Waves and Oscillations
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Lecture - 13
Q-Factor of Forced Oscillator

Welcome again and we will continue our explorations with the Forced Oscillator. And, in particular today's module, we will look at Q values once again for forced and damped oscillator.

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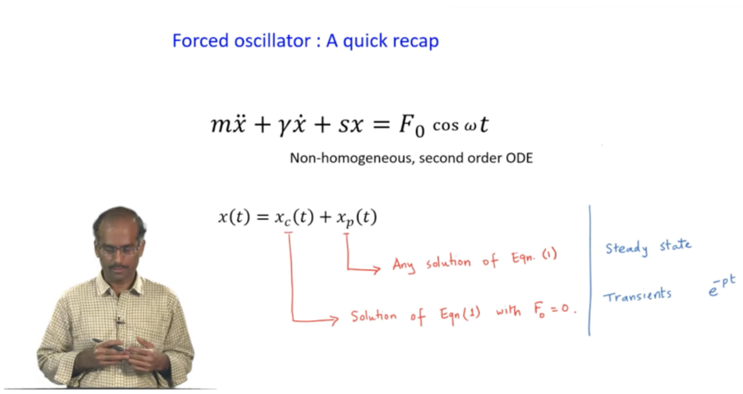

So, we wrote down the equation of motion for a forced standard damped oscillator turns out that, it is a non-homogenous second order and ordinary differential equation, that's the kind of description. So, it is a class of differential equation. And, we know from mathematical considerations that equations of this type have two parts to their general solution which is shown here.

So, $x(t)$ is the general solution it has two parts x of x_c which is called the complementary solution, which is simply the solution of this equation with the right-hand side being 0, corresponding to the statement that there is no forcing F_0 is equal to 0. And, x_p is what is often called the particular solution, which could be in general any one solution for this full equation.

Now of course, there are many ways of getting this particular solution, again we will not be getting into how to do that, if you pick up any differential equations book, you should be able to see at least 2-3 different ways of finding any possible solution for this class of differential equations. We are essentially physically motivated. So, if you look at as a physical problem you could attach physical meanings to both these parts x_c in a more physical language would be called the transient, the kind of dynamics that would vanish after sufficient amount of time, and particular solution would be what is called the steady state solution. So, you are interested in solution which lasts for infinite times.

So, steady state is what you would reach as time tends to infinity. So, as far as we are concerned, we are interested in the steady state solution for this problem. Now, let us since we have already worked out the solutions for this and it turns out that the steady state solution is simply oscillatory solution with the same frequency as the driving frequency. So, my driving frequency is this quantity which is ω. So, you are driving an oscillator with the frequency ω and the oscillator responds by oscillating with the same frequency ω, once you give it some time to relax.

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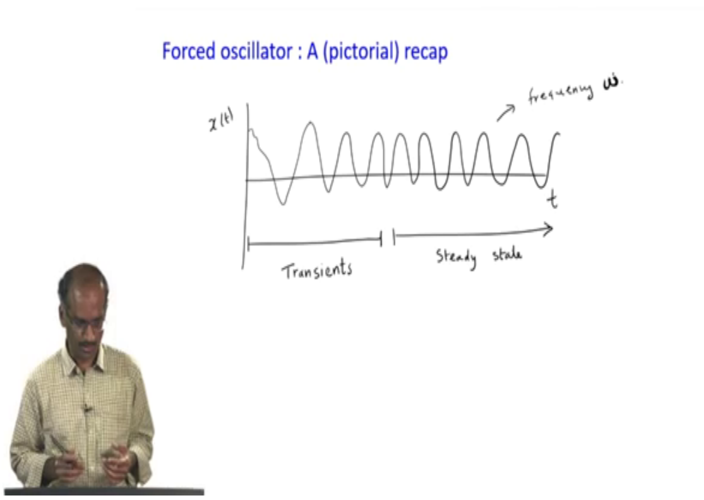

So, just to show how possibly it might look physically, let us just quickly plot the solution x as a function of t plot it as t. So, what you might c is may be something that initially looks very different and so on. But, finally, you will notice that it is settling to some oscillation whose frequency is same as ω. So, for instance from here to here, maybe you could say that these are transients corresponding to the so, called complementary solution and from here onwards you are in the regime of steady state I have not drawn it very well. So, it represents oscillation with frequency ω. So, this is what happens in a forced oscillator.

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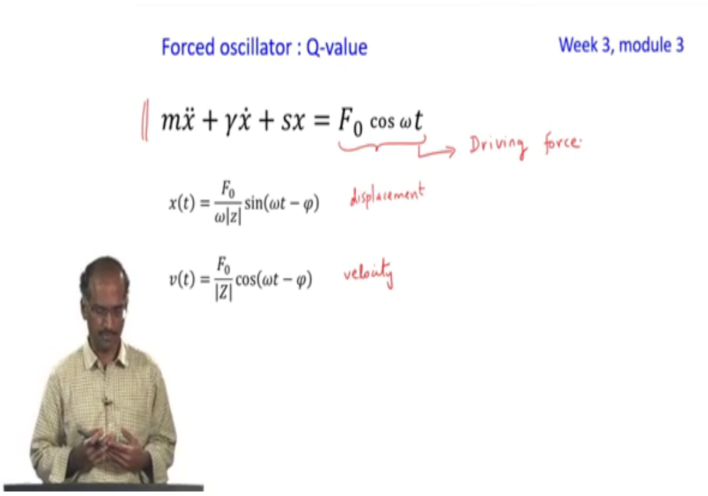

So, we will come to what we are going to do today. So, we look at finding Q values for the forced oscillator and we already made this quantity Q-value, when we looked at the damped oscillating systems. So, here again I have the equation of motion here written down and this is displacement as a function of time, and this is velocity as a function of time. So, the first thing I want to even before I get to Q-value. The first thing that I would like to do is to compute the power supplied to the oscillator by the driving force.

So, if I am driving by this force $F_0 cos(\omega t)$, how much power am I giving to the oscillator? So, that is the quantity that we will first compute and then we look at the Q value.

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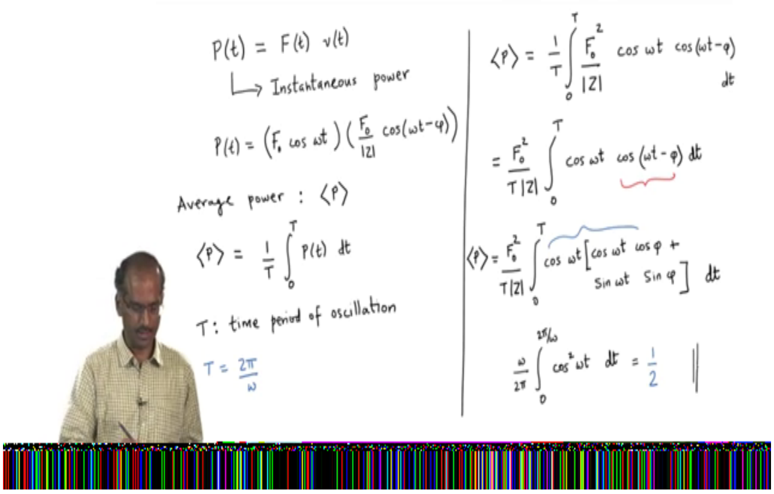

So, the instantaneous power at time t would be force multiplied by the velocity. So, force imparted at time t and the velocity of the system at time t. Now, we know the quantities on the right-hand side, we just need to put them together here. So, $P(t)$ would be $F(t)$, which is $F_0 cos(\omega t)$, that is my driving force multiplied by velocity as a function of time. So, again we go back to our velocity which is F_0 by modulus of Z into $cos(\omega t)$ minus 5.

So, I have formal expression for instantaneous power, but maybe what I would like is the average power. So, this is the power at a particular instant of time t, but a more useful quantity is what is the power supplied over an entire cycle of the oscillator or one full time period. So, the average power I just need to integrate this over entire time period and divide by the time period. So, that's my formula for average power. So, it would be 1 divided by the time period. So, integrate this quantity $P(t)$ ok.

So, you have these small chunks of time add the power supplied at every infinitesimal time differences over the whole time period going from 0 to t. This would be my average power and of course, it goes without saying that the quantity T is the time period of oscillation. And, remember that in steady state the angular frequency is ω, which is the frequency of forcing hence this capital T should be equal to 2π by ω.

So, now, we have the problem set up we just need to do one integral to be able to calculate this average P. And, of course, this integral is not too difficult to do. In fact, you can split this integral into 2 parts if we expand this quantity $cos(\omega t)$ minus ϕ. So, let me do that average P would be F_0^2 by T into this and 0 to capital T $cos(\omega t)$ into $cos(\omega t)$ $cos(\phi)$ plus $sin(\omega t)$ $sin(\phi)$ into dt. So, it is now split into 2 parts. And, the first integral essentially is $cos^2(\omega t)$ dt integral. And, the second one will be $cos(\omega t)$ $sin(\omega t)$ integral.

So, for example, the first integral to do the first integral you need to be able to do this integral $cos^2(\omega t)$ dt where this one goes from the limits go from 0 to capital T and capital T is 2π by ω. So, this is one integral for which we need to know the answer.

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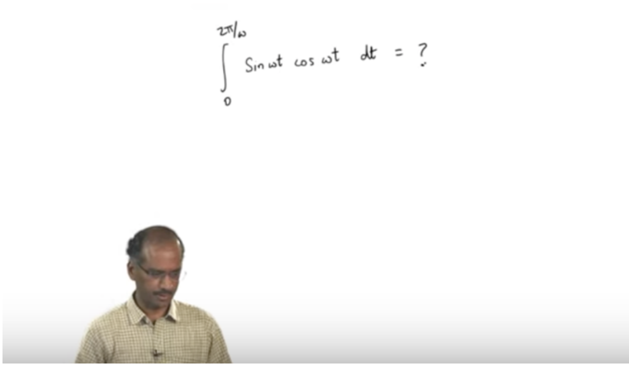

This second integral for which we need to know the answer is 0 to 2π by ω $sin(\omega t)$ $cos(\omega t)$ dt. So, this is another integral which we need to do, it would be more easier, if we also divide it by T we are doing 2π by ω. So, I will encourage you to try and do the integral yourself it is it is a fairly simple and straightforward integral to do, and if you do it correctly the first of these integrals should give you the value half.

So, clearly the contributions come only from the first part of the integral which is this one. So, now, you assemble together all the results that we have we will finally get to this point where we have evaluated the power supplied to the oscillator over the entire time period ok. So, now I have computed the average power supplied to the oscillator, where does this go of course, the oscillator is oscillating, but in the process of oscillation it is overcoming the frictional force. And it is expending it is expending this power in overcoming that frictional force. So, now, we could ask the question, how much of the power is used up in overcoming the frictional force, again we will average it over an entire time period or an entire cycle.

So, we are more interested in what is the average power that the oscillator uses up in overcoming the frictional force. I want to compute this quantity average power consumed by the oscillator in one complete cycle. So, it means that the oscillator is overcoming friction and is doing work. So, essentially what I want to know is, what is the rate of doing work against the friction.

So, that will involve the frictional force and we know what the frictional force is. So, the frictional force here is $\gamma\,\dot{x}$ and I want to know what is the rate of work done by this done against the frictional force. So, that will be this multiplied by velocity. More correctly work done would be simply frictional force multiplied by dx and of course, we want to know the rate of doing work, hence it is dx by dt. So, this would be $\gamma\,\dot{x}$ multiplied by $\dot{x}$ which is equal to $\gamma\,\dot{x}^2$.

So, what we have is an expression for again instantaneous power that is consumed by the oscillator. We will do the same trick again you compute this quantity over the entire cycle. Let me use the symbol average power with the d in the subscript to indicate the average power that is dissipated by the oscillator. And, this would be again 1 over T going from 0 to T you integrate this quantity $\gamma\,\dot{x}^2$ dt, $\dot{x}$ is something we already know $\dot{x}$ is simply the velocity which is given here. We just need to plug in this expression for velocity do the integral and we should be able to get it, which is what we will do now.

And, so, I will have γ I can take out 0 to T $\dot{x}^2$ that would be F_0 by mod Z $cos^2(\omega t - \phi)dt$. So, this can be further simplified average P average power dissipated is equal to γ F_0 by T into modulus of Z 0 to T $cos^2(\omega t - \phi)$. If, you remember this is the integral that we already did. So, we should be able to just use this result back in here. And, if I do that the final result would turn out to be the following it will be γ F_0^2 ok, I missed out power 2 here it should have been square here as well F_0^2 divided by mod is Z^2 and the integral value with 1 by T will give me half so, that is 2 here.

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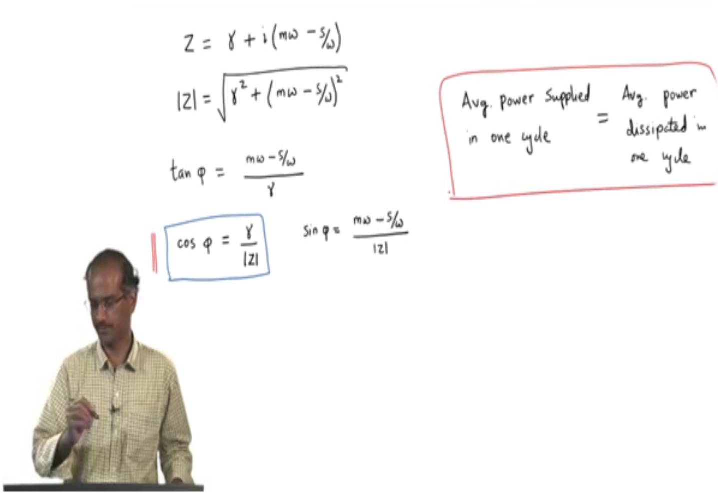

Let me remind you that our basic definition of impedance Z was this quantity γ plus i, multiplied to m ω minus s by ω. And, from here modulus of Z of course, was γ^2 plus this and we also wrote down a relation for $tan(\phi)$ which is m ω minus s by ω divided by γ.

So, if this is $tan(\phi)$ we could also write expressions for $cos(\phi)$ and $sin(\phi)$ in which case $cos(\phi)$ would be equal to γ by mod Z. And, of course, $sin(\phi)$ would be equal to m ω minus s by ω whole thing divided by mod Z, what we really need is this quantity an expression for $cos(\phi)$. Let us go and look back at the formula that we have for the average power dissipated. Let me rewrite it as F_0^2 divided by 2 times mod Z multiplied by γ times mod Z. And, we know that γ times mod Z is simply $cos(\phi)$.

So, I can go back and write this quantity as F_0 by twice mod Z multiplied by $cos(\phi)$ and that is simply our average power dissipated. Now, this is remarkable, because if you compare this result with the with what we have here these two results you will see that they are exactly identical. So, you are supplying some power to the oscillator by a mechanism of the external forcing, the average power supplied to the oscillator over an entire cycle over one complete cycle is equal to the average power that it dissipates over the same cycle. So, in other words it does not really hold the power, it does not store it for future use. So, the average power given to the oscillator is immediately spent up over the same time period. So, this two quantities are identical. So, this is our central message in this part of the section.

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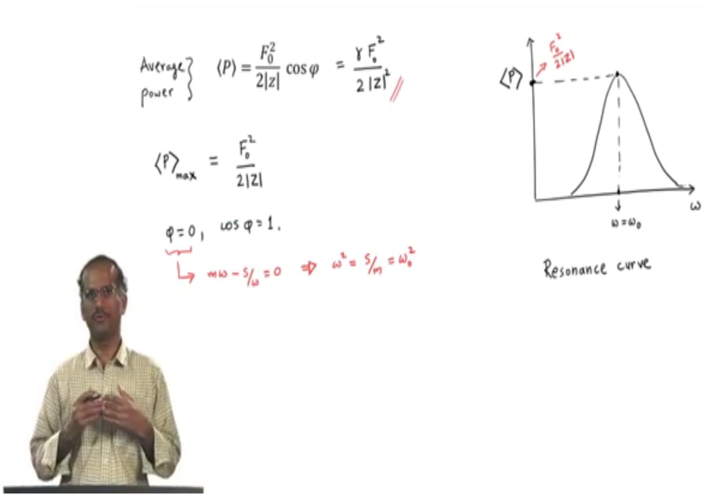

Now, we will use this quantity average power to define Q-value. Before, we define the Q-value let us sketch this function as a function of the driving frequency ω, because that is going to tell us how is how much of the power is being absorbed at different frequencies. I am going to choose some value here as ω equal to ω_0. In fact, you will notice that when ω equal to ω_0 that would be the case when $cos(\phi)$ is equal to 1 as we shall see now.

And, clearly from the formula that is written down here the average the maximum of the average power which I indicate by average P max here would simply be equal to F_0^2 by twice mod Z, corresponding to ϕ being equal to 0 in which case $cos(\phi)$ is equal to 1. Now, you could ask what is ϕ being equal to 0 would mean. So, ϕ being equal to 0 would imply that m ω minus s by ω is equal to 0 and this implies that ω^2 is equal to s by m which is equal to ω_0^2which is what we call the natural frequency of the system. So, when the driving frequency is equal to the natural frequency of the system, the average power supplied to the oscillator is maximum.

So, at ω equal to ω_0 this function is going to have it is maximum value. And, because the denominator here is quadratic the function would essentially drop off like this. This tells us physically that maximum power is absorbed by the oscillator, when the driving frequency matches the natural frequency of the system. And, also it tallies with the earlier resonance idea that we saw that at the natural frequency, when you drive it at the natural frequency the amplitude of velocity is the largest. And, the amplitude of displacement is nearly the largest when you are driving it at the natural frequency.

And of course, if a lot of power is being absorbed by the oscillator at ω equal to ω_0, it also dissipates the same amount of power over every cycle that is the general result that we saw just now. So, if you want a large sort of response from the oscillator, you should drive the system at the frequency corresponding to the natural frequency of the system. Sharper the curve more the power is given to the oscillator, and it is likely to sustain oscillations for a longer time. So, hence you could say that sharper curves of this type would correspond to larger Q-values without doing any calculation, let us see if that is true.

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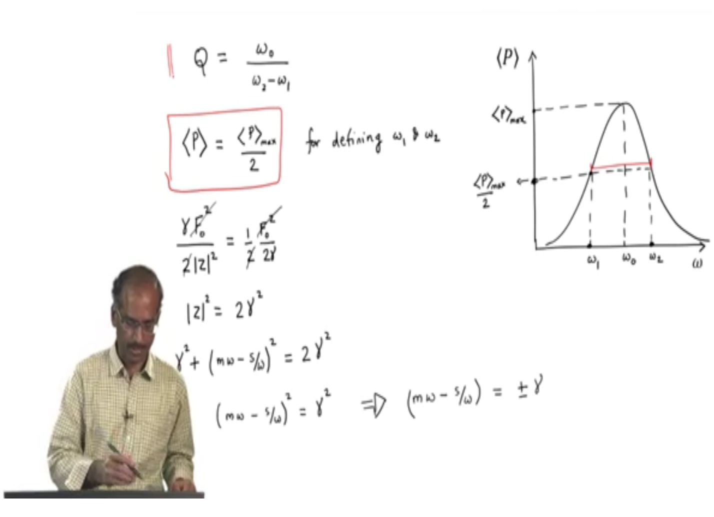

So, I have the same curve here average power as a function of driving frequency. And, what I would like to show is the following that Q is equal to ω_0 divided by ω_2 minus ω_1, where these 2 quantities ω_2 and ω_1 not defined as follows. So, let us define a value which is half the value half the peak value, which might be somewhere here.

So, this quantity is half of the maximum value and the 2 values of ω_1 and ω_2 are so, defined that ω_2 is this ω_1 is this. So, in general this curve is a symmetric curve about ω_0, but it is my poor drawing that it looks as though it is not symmetric um. So, my Q-value definition says that it is simply ω_0, which is the value of driving frequency corresponding to the peak power being supplied to the oscillator, divided by this width which is often called the full width at half maximum. That is the definition of Q-value in this case.

Now, let us see how we get to this our starting point is the definition that skew value is defined in terms of these 2 ω_1 and ω_1, which is which corresponds to 2 values of frequency at the same value of average power. And, the same value being average P max divided by 2 ok; so, at this value there are 2 possible values for frequencies. So, you draw a horizontal line from here, it cuts the resonance curve at 2 points. So, you draw 2 vertical lines you will get this ω_1 and ω_2 corresponding to that. So, this is our definition for defining ω_1 and ω_2. And, once you know ω_1 and ω_2 it defines the width and that would tell us what Q factor is.

All we now need to see is whether this definition of Q factor or Q-value coincides with the definition that we already had earlier on. Let us start by implementing this condition. So, the left-hand side is average power we derived this result. So, that will be $\gamma\ F_0^2$ by 2 mod Z^2 and the right-hand side is the maximum of the power supplied divided by 2. So, that is half into F_0^2 divided by 2γ. So, remember that we had this relation for maximum power supplied. So, this is the condition that we have and if you cancel out various things here, because for example, this would cancel out 2 would cancel out. So, things would cancel out and this equation would simply reduce to average of Z^2 is equal to 2 times γ^2.

And, now let us go back to the definition of module of modulus of Z^2, which we can just get from this relation here. If, I put in that relation I would get from here we are just 2 steps away from what we want, if I further simplify it will be m ω minus s by ω whole square, that is equal to 2 γ^2 minus γ^2 and simply this. And, this would imply that m ω minus s by ω is equal to plus or minus ω.

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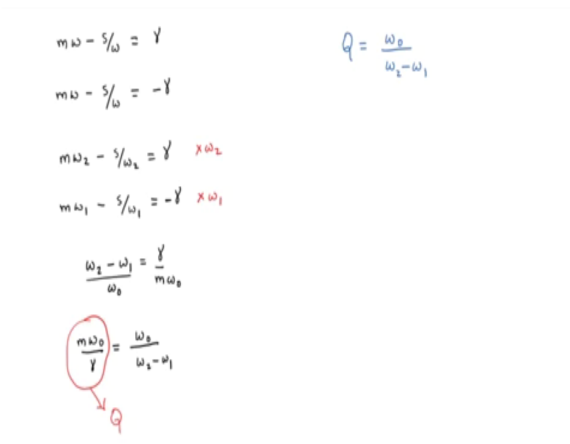

So, let me write it out as two different equations m ω minus s by ω 1 possibility is that could be plus γ, other one as m ω minus s by ω is equal to minus γ. We know that there are 2 solutions corresponding to ω_1 and ω_2. So, if I put in the 2 um. So, let us say that one of them corresponds to ω_2. So, I will have m ω_2 minus s by ω_2 is equal to γ and the other one let us say corresponds to ω_1 solution. So, I will have m ω_1 minus s by ω_1 is equal to minus γ.

Now, I want to eliminate s here s is the stiffness constant, and the way to do is simply multiply this equation by ω_2, multiply this equation by ω_2 and multiply this equation by ω_1, subtract 1 from the other and we will be able to eliminate s from the 2 sets of equation. And, once we do that I will leave it to you to do that small piece of mathematical step. And, once we do that we will get ω_2 minus ω_1 is equal to γ by m. And, now it looks like we are getting close to the result. Now, I will divide both sides by ω_0.

And, just invert it in which case you will get m ω_0 by γ is equal to ω_0 by ω_2 minus ω_1. And, what we know from what we had already studied is that, this quantity here is simply equal to the Q-value. So, I have my final result that Q, Q-value or Q factor is equal to ω_0 by ω_2 minus ω_1, and this is a very practical definition that you can get whenever you look at the resonance curve in the form of average power supplied as a function of the driving frequency. So, this is ω_2 and ω_1 just to remind you once again are the 2 values of driving frequencies at which the average power is half the peak power.

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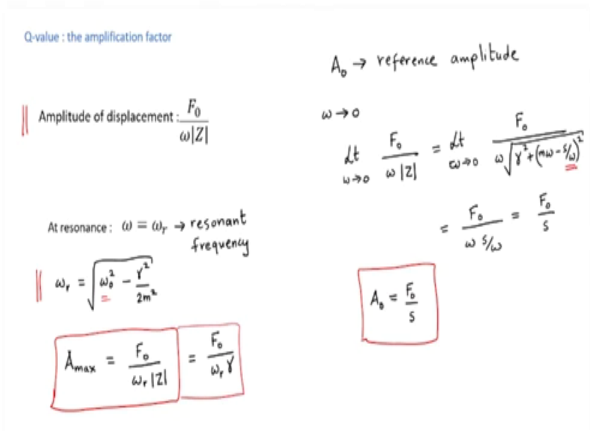

So, there is one another way of looking at Q-value in terms of what is called the amplification factor and we will soon realize what it means but let me go through the procedure itself. So, to begin with you might recall that the amplitude of displacement for a forced and a dissipating oscillator is given by this expression. So, we derived it in one of the earlier modules and at resonance let us call the resonant frequency as ω_r. So, ω_r is my resonant frequency. And, we also saw that resonant frequency what I call as ω_r is simply equal to this expression, and we repeated that it is slightly lower than the natural frequency, which is given by ω_0 or ω naught here ok.

So, now the maximum amplitude or let me call that quantity A max. The maximum amplitude that you can ever get would be F_0 divided by you replace ω by ω_r times mod Z ok. So, this is the maximum amplitude you could ever get for the set of parameters that you have chosen for your oscillator. Now, let us also define another reference quantity. So, all I am going to do is to measure A max against a reference amplitude and that reference amplitude, let me call it some A_0. And, how do we get this reference amplitude? So, what I am going to do is to look for amplitude in the limit that drive driving frequency tends to 0. So, you take the limit the driving frequency tends to 0 for the amplitude of displacement.

Now, when you take the limit ω tending to 0, among the various terms in under the square root the one with the ω in the denominator this term, that would be the most dominating term because you are taking the limit of very small ω. In which case this expression would simply be F_0 by ω into and in comparison, with s by ω, γ^2 would be small. So, we can neglect that. So, I am going to have s by ω here, ω and ω would cancel. So, this would finally, give me F_0 by s. Remember that s is a stiffness constant.

So, in the limit of ω tending to 0 my reference amplitude is a 0 which is $F_0$by s. And, this reference amplitude as you can see is independent of ω itself. So, now, I have two things; one is this quantity A max and the reference amplitude, which is F_0 by s, what I am going to do is to take the ratio of A max to A_0.

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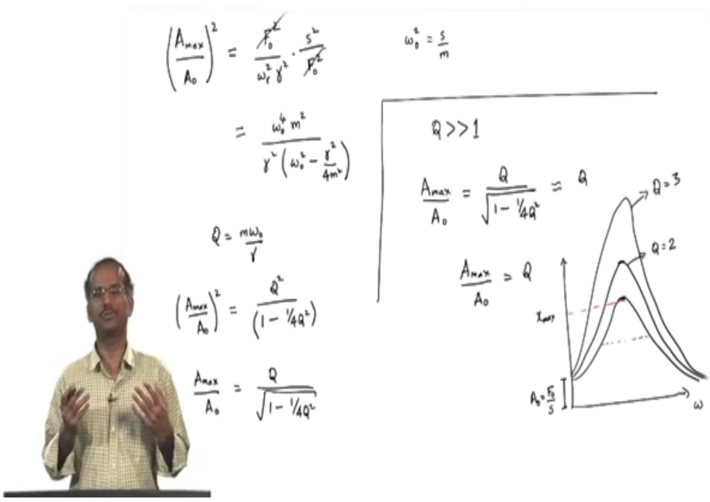

So, let me start by writing A max by A_0 whole square and I will just copy the expressions, that I have written down here. In which case and before I take the ratio this can actually be further simplified. So, that this will be ω_r into γ essentially, we have just put in the condition for the resonant frequency. So, now, I need these two quantities A max and A_0, I am going to take their ratios ok. So, I have this ratio between A max and A_0. Now, we will manipulate it a bit to write it in terms of Q-values. The next step is to cancel off F_0^2 and remember that ω_0^2 is equal to s by m. I can replace s^2 by ω_0 to the power 4 into m square divided by γ^2. And, ω_r^2 is the expression that we already have here, which is ω_0^2 minus γ^2 by 4 m square.

So, from here on you will see that in this expression the only quantities that are involved are the quantities which we normally use to define our Q-value, which is m ω_0 by γ. So, it becomes very easy to manipulate this to write it in terms of Q values. So, in a very straightforward way you could write A max by A_0 to be equal to Q square divided by 1 minus 1 by 4 Q square.

Further you take off the square root from here and that will give me Q divided by square root of 1 minus 1 by 4 Q square. And, now we will take the limit that Q is much greater than 1 meaning that we are working in the limit or in the regime of parameters where Q-value is very large, in which case this expression here becomes approximately equal to Q.

So, this is our required expression that A max by A_0 is Q and remember that A_0 was F_0 by s. And, now we can sketch this resonance curve in terms of the Q values. So, for instance here I am plotting x max as a function of the driving frequency ω, and it might look something like this. So, this value could be the reference value which is A_0 equal to F_0 by s, and every other peak here is written in terms of this reference value A_0. For instance, this curve could be the case of Q equal to 2, so, which would mean that the peak here is twice this value and so on.

So, you can we can draw some more curves like this, and this could probably be Q equal to 3 and so on. So, you would notice that larger the value of Q sharper is the resonance and the resonance width, which is the value of this width at half the peak with respect to let us say this point with respect to this peak; it gets narrower and narrower as Q increases.

So, this is another way of understanding the Q Q-value as telling us about the amplification factor, but ultimately it relates to the same meaning in terms of the original meaning that we had seen earlier, it simply at some level tells you how much of energy that you put in is sustained by the oscillator ok. If, the oscillator can sustain it for long enough time, then you would say that it is Q-value is large.

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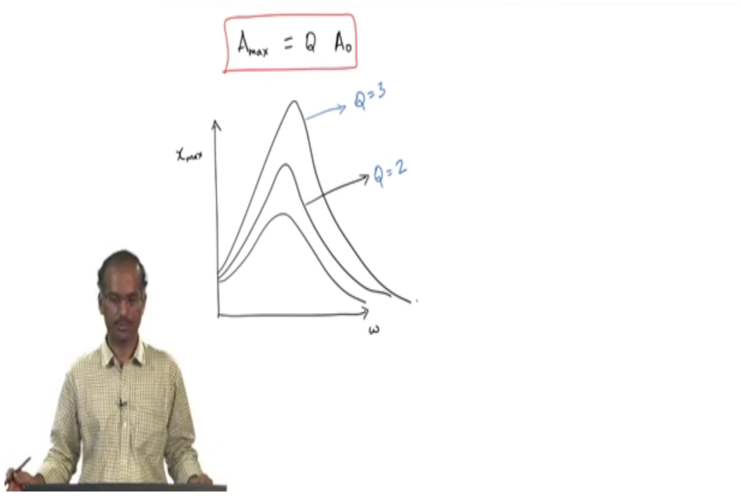

So, just to summarize what we have been discussing in today's module. We obtained this first and important result that the average power supplied in one cycle is equal to the average power that the oscillator dissipates over the same cycle. Effectively it does not store the power for future use, and we were able to show that the Q-value which we already saw in an earlier module can now be defined in terms of the width of this resonance curve. And, Q-value is equal to ω_0 by ω_2 minus ω_1, where ω_2 and ω_1 correspond to 2 values of driving frequencies at which average power supplied is half the peak value.

And, finally, we were able to show that Q factor is simply the ratio between the maximum amplitude divided by and initial amplitude.

And, we will see some applications of all these concepts in the next model.