Waves and Oscillations
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Lecture - 15
Forced Oscillator: Problems

Welcome to the 5th module of this 3rd week. This entire week we had been looking at the Forced Oscillator. So, we will do some problems today, some selected problems. Let me straight away get into the first problem.

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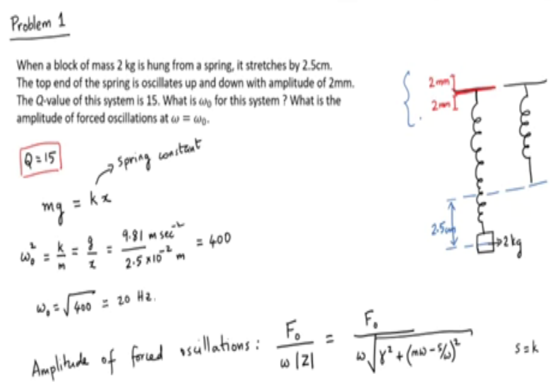

So, in this case a block of mass 2 kilogram is hung from a spring. So, the situation can be picturised like this. So, I have a spring like this and what is done is to this spring a block with mass 2 kg is attached to it. So, when you do that, the spring will extend and the problem states that it stretches by 2.5 cm. So, this is the block of 2 kilograms and so the stretching is by an amount 2.5 cm.

And you are also told that the top end of the spring which is this part here which I am marking now in red is also undergoing harmonic oscillation and it has an amplitude of 2 mm. So, this will be 2 mm. So, it goes up and down 2 mm on either side from the equilibrium position and we are also given that the Q value of this system is 15. So, the first question is find what is ω_0? ω_0 is the natural frequency. So, we would like to know what is the natural frequency of the system. So, this can be done from the information which is given in the first part of the problem. So, we are told that you have a spring and you add the mass, the spring stretches by about 2.5 cm.

Then the stretching which is let us say x would be given by the following. So, the weight which is mg is equal to kx,

$$mg = kx$$

from this equation we can find ω_0 simply because $\frac{k}{m}$ which is equal to ω_0^2 that would be equal to $\frac{g}{x}$ and g is let us say 9.81 m/sec^2 square divided by x which is the amount by which it stretches which is 2.5 cm. So, that would be 2.5×10^{-2} m. So, we need to take the square root that would give us a value for ω_0

So, approximately this value would be about 400 Hz or actually I should take square root of this. So, ω_0 would be about square root of 400. So, approximate value is about 20 Hertz. So, that answers the first part of the problem. The second part is what is the amplitude of the forced oscillations at this frequency which is 20 Hertz. What we know from everything that we studied is the amplitude of forced oscillations is

$$\frac{F_0}{\omega |Z|}$$

which is the impedance. So, that would be

$$\frac{F_0}{\omega\sqrt{\gamma^2 + \left(m\omega - \frac{s}{\omega}\right)^2}}$$

Now we need to evaluate this at ω_0 which is not too difficult.

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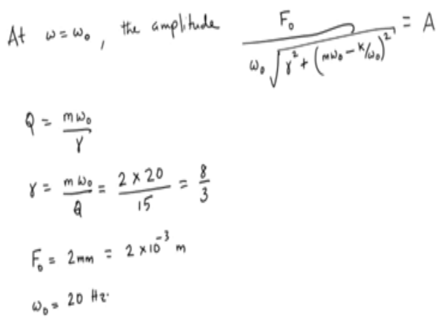

So, at $\omega = \omega_0$ the amplitude would be

$$\frac{F_0}{\omega_0\sqrt{\gamma^2 + \left(m\omega_0 - \frac{k}{\omega_0}\right)^2}}$$

So, in fact the s stiffness constant here is just the spring constant k. So, remember that this s that I wrote down here should actually be k which is the spring constant. So, here mass is already given. So, I know the value of mass spring constant is something that we know actually we know the value of $\frac{k}{m}$ itself. So in fact everything else is known except two things. One is we do not know what is F_0 and we do not know what is γ. γ is easy to calculate simply because we know the value of Q factor which is given as Q = 15. So, Q which is equal to $\frac{m\omega_0}{\gamma}$.

So, you can see that we have calculated ω_0 already, m is already known and γ is the only unknown here. So, γ will be equal to $\frac{m\omega_0}{Q}$. So, that would mean that mass is 2 kg. So, that is 2 multiplied by ω_0 is 20 that is 20 divided by Q which is 15. So, this will be 4 by

3 4 into 2. So, it is about $\frac{3}{4}$. So, I know γ now, I know m, ω_0 is known, k is known. What about F_0? F_0 is the amplitude of the forcing and the forcing here is such that you could see that the top of the pendulum oscillates up and down and the amplitude is 2 mm. So, that is our F_0.

So, F_0 is 2 mm which would be 2×10^{-3} meters. Now we have everything that we need. So, we just need to plug in all these values into this expression for amplitude which I will call it A. We know F_0, we know γ, let me also for completeness write ω_0 that we just calculated which is 20 Hertz. So, with all this information you should be able to calculate the amplitude and I will leave it as an exercise for you to put in the numbers and calculate the amplitude. So, that completes the first problem that we started with.

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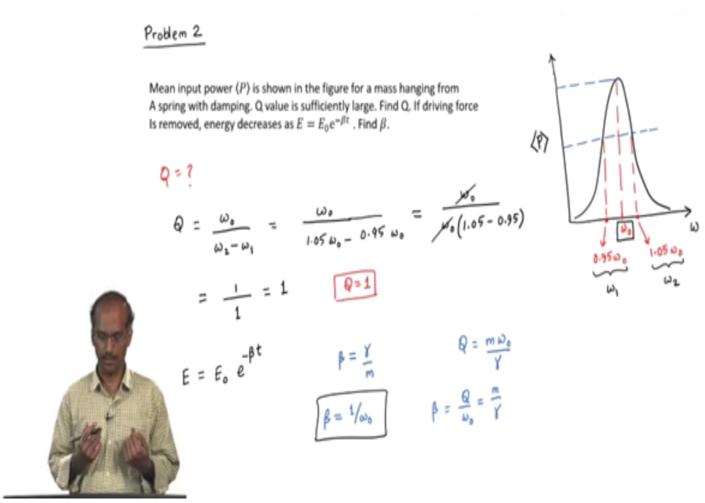

So, this problem is based on the figure which is given here what is shown is mean input power as a function of ω which is the frequency of the driving. And we have told that this is for the case of a spring which is also damped a mass hanging from a spring and it is a damped and we can assume that Q value is sufficiently large and the question is find the Q value itself. And there is a second part to this question which is that if you remove the driving part, the energy of course will start damping and we are given how the energy is damping.

So, the question is find the value of β that appears in this equation. So, let us begin with the first part of the problem to find the value of Q and this is among the easiest of problems because if you remember what we did in the earlier module, Q value is simply ω_0 by ω_2 minus ω_1.

So, ω_0 here is we are not given a number. It is just ω_0. So, that is this quantity here. So, we do not know its numerical value, but it does not matter, but it is ω_0, but we know what is ω_2 and ω_0 ω_1. So, this is $\omega_1$and this is ω_2. So, all I need to do is to simply substitute those values ω_0 divided by ω_2 is 1.05ω_0 minus 0.95ω_0. So, this would be ω_0 by I can take ω_0 common here. That will be 1.05 - 0.95. So, ω_0 and ω_0 cancels. So, you can see that we do not need to know the value of ω_0 in this case. So, the answer is 1 divided by 1. So, the Q value is 1. So, that is the result. So, the Q value in this case is 1.

Now there is a second part to this problem. The driving force is removed and as you would expect if the driving is removed, the oscillator is going to start damping and the energy damping is given here that E as a function of time goes as $E_0e^{-\beta t}$.

So, if you go back to what we studied for the damped oscillator, you will realize that what is given as β here when you relate it to actually the parameters of the problem that would be $\frac{\gamma}{m}$ itself. Q is $\frac{m\omega_0}{\gamma}$ and this can be written as

$$\frac{Q}{\omega_0} = \frac{m}{\gamma}$$

in which case this is simply β itself and we know that $Q = 1$. Therefore, the value of β in terms of ω_0 would be $\frac{1}{\omega_0}$. So, that would be the result unless the value of ω_0 is given, we will not be able to obtain a numerical value for β. So, it is written in terms of the natural frequency of the system.

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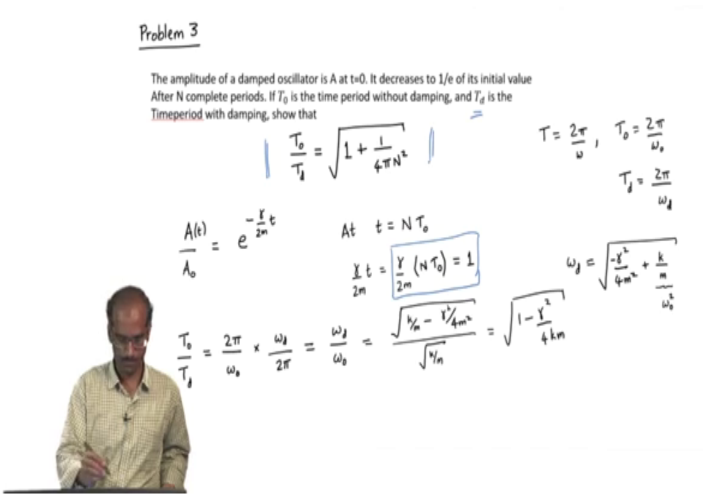

Let us start with the 3rd problem. So, in this case the amplitude of the damped oscillator at $t = 0$ is given as A and it decreases to $\frac{1}{e}$ of its value after n time periods and if T_0 is the time period without damping and T_d represents the time period with damping. Now we need to show that in such a situation the ratio $\frac{T_0}{T_d}$ is given by this. So, let us use the first statement of the problem which is that $\frac{A}{A_0}$. So, A_0 is the initial amplitude and A is amplitude at some time let us say t; $\frac{A}{A_0}$ is of course $e^{-\frac{\gamma}{2m}t}$. This is nothing new. This is something that we studied for the case of damped oscillator.

Now, when after N complete periods the amplitude A simply falls to $\frac{1}{e}$ of its value which means that let us say that at t it is NT_0. At this time this is N times the period without damping at this time what we have is that $\frac{\gamma}{2m}t$ which is equal to $\frac{\gamma}{2m}NT_0$. This is equal to 1 only then the amplitude can fall by $\frac{1}{e}$ of its original value. So, we have now a relation that connects γ, m, N, T_0. So, all I need to do is take the ratio $\frac{T_0}{T_d}$. So, T_0 is; remember T_0 is $\frac{2\pi}{\omega_0}$ multiplied by T_d is $\frac{2\pi}{\omega_d}$.

So,ω_0 is the natural frequency and T_d is the time period with damping ω_d is the angular frequency with that damping. So, remember that in general time period is $\frac{2\pi}{\omega}$. Therefore, we have used the fact that T_0 is $\frac{2\pi}{\omega_0}$ and T_d is $\frac{2\pi}{\omega_d}$. So, that is what I have done here. So, 2π and 2π will cancel. This will give me $\frac{\omega_d}{\omega_0}$.

Now what we need to do is to simply substitute for ω_d. If you remember again ω_d is the frequency of the damped oscillator and that would be equal to

$$\sqrt{-\frac{\gamma^2}{4m^2}+\frac{k}{m}}$$

Of course, this has to come with the minus sign. So, we just need to substitute these things $\frac{\sqrt{\frac{k}{m}-\frac{\gamma^2}{4m^2}}}{\sqrt{\frac{k}{m}}}$.

Because this quantity $\frac{k}{m}$ is equal to ω_0^2 and from here on it is some simple manipulations is what we need to do. So here if you take $\sqrt{\frac{k}{m}}$ outside, you will get $1-\frac{\gamma^2}{4km}$. So, just take $\sqrt{\frac{k}{m}}$ outside. So, the $\sqrt{\frac{k}{m}}$ in the numerator and $\sqrt{\frac{k}{m}}$ in the denominator would cancel and the rest would be this.

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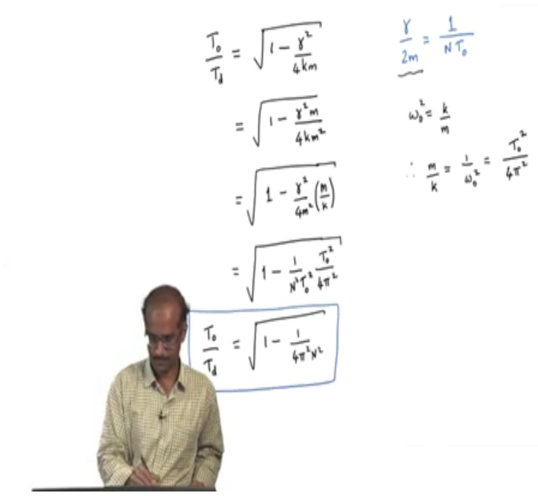

And now to get the desired form just multiply and divide by m here. So, you will get $\frac{\gamma^2 m}{4km^2}$. So, we need to eliminate γ, m, k and for that we have this relation which we already calculated. So, it says that $\frac{\gamma}{2m} = \frac{1}{NT_0}$.

Let me write it out here. $\frac{\gamma}{2m} = \frac{1}{NT_0}$. So, I will have

$$1 - \frac{\gamma^2}{4m^2}\frac{m}{k}$$

So if you look at $\frac{\gamma^2}{4m^2}$, it is simply the square of this relation square of this quantity. Hence I will have $1 - \frac{\gamma^2}{4m^2}$ will be $\frac{1}{N^2T_0^2}$ multiplied by again $\frac{m}{k}$ is related to ω_0. That is because ω_0^2 is $\frac{k}{m}$. Therefore, $\frac{m}{k}$ would be $\frac{1}{\omega_0}$ and this quantity is equal to ω_0 is $\frac{2\pi}{T_0}$. Therefore, this will be 4π square by T_0^2.

So, I can now substitute this as $\frac{T_0^2}{4\pi^2}$. So, clearly T_0 square will cancel. So, I am going to get $1 - \frac{1}{4\pi^2 N^2}$. This is $\frac{T_0}{T}$ and this was the required expression and this is the answer that needed to be shown.

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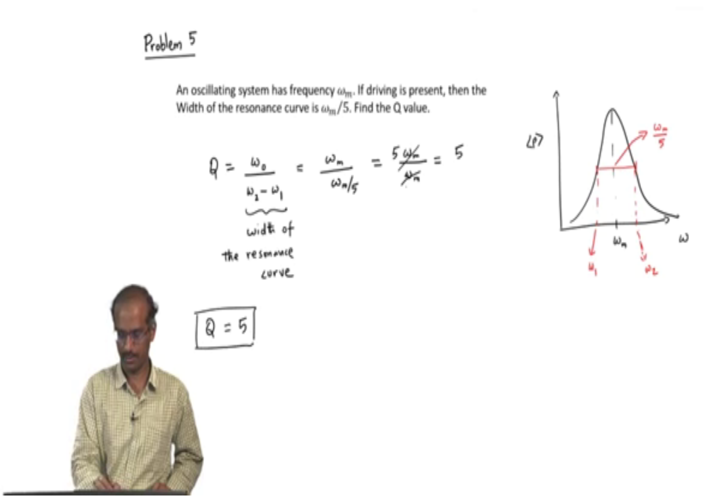

The 5th problem is really the simpler problem. It is like the simplest problem. So, it tells us that an oscillating system has resonant frequency ω_m and if driving is present, the width of the resonant curve is $\frac{\omega_m}{5}$. Find the Q value. So, roughly to sketch the information here it says that if you have $< P >$ as a function of ω and maybe here is your resonance curve and this is ω_m which is the resonant frequency and the width here this width is equal to $\frac{\omega_m}{5}$ and we need to find the Q value.

So, the Q value is simply the width is the resonant frequency divided by the width of the; width of the resonance curve which is this quantity. So, in this case we do not need to know individually what is the value of ω_1 and ω_2. Basically I do not need to know what is this value ω_1 and the value ω_2 because I directly have the information of the width.

So, Q is equal to ω_0 divided by ω_2 minus ω_1 and ω_0 is given as ω_m which is the resonant frequency divided by the by this quantity ω_2 minus ω_m ω_1. This is called the width of the resonance curve. We know what it is that is simply $\frac{\omega_m}{5}$. So, that would be ω_m divided by ω_m. So, that is equal to 5. This will cancel. So, the value of Q for the system is 5. This is the required answer.