Waves and Oscillations
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Lecture - 16
Coupled Oscillators: Part 1

Welcome to the 4th week, this is the first module and at this point we are going to sort of shift gears and get into another level of complexity as we deal with more of oscillations. So, in the last 3 weeks we saw what we called simple oscillations of let us say a single particle, then we added another realistic effect which is to add damping to it. We saw what are damped oscillator is and then we also added another realistic effect which is to give it continuous supply of energy so, that it keeps oscillating. So, all this we did for a single particle, but if you think carefully about what you actually see around you, many of what you see as oscillations you would see that very rarely you would come across oscillations of a single particle.

Often what you see is a collection of particles working together, oscillating together. So, you for example, I even just pushed the air here ok, it is going to set off oscillations in some way and the disturbance that I created just by pushing the air here moves forward. In fact, that is what happens when I speak or when anyone speaks ok, you are able to hear me speak simply because the sound waves which are essentially disturbances, pressure variations in the medium of air are created and they get propagated.

So, here a lot of these molecules are coupled together, somehow they act together and help in propagating wave forms. So, let us today start with how we describe these coupled phenomena. In principle we want to study what happens if I couple a large number of particles, maybe 10^{23} atoms Avogadro kind of number, it is going to be fairly complicated. So, let us begin by first starting to learn what happens if I couple just two of them.

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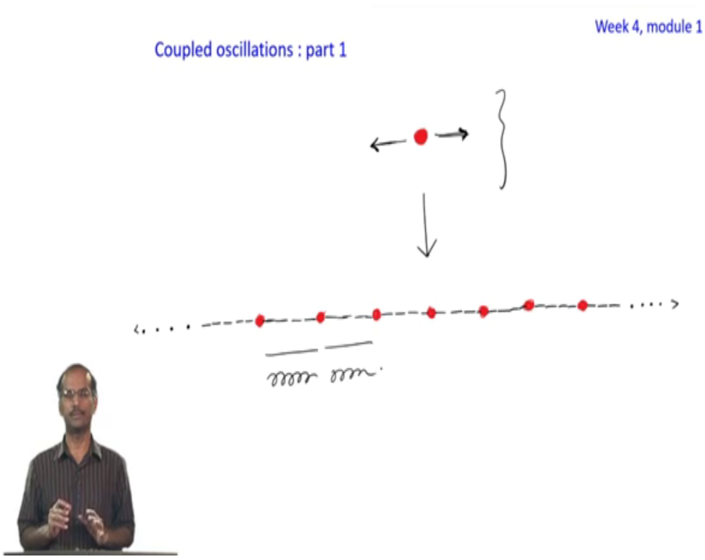

We had already seen everything about or at least the basics of single particle oscillations. And, now we are going to progress towards studying coupled oscillations as you can see of many particles coupled together. And this coupling can take very many forms, it could just be a string that is coupling two particles here or it could be a spring. Or maybe there is nothing physical like a spring or a string, it could just be the effect of the potentials, potential at each site for instance.

So, the coupling can take very many forms, it does not have to be always a spring or a string or something that is visible to us ok. So, keeping all these generalizations in mind, let us try and formulate and see whether we can solve coupled oscillations.

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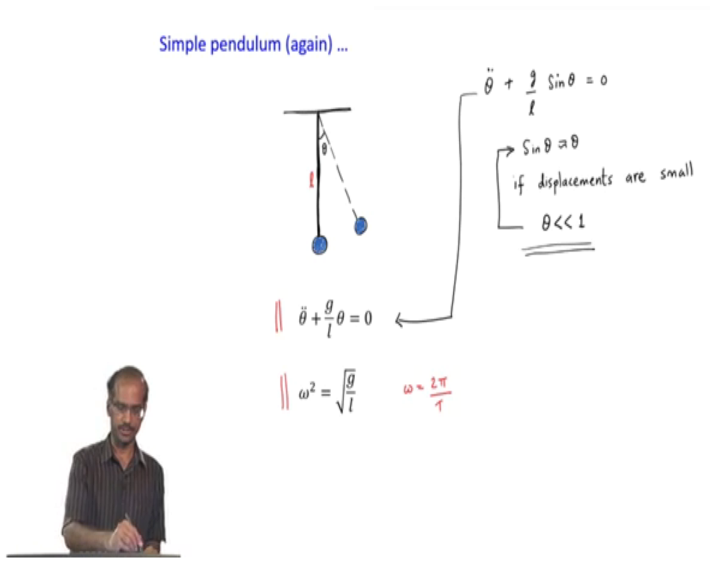

And, as we do that it is a good idea to again look back at one system which is a sort of paradigm for oscillations that is the simple pendulum. So, we have already done this in a bit of detail in the very first week. What I have is a simple pendulum and here I have the equations of motion written for this simple pendulum, we solve these things. And, once I have the equations written for it, I can immediately extract the angular frequency and the angular frequency depends only on the acceleration due to gravity which is g and l which is the length of the pendulum.

And from this its straightforward for me to get what is time period because, I know that ω is 2π by time period. If I substitute I will get the standard formula for the time period of a simple pendulum ok. Now, let us go one step further. So, as I said we are going to couple two oscillators. Now, we can ask the question what if I couple two pendulum?

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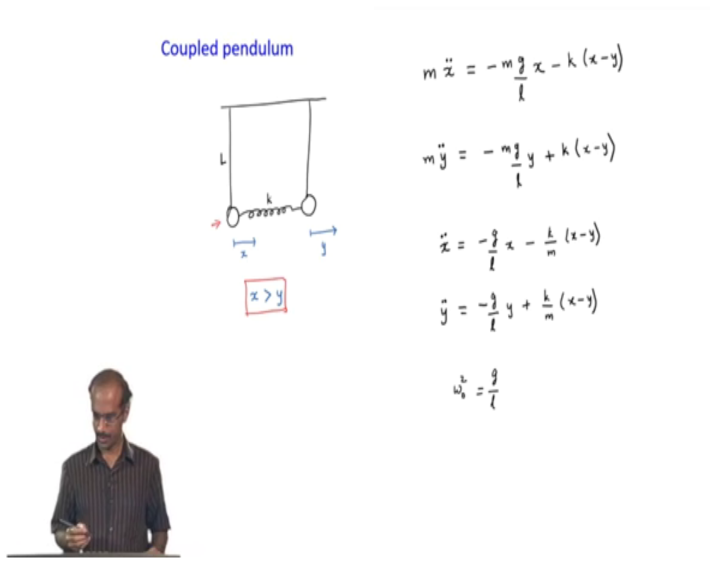

So, here is what I plan to do. So, let us say that I have one pendulum here, another pendulum here and I am going to couple these two pendulum through a spring. And we shall assume that L is the length of the string here and m is the mass of these bobs that you see here and k is the spring constant for the spring. And as usual we are going to make very idealistic assumptions which may not be true in practice, but nevertheless are useful to obtain some good understanding of ideal systems. So, what are the idealistic assumption? So, we are going to assume that spring does not have a weight, strings also do not have weight and the bob here is a point particle and so on.

These are the standard assumptions and most importantly we will continue to stick to small angle approximation. In the sense that if you remember in the case of simple pendulum more correct equation of motion was,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

And we said that $\sin\theta$ is approximately equal to θ, if displacements are small and this can be mathematically written as theta being much smaller than 1. So, we are going to stick to this so, called small angle approximation even in the case of coupled pendulum. So, to begin with we need to write the equations of motion.

Let me first say that the left hand side pendulum has x pendulum and let us say that I am displacing it by an amount x. And the pendulum on the right hand side I am displacing it by an amount y. So, I am going to make a general assumption that x is greater than y. So, let me first write the equation of motion for the x pendulum

$$m\ddot{x} = -\frac{mg}{l}x$$

This assumes that you have nothing else, in the sense that there is no y pendulum if there was only x pendulum here then this is the equation of motion, but we know that there is a y pendulum and there is a spring.

So, somewhere they should make it, they should make their appearance and here it makes its appearance in this way.

$$m\ddot{x} = -\frac{mg}{l}x - k(x - y)$$

So, that is a second term which corresponds to the effect of having a spring coupling the two bobs. Let me explain to you how this term comes, but before I do that let me also write the equation for the second bob corresponding to the y pendulum;

$$m\ddot{y} = -\frac{mg}{l}y + k(x - y)$$

Now, you notice that I had made this assumption that x is greater than y in which case you can imagine that, if your x bob which is this one is being pulled in this direction. And if x is the displacement of the x bob and y is the displacement of the y bob, the natural length of the spring has changed now.

So, it is going to act against the acceleration of the x bob. So, the change in length is $x - y$ and since it is acting against the acceleration of the x bob you have this crucial minus sign here. On the other hand under the same condition it is going to help the acceleration or favor the acceleration of the y bob which is why you have a plus sign here. And, just to repeat so the compression of the spring by my assumption is such that it is helping the y bob to accelerate whereas, it is acting against the acceleration of the x

bob. So, if I had made this assumption as y greater than x then these signs would sort of exchange with one another.

So, with that in mind we will now have to find a way of solving this equation. Now of course, if $k = 0$ it is like two independent pendulum and two independent pendulum we know the solution. We have studied that already, is just that there are two independent pendula and they do not interact with one another. So, what the coupling does is to make them interact with one another, they are no more independent. So, you disturb one, it is going to affect the other; to begin with first let me rewrite this equation slightly differently.

So, I will write it as,

$$\ddot{x} = -\frac{g}{l}x - \frac{k}{m}(x - y)$$

$$\ddot{y} = -\frac{g}{l}y - \frac{k}{m}(x - y)$$

So, since each of them is a pendula that is the main oscillating object and we know that for a single pendulum this is my ω^2. I am going to call the quantity $\frac{g}{l}$ as ω_0^2, in some sense natural frequency of a simple single pendulum ok. So, with all this we can rewrite the equations as follows ok.

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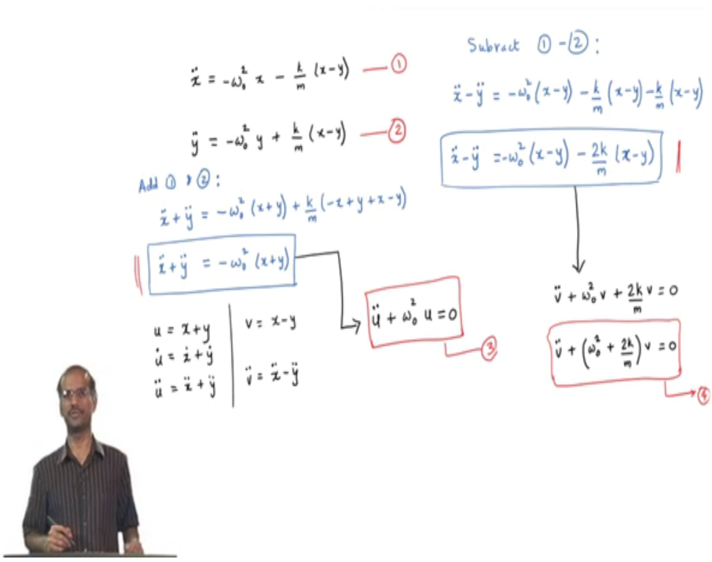

So, I have just substituted ω_0^2 for $\frac{g}{l}$. Now, the question is how do I solve it? You know it is still a second order ordinary differential equation, like the one we had for the case of simple pendulum and many other systems, but the crucial difference because of coupling is the following. So, if you look at the first equation here this one. So, it is an equation for x displacement of the x pendulum, but you notice that the x pendulum has contribution from the y pendulum, that somehow appears in the equation.

And similarly you look at the equation for the y pendulum, it has contribution from the x pendulum. So, in that sense they are coupled; earlier we could easily solve it because there are no such couplings. Now, you need to know y to be able to solve x and to solve y you need to know x. So, it is a kind of chicken and egg problem. How do we solve it? So, let us see let us try a few things. So, the first thing I am going to do is to add and subtract the equation, you will see why I am trying to do that. So, let us call let us say that this is equation 1 and this is equation 2. So, let me first add the two equations, if I add the two equations so, I have one equation which looks like this.

So, before we understand this, let me also do the second part that I said. So now, I have added equation 1 and 2. So, that is what I have done here add 1 and 2, that is what gave you this; now let me subtract the two. So, I want to do the following subtract so, I want to subtract 2 from 1. So, that would give me, now I have simplified and written this equation. So, I have the second equation which is resulting from subtracting equation 2 from equation 1. Now, if you look at the structure of these two equation, it will be clear what we will do next. In the first equation that I have here, the variable always comes as $x + y$ or $\ddot{x} + \ddot{y}$.

And in the equation here it always comes as $x - y$ or $\ddot{x} - \ddot{y}$. So, which means that I can in fact, invent a new variable which is for example, could be $x + y$ let me call it u. And if u is $x + y$, $\dot{u}$ will be $\dot{x} + \dot{y}$, similarly $\ddot{u}$ will be $\ddot{x} + \ddot{y}$. And, we can also following the same recipe, you can also call v is equal to $x - y$ and $\ddot{v}$ will be equal to $\ddot{x} - \ddot{y}$. Now, it is clear what I need to do, simply substitute these things into these equations; let me write that. So, if I take the first of this equation so, this one now would become

$$\ddot{u} + \omega_0^2 u = 0$$

whereas, this equation here would become $\ddot{x} - \ddot{y} = \ddot{v}$

So, that would be

$$\ddot{v} = -\omega_0^2 v - \frac{2k}{m} v$$

of course, this can be further simplified. So,

$$\ddot{v} + \omega_0^2 v + \frac{2k}{m} v = 0$$

Now, examine these two equations; they look like equation of a one dimensional oscillator and the frequency is ω_0^2 in the first case in this one. And the are the square of frequency in this case is $\omega_0^2 + \frac{2k}{m}$. Now, you see that the equation for u depends only on you, v does not enter the picture here and similarly the equation for v depends only on v and it does not involve u.

So, we seem to a magically uncouple the two oscillators. And what was our formula for doing that? We simply went from one set of coordinate system to a different set of coordinate system.

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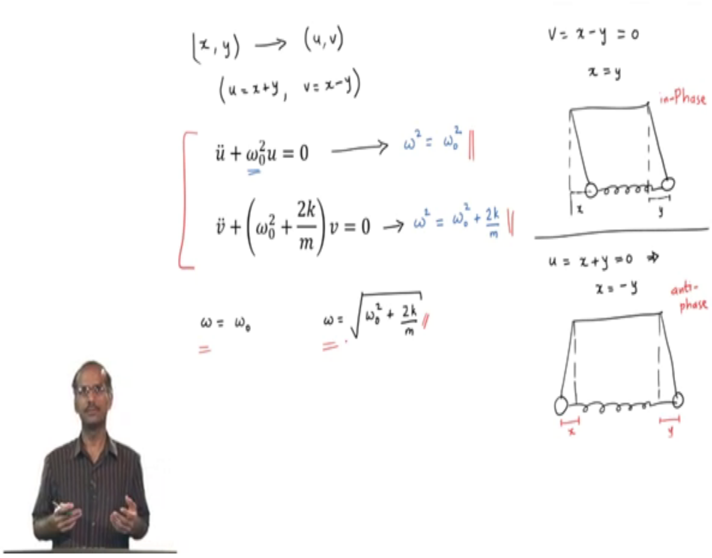

So, originally our coordinate system was defined by x and y and we went from x and y to u, v. And the transformation was $u = x + y$ and $v = x - y$. And this appears like a very magical transformation which transformed a coupled set of equations given by equations 1 and 2 to an uncoupled set of equations given by this equation 3 and 4.

So, let us call it equation 3 and this is equation 4. In the new coordinate system that we wrote down which is this $x + y$ and $x - y$ coordinate system, the equations of motion are here in front of you. And as I pointed out you could say that each one is now an independent equation, it does not couple to the other.

So, the equation for u does not involve v and the equation for v does not involve u. So, its made our life easier because we know how to solve these equations. So, in the case of this equation, the frequencies are ω^2 is simply equal to ω_0^2, the quantity that sits here. And, in the case of this equation, the frequencies are ω^2 is equal to $\omega_0^2 + \frac{2k}{m}$. And you will notice again that as a cross check you can set $k = 0$; meaning that spring constant is

0. There is no spring at all in the first place. In such case you just get back two independent pendula, both of them have frequency which is equal to ω_0.

So, it is consistent to that extent, let us set $v = 0$ in which case v being equal to $x - y$ and that being equal to 0 would imply that x is equal to y. So, the constraint here is that x and y, the displacements of each of the pendula should be exactly equal and in phase. So, what we are looking at is that two pendula are essentially oscillating in phase like this and the spring here does not play any role. So, it is like these two pendula which move like this in phase; so, the spring that is in between them is neither extended nor compressed.

So, there is no effect of the spring at all in this particular mode of the couple pendulum. So, this is the case where x and y are equal; so, x and y just to indicate what they are. So, this is x and this is y both the displacements are equal. So, that is the physical meaning of the first set of first equation. Now, the second equation which is the equation for v in which case u could be 0. It should mean that $x + y$ is equal to 0 which implies that $x = -y$. So, that would correspond to the pendulum having the following pattern of oscillation.

So, you could see that the pendula basically move like, this its what is called anti-phase. So, in the other case we saw that they were in phase, both of them move in this direction like this and in this case they are in sort of anti-phase. So in fact, this is called in phase motion, both the bobs are in phase and this is called anti phase motion. Because, as you can see when one of the bobs is trying to go in this direction, the other one actually goes in the opposite direction. So, there is a phase difference of π between these two oscillators. In this kind of a motion what happens is that the spring is always either compressed or its extended; its never it never has its natural length.

Unlike in the first case, in the case of in phase oscillation the spring is never extended or compressed. So, the effect of spring is absent which is why you see that the frequency is simply the frequency of the single pendulum. Both the pendula even though they are coupled by the spring, they do not feel the effect of the spring in the first place. On the other hand in the anti-phase mode, the spring is always either like you can see the way I have drawn its either extended or compressed. So, the effect of spring is there and which is why you have this k coming here. So, in this case the frequency depends on the spring constant k.

So, these two patterns of oscillations: in-phase and anti-phase, they are often called modes of oscillation. And these two frequencies are called the normal mode frequencies. And these two patterns are called normal mode patterns or simply normal modes or normal mode oscillations. To summarize this part we started with trying to couple two pendula, we used a spring to couple the two pendula as we have done here. And we wrote down this equation of motion and we realized that the problem in solving this equation of motion was that the first of the equations or the equation for x involved the variable y and the equation for y involve the variable x.

So, its coupled to one another and what we did was to add and subtract the two equations and we got two other equations which is this equation 3 and 4. And magically it looks like they simply got uncoupled. In the new variables u and v, we have two equations of motion which are which do not interact with one another which is very good for us, because we know how to solve them. And straight away I could extract the frequencies without any further work and we could also physically interpret these modes of oscillations. As I said these patterns, the collective pattern of oscillation is called the normal mode. There are two normal modes, there are two particles basically two bobs.

So, there are two normal modes and corresponding to each of the normal mode or each pattern of oscillation, that it exhibits you have two frequencies. The two pattern of oscillation is one is where both of them are in phase like this, they oscillate like this. And the other one is where they oscillate like this, they maintain a phase difference of pi. In the modules ahead we will see even more complicated versions of such coupled pendulum, until the point when we couple a large number of these.