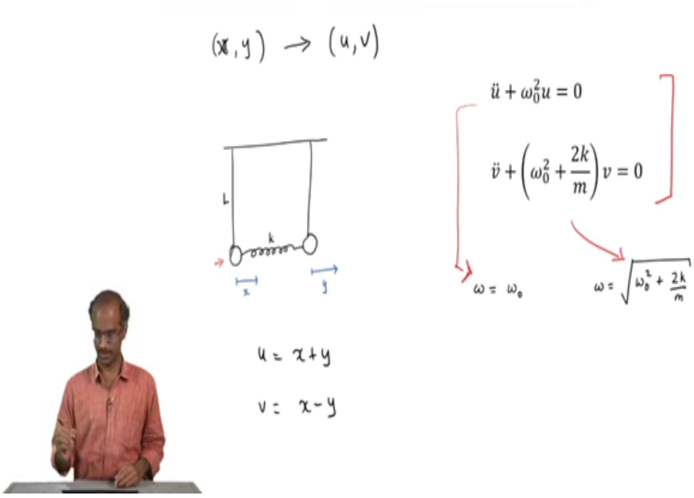

We looked at this example of two pendula coupled together by a spring of spring constant k. The new thing that is entering the picture is that, when you disturb one of the particles in this system the other one is automatically going to get the disturbance sooner or later. So, both will start oscillating. So, you cannot treat one of them in isolation without considering the other. So, in other words what we have is a total system which is made up of say, two particles like we have it here in the slide or maybe it has some large number of n particles, in either case you cannot treat it as n individual particles. It has to be treated in its totality. And what we did was to write down equations of motion for each of these particles.

So, we said that the displacement of the first particle be called x and the displacement of the second particle be called y and we wrote down the equations of motion corresponding to each one of this. So, in general we assume that, x is greater than y and we have these two equations of motion written together. And as usual we have identified this $\frac{g}{l}$, g is the acceleration due to gravity and l is this quantity which is the length of the string in the pendula, ω_0^2 which is like the natural frequency of the individual system is $\frac{g}{l}$.

Now, with this the trick we did was to add the two equations and subtract the two equations and when we added the two equations and subtracted the two equations, we went to a new set of equations. And, to be able to do that all we did was to go from let us say this coordinate system which is described by (x, y) to a coordinate system which will be described by (u, v) and the actual transformation was that u is $x + y$ and v is $x - y$. So, when you do this transformation what you will see is that, this particular set of two equations will transform into these two sets of equations.

And as you can see these two equations are now independent equations, in the sense that the equation for u does not involve v and the equation for v does not involve u. So, it is like the 1-dimensional oscillator equation of motion that we saw earlier. So, we can straight away write down the frequencies. So, in the first, for the first case the frequency was this we did this identification, and for the second case the frequency is given by this.

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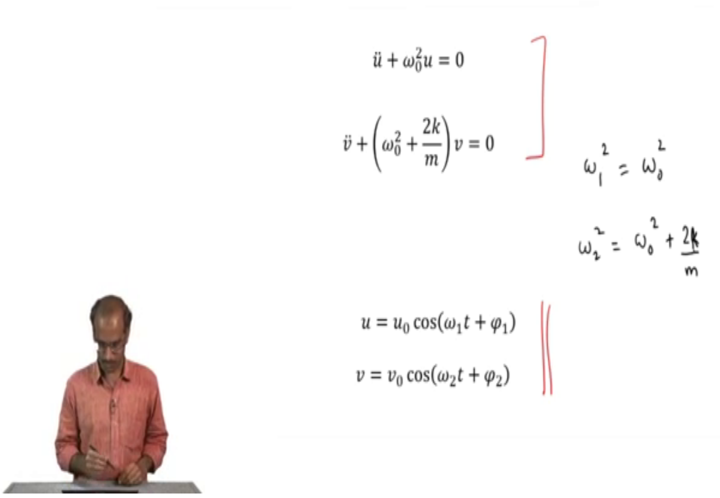

So, given these two equations of motion, we can straight away write down the solution because we dealt with this in the very first week itself. The solutions are simply *sine* or *cosine* functions and in this case I will simply choose to use cosine function. So, I have written down the solutions here and there are these two frequencies ω_1 and ω_2, and ω_1 is related to ω_0. So, ω_1^2 will be ω_0^2 whereas, ω_2^2 will be $\omega_0^2 + \dfrac{2k}{m}$, and ϕ_1 and ϕ_2 are the two phases.

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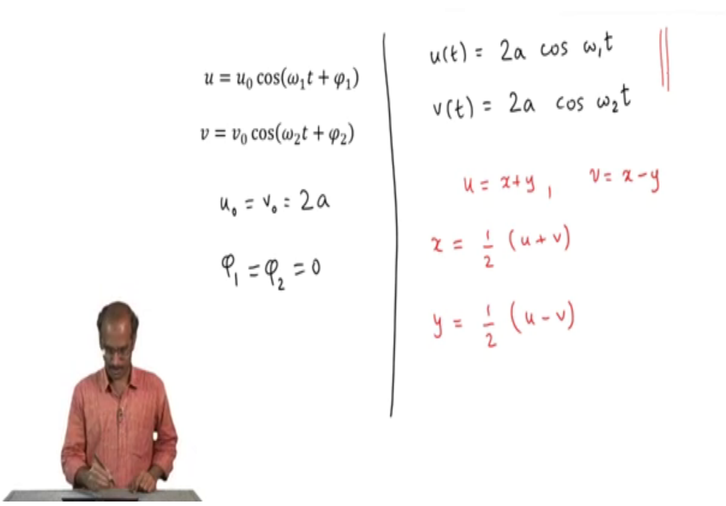

To go further I am going to make some choices for the amplitudes and the phases. So, let me take the amplitudes u_0 and v_0 to be simply equal to 2 times a. So, I just want to make the amplitudes equal, so that it makes analysis easier and also again for the same reason I will also take ϕ_1 and ϕ_2 to be equal to 0, in which case these solutions would become $u(t)$ is $2a\cos\omega_1 t$ and $v(t)$ is $2a\cos\omega_2 t$.

So, if you remember we had said that u is equal to $x + y$ and v is equal to $x - y$ and from this I can write an expression for x and y, x will be equal to $\frac{1}{2}(u + v)$ and y will be equal to $\frac{1}{2}(u - v)$. So, let me substitute these two expressions in this expression for x and y in which case I will get the following.

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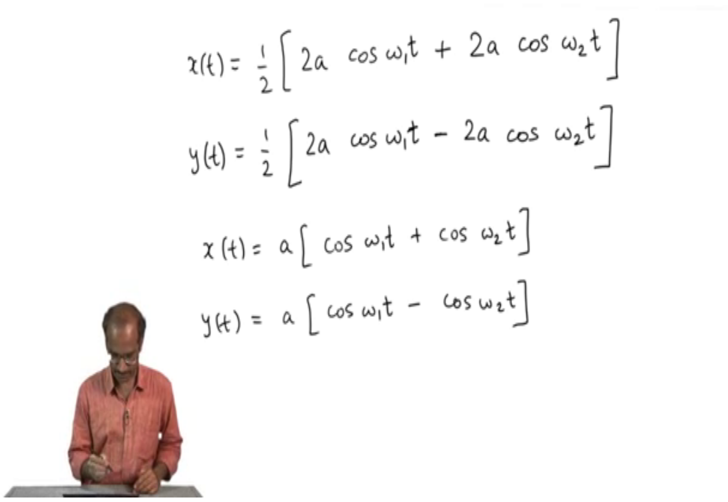

I have these two expressions for x and y as a function of time and so, 2 can be cancelled throughout. So, x will simply be equal to $a(\cos\omega_1 t + \cos\omega_2 t)$. So, if I simplify it $x(t)$ would simply be equal to $\cos\omega_1 t + \cos\omega_2 t$ and similarly I will get another expression for y which will be $\cos\omega_1 t - \cos\omega_2 t$. Now, we can use the $\cos(A + B)$ and $\cos(A - B)$ formula and let us see based on using that formula what kind of dynamics we can interpret out of this.

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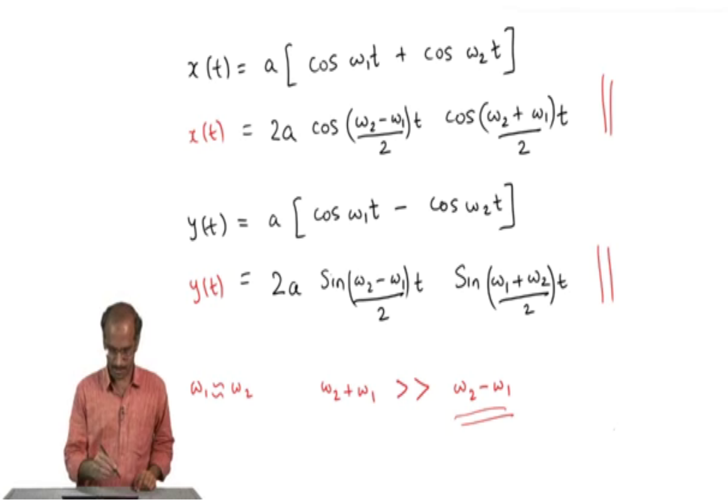

So, using cos a plus cos b formula the trigonometric identity, I would get the following relation and I can write a similar expression for $y(t)$, which will give me the following result.

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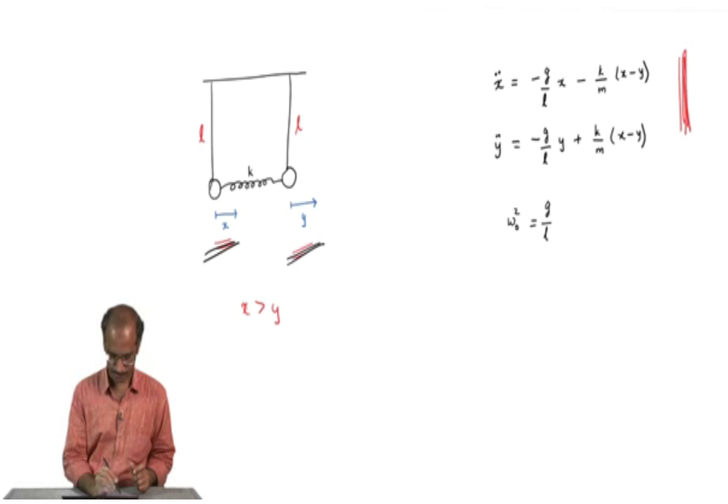

So, now we have expression for, if you remember, x and y are simply the displacements of the first and the second pendula. So, finally, I have managed to get the expressions for the displacement of each of those pendula this and this. We should plot both of them together and see what it reveals. These look like the kind of expression we obtain for the case of the beats phenomena. So, there is one component which is going to be a fast oscillation especially if we make the assumption that ω_1 and ω_2 are nearly equal to one another, $\omega_1 + \omega_2$ be much greater than $\omega_2 - \omega_1$.

So, if I assume that ω_1 is approximately equal to ω_2, in such a case $\omega_2 + \omega_1$ will be much greater than $\omega_2 - \omega_1$. So, with that assumption put in we would get what would look like beats.

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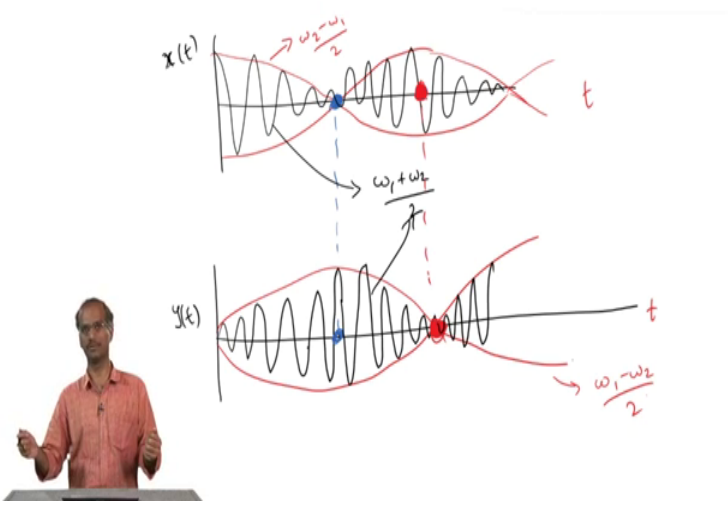

So, let us sketch that the fast oscillation would correspond to the higher frequency $\frac{\omega_1 + \omega_2}{2}$ and the slower oscillation which is shown in red would correspond to $\frac{\omega_2 - \omega_1}{2}$. So, if you look at the y oscillation again it has a component which is faster, which is $\frac{\omega_1 + \omega_2}{2}$ and there is the slower there is the slower oscillation which is given by this red profile which will be $\frac{\omega_1 - \omega_2}{2}$.

Now, what you will see is interesting. When you compare the displacements of x and y side by side you will notice that, at this point when the displacement of the x oscillator is nearly 0, the y oscillator precisely at the same time has maximum amplitude oscillations.

In other words, when you physically translate it is equivalent to saying that you have given energy to this system of 2 pendula connected by a spring and you set it to oscillations.

What is going to happen is, there will be times when one of the oscillators, let us say, this one is going to show you 0 displacement essentially it is going to remain at its equilibrium position and at precisely the same time the y oscillator is going to show large amplitude. And that is not the end of the story, it is going to go further and at some other time x oscillator is going to have larger amplitude somewhere, let us say, here and precisely at the same time the y oscillator is going to have 0 amplitude or 0 displacement. So, the totality of the picture is as follows.

So, both of them are, say, two oscillations and at some point when the x oscillator has 0 displacement y oscillator has maximum displacement and conversely when x oscillator has maximum displacement y oscillator has 0 displacement. Now you can imagine what does it mean to say that the displacement is 0 the particle is not oscillating at all and in that case you could also verify that the velocity would be 0. So, which means that the kinetic energy of the particle at that point is also 0.

So, at these blue points the energy of one of the oscillators in this case, the x oscillator, is nearly 0 and the y oscillator has all the energy. So, initially you gave energy to both of them, but a point in time has come when x oscillator has nearly 0 energy, but y oscillator has all the energy, but at a later instant in time what you see is that, if you look at the red points here and the red point here you notice that, now all the energy is with the x oscillator and the y oscillator does not have energy. So, this scenario is going to repeat itself in time again and again. So, the energy is going to keep shuttling between one oscillator to the other and back to the first oscillator and so on.

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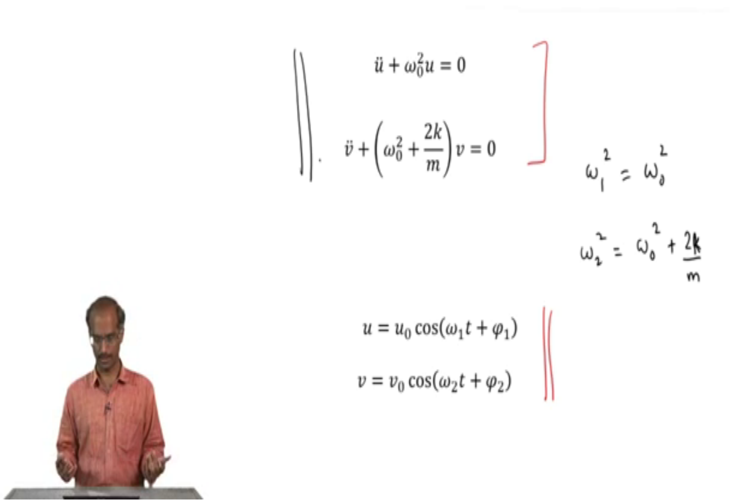

When we finally, wrote down the equations for these two systems in terms of the transformed coordinate system, that is, these two equations, they are exactly like the one-dimensional oscillator equations that we had seen much earlier on and that is the case for which we argued that, since there is no mechanism to dissipate energy, energy is not lost, energy is a constant of motion. So, here, the same story applies that the total energy is still a constant, but it keeps moving between the first oscillator and the second and back to first oscillator and so, on.

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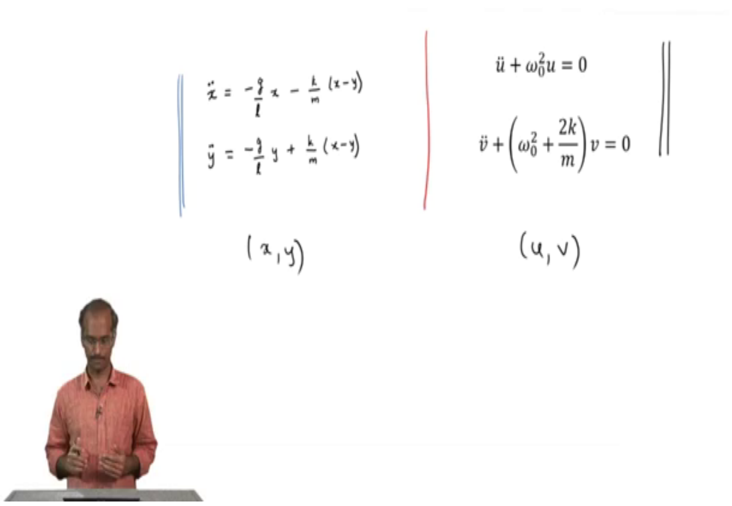

In the first set of equations where we describe these coupled oscillators in terms of x-y coordinate system and the second one we just use the transformed coordinate systems in terms of u and v. So, if you look at it from the perspective of u and v oscillator, the energies cannot keep shuttling between the u and the v oscillator because u and v are two independent oscillators. They do not interact with one another that is what we see when we look at these two equations, again remember that each one of them is like an independent one-dimensional harmonic oscillator their frequencies are different.

So, u is an independent oscillator, v is another independent oscillator whatever energy you initially put in u and v will remain in u and v modes forever.

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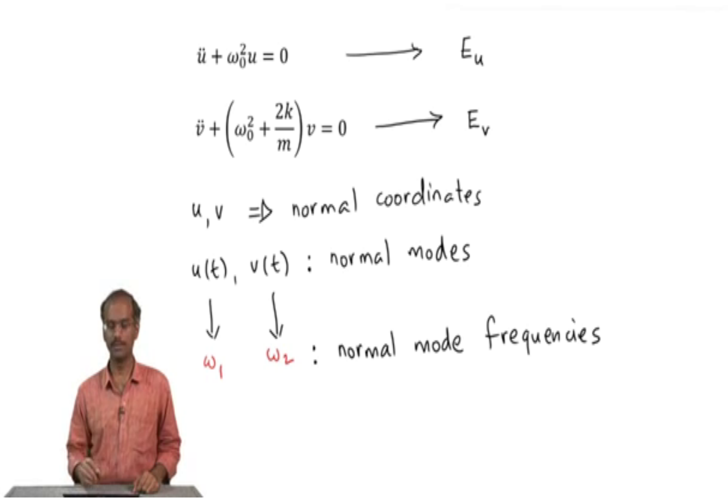

So, let us write down all these consequences. The two new coordinates that we wrote down u and v would be called the normal coordinates or normal modes and as we had already seen these new coordinates u and v are related to the old coordinates x and y, the difference being that in the x and y coordinate system the two oscillators are coupled, but in u and v they are uncoupled.

So, it is a special coordinate system, we will give a name for that. So, u and v together would be called normal coordinates or $u(t)$ and $v(t)$ would be called normal modes or the pattern of oscillation given by $u(t)$ and $v(t)$ would be the normal modes. And similarly we also saw that there were two frequencies, one associated with the u mode and the other associated with v mode. The two frequencies are ω_1 and $\omega_2$corresponding to u and v mode and these two are called normal frequencies or normal mode frequencies. And these frequencies are clearly different from the frequencies of the one-dimensional oscillator.

So, in this case, it so happens that u mode or one of the modes corresponds to the frequency of one individual pendulum and the second one of course, takes into account the presence of the spring in between the two pendulums. And, I can associate let us say energy E_1 with the u mode or let me call it E_u and energy E_v with the v mode.

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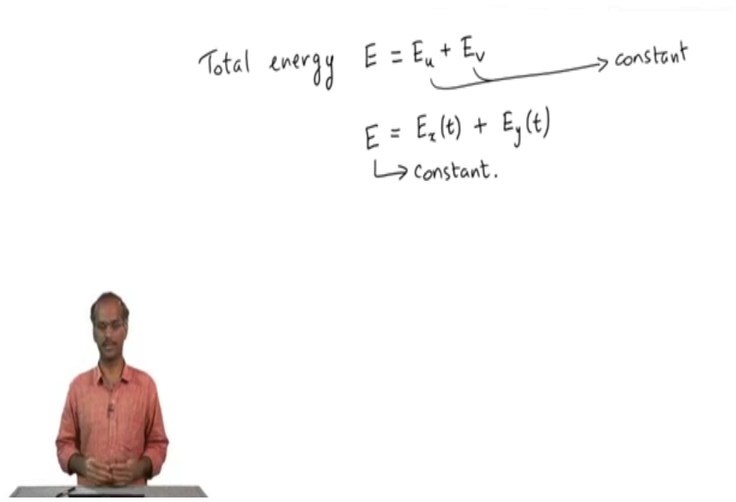

Total energy of the entire system will not be a function of time and the constant value would be $E_u + E_v$. On the other hand, I can do the same thing with x and y mode as well. So, E_x and E_y would be the total energy which would be a constant but very crucially each of this E_x and E_y themselves will not be constant. E_x would be a function of time and E_y would be a function of time. But, the sum of E_x and E_y would be a constant independent of time whereas, when you look at it from the point of view of the normal modes, and their associated energies. Again let me repeat the associated energies being E_u and E_v. Each of them is a constant and the total energy is also constant.

So, in the normal mode coordinate system there is no interchange of energy between the two modes, each one acts like an independent oscillator. You will realize, that our method of analysis dependent on being able to see that by adding and subtracting the two equations of motion, we were clearly able to take coupled equations of motion into two uncoupled equations of motion that was a trick that we employed. Now, it is not clear that if I write out more complicated, I mean, if I analyze more complicated couple d systems this trick would always work. So, we need to find a general way of dealing with such coupled system and this is what we will do in the next module.