Waves and Oscillations

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Lecture - 12
Resonances

Welcome back and this week we will continue with our study of forced oscillator. And, in this module in particular we will look at Resonances and as usual we will first begin with the quick recap of what we did in the previous module.

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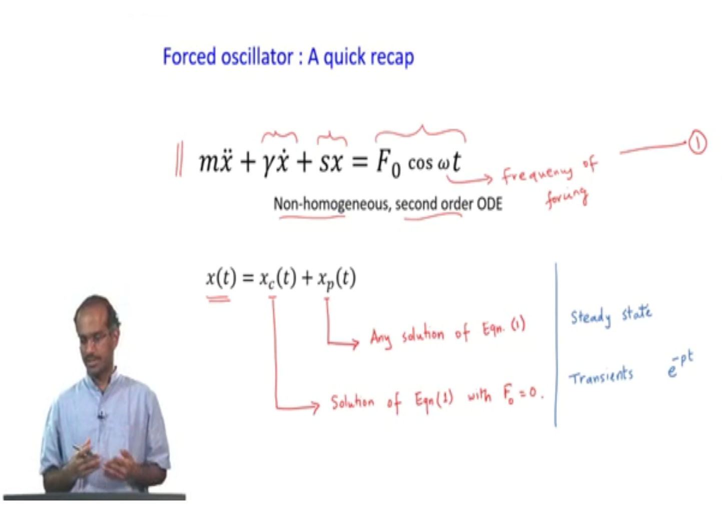

So, we started by writing down the equation of motion for a forced oscillator, which is this once again to remind you, the forcing or the external forcing really come from this term here and ω is the frequency of forcing. And, this γ as usual is the term that represents dissipation in the system. And, of course, s is the stiffness coefficient that provides the ingredient for oscillations in the first place.

So, together this entire equation in mathematical terms would be called non-homogeneous, second order because it involves $\frac{d^2x}{dt^2}$ and it is an ordinary differential equation, meaning that it is not a partial differential equation ok. And, what we know purely as a mathematical problem for class of these kinds of differential equations is that, when we try to right out the solution. The solution has actually two different parts to it; one is what is called the complimentary part, which I have represented by x_c here.

So, this complimentary part would be the solution of the corresponding homogeneous part, which would mean that it would be the solution of let us call this equation 1, it will be solution of equation 1 with F_0 being equal to 0. So, which is simply the standard damped oscillator that we have already seen and we already know those solutions. In particular the solutions there come with the exponential damping factor. And, there is a second part of the solution, which is called the particular solution, which is denoted by x_p here.

So, this particular solution is any solution of equation 1. The general solution which is denoted by x as a function of t here consist of these two parts; the complimentary part and the particular solution, while this is the mathematical sort of description of the solution. We also know that from very physical considerations complementary part corresponds to damping solutions, which means that they would die away after sometime. So, physically they are called transients. So, you would remember that they all come with this e^{-pt} multiplied to something. Particular solution would be what is called the steady state.

The reason for that is if I have an oscillating system and it has its own natural frequency of oscillation given by the parameters. Now, if I try to externally oscillate it by giving something like $F_0 \cos \omega t$ or some other external driving. So, what would happen is that in general the system would adjust it is rhythms. So, that ultimately after sometime, it would oscillate at the same frequency with which I am driving the system. So, in that state the transients have died out, what is left is steady state oscillations.

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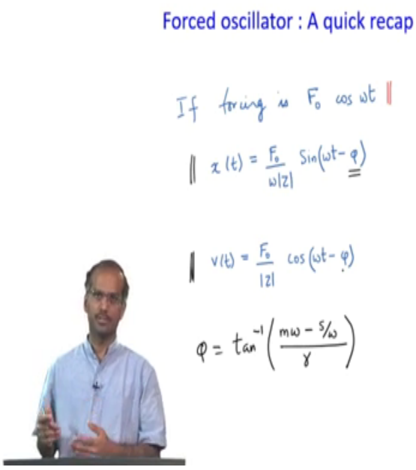

So, in this specific case where the forcing is $F_0 \cos \omega t$ like the way it is written here. In that case we worked out the solution for displacement and for velocity as a function of time. If, you compare with the forcing which is $F_0 \cos \omega t$, you would notice that $x(t)$ displacement as a function of time comes with the a $\sin \omega t$. So, clearly there is a $\frac{\pi}{2}$ difference phase difference between these two functions, between the forcing and the displacement. In addition there is also an additional phase difference of ϕ.

So, you can always adjust ϕ to be 0, but still there would be a phase difference of $\frac{\pi}{2}$ between the forcing and the displacement. And, the other hand if you look at, if you look at the velocity as a function of time. So, there is a there will be a phase matching between the forcing and the velocity provided $\phi = 0$. If, $\phi \neq 0$ there will be a phase difference of ϕ between the forcing and the velocity. So, you can take forcing to be a $F_0 \sin \omega t$ or in general some arbitrary time dependent function, which can be resolved in terms of cosines and sins, but ultimately the qualitative results would remain the same.

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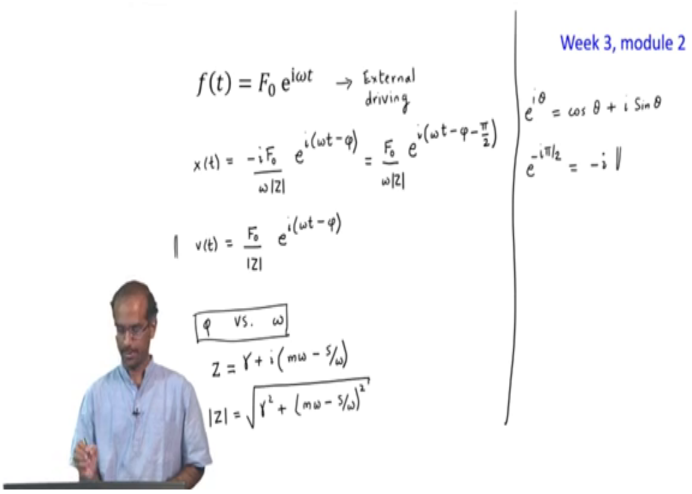

Today, we will start by looking at some unusual novel phenomena that would appear, when you have both damping and forcing in a oscillatory system. But, before we do that let us take a closer look at this equation ok. So, I have the solutions written down here. And, if you look at the first one which is the displacement as a function of time, you will notice that there is a $-i$ in front of it. And, again from the fact that let me write $e^{i\theta}$ we know that this is equal to $\cos\theta + i\sin\theta$.

And, if I need $-i$ it will have to be $e^{i\frac{\phi}{2}}$ will be equal to $-i$. So, all you need to do is to substitute θ by $-\frac{\pi}{2}$ and you will get this result. So, which means that now I can rewrite this displacement as a function of time, by absorbing $-i$ in the exponential. So, it would be $\frac{F_0}{\omega|Z|}e^{i\left(\omega t-\phi-\frac{\pi}{2}\right)}$. So, remember that earlier on we had said that between the forcing and displacement, there is always a phase difference of $\frac{\pi}{2}$ even if $\phi = 0$.

So, clearly that comes out and it is consistence with that expectation. We would like to look at the phase factor ϕ as a function of driving frequency ω. To do that let me once again and also to remind you let me write the expression for Z. So, we want to look at this and to do that I need know what is Z. And, Z is $\gamma + i(m\omega - \frac{s}{\omega})$. And, $|Z|$ is equal to $\sqrt{\gamma^2 + \left(m\omega - \frac{s}{\omega}\right)^2}$.

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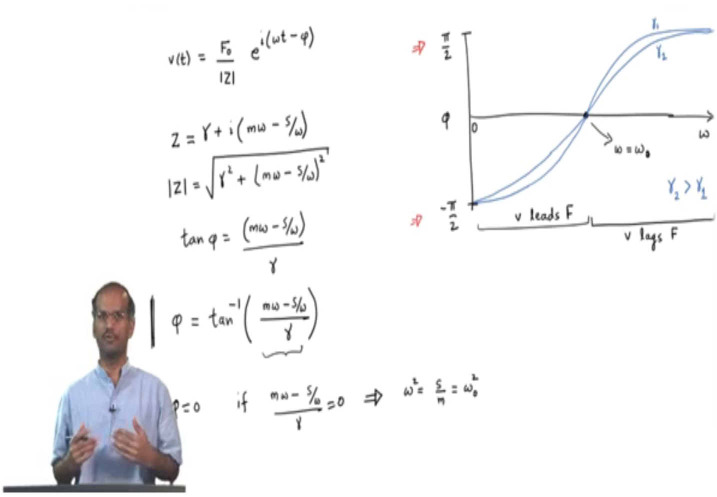

So, let us see first what could be the range of ϕ. So, if ω goes from let us say 0 to. So, this my ω axis and if it goes from 0 to some really large number ok. In principle theoretically you can think of it as infinity. In that case what would be the extremities of ϕ itself. So, let us say let us put ω here in this equation, if you do that as ω tends to 0 the dominant term would be $-\frac{s}{\omega}$. So, it would give me $-\infty$. So, $\tan^{-1}(-\infty)$ corresponds to $-\frac{\pi}{2}$. So, phi which will be on the y axis will run from $-\frac{\pi}{2}$ to some large value, which will be determined by the value of ω at the other end.

So, as ω tends to infinity this expression will again be dominated by $m\omega$, because $\frac{s}{\omega}$ would tend to 0 as ω goes to ∞ and $m\omega$ would tend to ∞. So, $\tan^{-1}(\infty)$ would be $\frac{\pi}{2}$. So, this goes from $-\frac{\pi}{2}$ to plus $\frac{\pi}{2}$. And, somewhere in between when the quantity inside this bracket here is equal to 0 than ϕ would be equal to 0. So, that would happen somewhere here and when that happens ϕ would be equal to 0. If $\frac{m\omega - \frac{s}{\omega}}{\gamma} = 0$ and this implies that $\omega^2 = \frac{s}{m}$.

And, we know that $\frac{s}{m}$ is what we attribute as a natural frequency of oscillation and we will denote it by ω^2. So, this is the point where ω is equal to let us say ω_0, which is equal to $\sqrt{\frac{s}{m}}$. So, we have everything in place all we need to do is to simply draw the figure. And, so, now, we know that the figure is such that it goes from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, what remains to be done is just to determine how it goes, I will leave it to you as a problem. So, you need to figure out whether from here it goes like this, or it goes like this, or it goes more like this.

So, here you just need to find out the slope of ϕ as a function of ω and find out it is value at $\omega = 0$. And, if you do that it will turn out that for small values of γ, it indeed is very close to 0; in which case the expected figure would look something like this. And, in principle one could also draw. So, for instance if I denote the first curve that I drew by γ_1 and this by γ_2, than then in this case γ_2 is larger than γ_1.

And, also it is easy to figure out that in this part where the value of ϕ is negative velocity leads F in phase. In other words the phase of velocity leads the phase of the external forcing. On the other hand in this region ϕ is positive it goes from 0 to $\frac{\pi}{2}$. In, that case velocity lags F or more precisely the phase of velocity lags that of the forcing term. So, it gives us a global picture of how the phases or the phase difference between velocity and the forcing plays out as you vary parameters ok.

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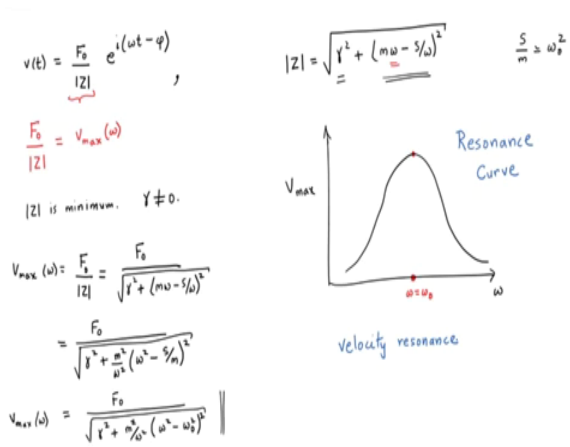

So, I have the solution for velocity as a function of time here. So, irrespective of what initial conditions we might choose and so on and so forth. The amplitude of the velocity would still be given by this quantity $\frac{F_0}{|Z|}$. F_0 is a constant you really cannot change it, because it is a given as part of the problem you cannot change the value F_0 in the middle of the problem. But, what you can actually change is this quantity $|Z|$ impedance, which depends on the driving frequency ω. So, you can actually tune the driving frequency.

So, in that case the question would be how does the amplitude of velocity?. The maximum of that behave as a function of the driving frequency. So, if this is my $V_{\max}$, how does $V_{\max}$ change as a function of driving frequency ω? So, that is the question. And, it is very easy to see from the expression for $|Z|$ that the maximum value of amplitude will, will come when the denominator is minimum, in a other words the maxima of $\frac{F_0}{|Z|}$ will happen when $|Z|$ is minimum. And, $|Z|$ itself will be minimum when of course, when both this γ and these 2 terms are equal to 0, but then we know that we assumed that there will be a dissipation.

So, $\gamma \neq 0$. In which case, $|Z|$ will be minimum when this quantity $m\omega - \frac{s}{\omega} = 0$. So, let us rewrite it slightly differently. So, $V_{\max}$ as a function of ω is $\frac{F_0}{|Z|}$, which will be equal to $\frac{F_0}{\sqrt{\gamma^2 + \left(m\omega - \frac{s}{\omega}\right)^2}}$. And, this by a small rearrangement, I can rewrite it as F_0 divided by γ^2 plus take $\frac{m^2}{\omega^2}$ out, in which case you will get $\left(\omega^2 - \frac{s}{m}\right)^2$.

And, if you remember $\frac{s}{m}$ is are ω_0^2 or what we call the natural frequency of the system, in the absence of dissipation, in the absence of forcing right.. Then, I will just change the notation in which case this expression would become $\frac{F_0}{\sqrt{\gamma^2 + \frac{m^2}{\omega^2}\left(\omega^2 - \omega_0^2\right)^2}}$. So, this is $V_{\max}$ and it is a function of ω. So, now, we have this in nice functional form we can now sketch this.

So, clearly the maxima of this quantity will happen when $\omega = \omega_0$. So, let us sketch that. So, I am plotting $V_{\max}$ as a function of ω. And, from the expression that I have here it is very clear that when let me call this point $\omega = \omega_0$. So, that would be the point at which the maxima of $V_{\max}$ will occur. And, for any other value of driving it is going to decrease and forgotten a 2 here. So, it is going to decrease quadratically. So, the functional form would look something like this. So, curve of this type is called the resonance curve.

So, it tells me the following important physical information. So, I have a damped forced oscillator, I keep driving the system and let us say that I varying the angular frequency of the external drive. So, at some point when the frequency of the external drive is equal to the natural frequency of the system, the response of the system is going to be large is going to be responding the greatest, when the external driving frequency matches the natural frequency of the system ok. For any other value of drive frequency, the response is going to be really-really small and it is going to drop off quadratically.

So, this is what would be a resonance curve for velocity one could ask similar question of displacement as well. So, we can think of displacement resonance. So, this is velocity resonance. So, physically when we say that velocity resonance is happening, it means that the velocity of the oscillating particle is going to be the is going to be a maxima ok.

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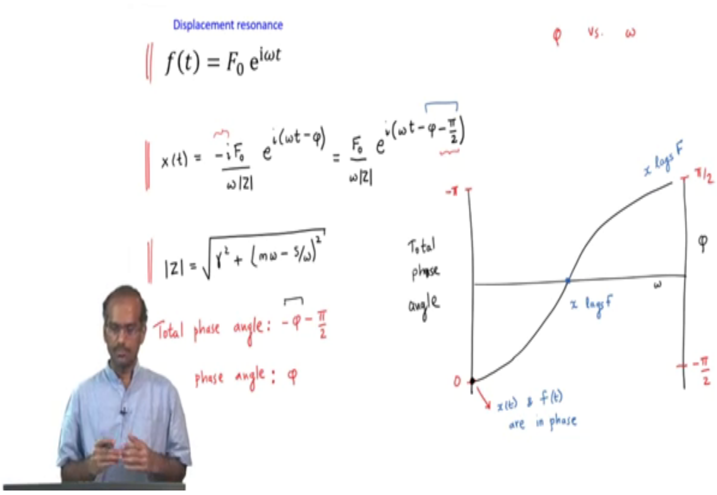

So, before we look at the displacement resonance let us as we did for case of velocity, let us first look at how ϕ changes with the angular frequency. So, ϕ is the phase difference. And, of course, I have the standard driving term here given by $F_0e^{i\omega t}$ and I have the displacement as a function of time. So, here I have absorbed this $-i$ which is here in the exponential and so, it shows up as a phase factor. Remember that in one of the earlier modules we said that when you multiply by i, it is equivalent to turning the vector by $\frac{\pi}{2}$.

So, when you multiplied by $-i$ is equivalent to adding this phase of $-\frac{\pi}{2}$. And, of course, $|Z|$ is what we already have all along right. So, now the total phase angle would be the following. So, as usual you subtract this $i\omega t$ and $i\omega t$ that goes away. So, the total phase angle is $-\phi-\frac{\pi}{2}$, where as just the phase angle alone would be simply ϕ itself.

So, now, let us since we want plot ϕ versus the angular frequency ω, let us first look at the range of these quantities. Let us sketch the total phase angle let me write it here total phase angle; I am going to sketch this as a function of ω on x axis. And, I can also plot ϕ just the phase angle on right hand side.

So, if you remember from our previous arguments ϕ goes from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, because the expression is still the same, whether it is velocity or displacement ϕ will still go from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. So, let us say that here it will go from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. On the other hand when ϕ is equal to $-\frac{\pi}{2}$ the total phase angle would be simply equal to 0, that is because you simply substitute $-\frac{\pi}{2}$ here in the value of ϕ, that will just give you 0 and when ϕ is equal to $+\frac{\pi}{2}$ it will give you $-\phi$.

So, the value here would be equal to $-\phi$. So, now, all we need to do is to simply plot the functional form of this $\phi = \tan^{-1}\left(\frac{m\omega - \frac{s}{\omega}}{\gamma}\right)$. So, here again I leave it as an exercise for you to do it yourself, but the curve should look something like this ok. And, here this the point where total phase angle is 0 right.

So, imagine if total phase angle is 0 this quantity here is 0, which means that displacement and forcing are in phase. So, that is the point where, displacement and forcing are in phase the phase difference between them is 0 and this would be a point where x would lag in phase with respective to that of F and again here x would x lags F.

So, with a little bit of thought you can figure out these differences in a phase between displacement and forcing. And, here I am not going to draw the case for two different values of γ, again I leave it to you as an exercise to draw similar function for two different values of γ, where 1 is greater than the other. Now, with this let us go to the question of displacement resonance. So, the question there is identical to what we asked when we looked at velocity resonance.

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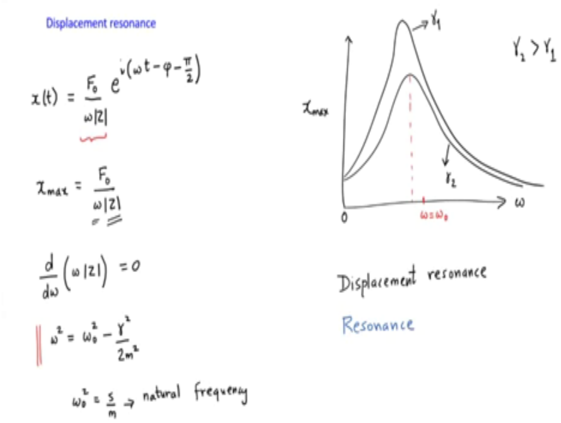

So, I have the solution for displacement as a function of time and you will notice that maximum amplitude would simply be $\frac{F_0}{\omega|Z|}$, let us write that $x_{\max}$ would be $\frac{F_0}{\omega|Z|}$. So, here again the question is same, if I were tuning ω if I were changing the frequency of driving.

How would $x_{\max}$ change? So, remember that unlike in the previous case here the question is slightly little more complicated, because you have a ω here, but there is also ω inside this $|Z|$. So, we need to minimize this function with respective ω ok. As usual the argument is that $x_{\max}$would be maximum when the denominator is minimum, the denominator here is $\omega|Z|$.

So, since we want to find the minimum of the denominator here, what we need to do is to differentiate the denominator with respective ω set it equal to 0 and find the value of ω. So, we can do this has a spot of one of the problem, but if you do it which is straight forward to do the result that you will get is that $\omega^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}$ ok where, ω_0^2 is $\frac{s}{m}$ which is called the natural frequency of the system without either damping or the forcing.

So, with this ingredient we are sort of ready to sketch the displacement resonance curve. So, here you notice that the resonance or the maximum value of a $x_{\max}$ happens at; not at ω_0^2, but at a value of a frequency, that slightly below ω_0^2. So, you could see that it is $\omega_0^2 - \frac{\gamma^2}{2m^2}$. So, what we are going to get is a series of a curves, that would look like this let me sketch them for you now. I have sketched the displacement resonance curve.

So, you will notice some important features; one is the maxima takes place maxima of displacement happens at a value of ω which is less than ω_0. And, it changes with γ as well as you increase dissipation the maximum value of course, decreases, which is what you would expect naturally from just the physical intuition. Just to interpret this curve physically it tells us that, if you have an oscillating system which is both damped and you are driving it with some angular frequency. And, if you are changing the angular frequency then the response of the system would be greatest, the response here is measured in terms of the amplitude of the oscillation.

The amplitude of the oscillation would be largest at a value of driving frequency, which is slightly lesser than the natural frequency of the system. So, this would be called displacement resonance or simply resonance. To summarize this module, what we saw is we looked at both the solution at we obtained for the driven damped oscillator, both the solution meaning that we looked at the displacement as a function of time and velocity as a function of time. And, we tried to closely see what happens if, if I tune the angular frequency with which I am driving the system.

So, you look at the phase difference between velocity and the forcing as a function of ω and that is what is sketched here in front of you. So, there is a range of frequency values below ω_0, which is the natural frequency where velocity leads F in terms of phase and if driving frequency is larger than the natural frequency V velocity lags driving frequency in phase. And, that is as far as velocity is concerned and we also looked at the idea of velocity resonance.

So, you ask what is the maximum amplitude of velocity, how does it change with driving frequency? Interestingly it turns out that when the natural frequency is equal to the driving frequency velocity is the amplitude of velocity is maximum. And, exactly identical questions can be asked about displacement as a function of time. So, we did that. So, we looked at how the phase angle varies as a function of ω. So, again that is the curve that is sketched in front of you here and we also looked at the idea of displacement resonance ok.

In the next modules we look at how we can quantitatively describe these resonances. So, we will be coming back to the idea of Q values or Q factors that we saw earlier.