Waves and Oscillations
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Lecture - 11
Forced Oscillator: Part 2

This is week 3 of this course, welcome to this session and we are going to look at Forced Oscillator this week. All the while in the first week, we saw about the simple harmonic oscillator and its properties. In the second week, we looked at the damped oscillator and various properties. Just to quickly recap what we had been doing in the last week.

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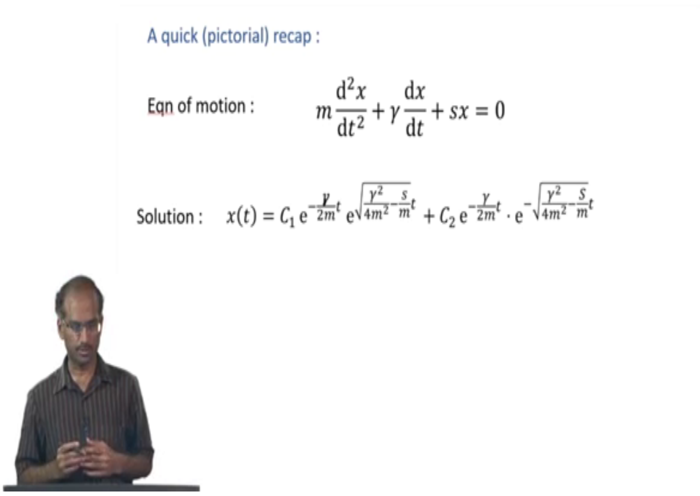

We wrote down the equation of motion for damped oscillator, which is this and to remind you again this γ here is the dissipation coefficient. And, we also obtained the solution for this equation of motion. And, depending on the value of the quantity here inside the square root, you either get damped oscillation or you simply get no oscillation at all.

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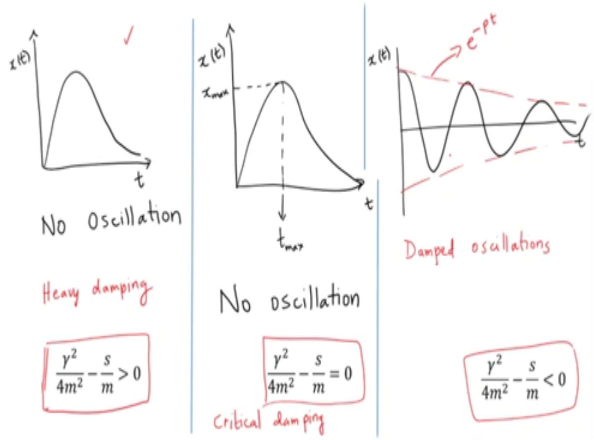

So, in other words I have tried to summarize what we saw? So, there was the first case which we simply called as case of heavy damping. So, that happens when this condition is satisfied, this one here and, in this case the displacement as a function of time is shown here. So, you give it a push it goes to certain distance and comes back towards the equilibrium position.

There is a second case when which is called the case of critical damping and it is characterized by this relation between the parameters. So, in this case again, there are no oscillations on the other hand if you had chosen your parameters which satisfy this relation, in that case you get damped oscillations and it is sketched here. So, you see that you do have oscillations whose amplitudes successively are reducing. So, what you have is damping energy is being lost from the system. So, these are the three possible cases, we saw in the case of damped oscillator.

It only means that left to itself the system would simply dissipate the energy to the surroundings, but you can maintain the energy by giving it continuous supply of energy, which is what you do for instance in a clock with a pendulum as such without an external source of energy either from a battery or in a more mechanical clock one actually turns the key and so on which stores the energy and supplies it. So, if you do not do any of that the clock would not work beyond a few seconds. But, when you supply energy you can make it work and it does give periodic oscillations, which is what the remit of this module, which is to look at the forced oscillator.

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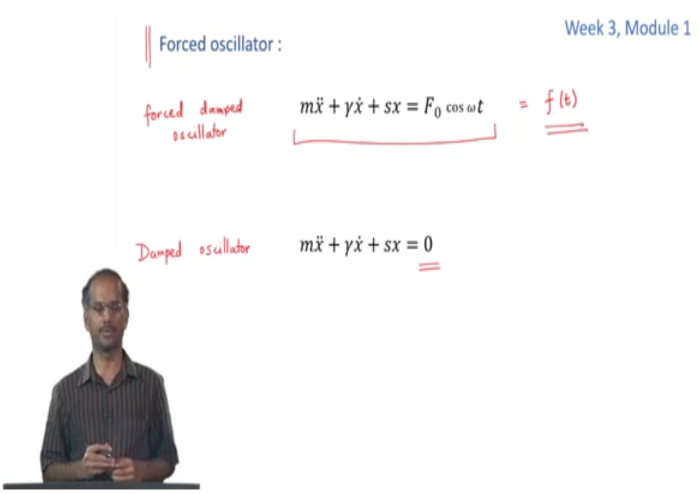

So, essentially we need to model the idea that we are continuously giving it energy. And, here if you look at the equation of motion that I have written down, it looks very similar to the damped oscillator except for this part on the right hand side. So, F_0 is the amplitude of forcing. So, this term on the right hand side, tells you that you are supplying energy as a function of time.

So, in this case the model that we have written down says that F_0 is the amplitude of oscillation and there is a cosine or $\cos\omega t$ attached to it. So, the external force that you are supplying oscillates in a sinusoidal manner. The difference between the damped oscillator and the forced oscillator is what is given here on the right hand side. So, this is the case of equation of motion for a damped oscillator. And, this one is the equation of motion for a forced and damped oscillator and for short, we will just call it the forced oscillator.

The first question that we will be asking is why did we choose this particular form of forcing, this particular form of external forcing, it could have been anything else. So, the reason is I could have chosen it as $F_0 \cos\omega t$ or $F_0 \sin\omega t$ or even much more complicated function. So, in general it could have been some function of time not necessarily cos or sine. But, the good thing about this is that if I choose any arbitrary function, I can always decompose it in terms of sin and cosine functions through what is called a Fourier transformation.

And, if I do that in general the central and the simpler problem that I need to solve is what is written here essentially this. So, if I can solve this I have essentially solved a problem for a general forcing which is $F(t)$ ok. So, it is enough if I know how to solve a cosine or sin kind of forcing.

So, that is the first aspect that we should keep in mind it is not very restrictive. In fact, we are trying to solve a fairly general problem. Now, we are interested in solving this equation.

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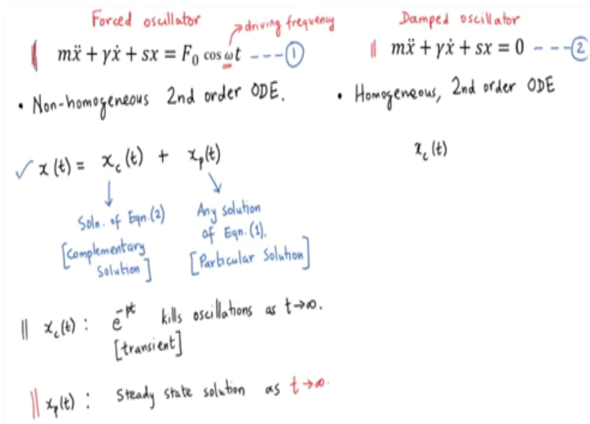

So, let us see how we go about solving this equation, which is essentially what we are going to do in this module.

So, I have both the versions of oscillator. So, this is damped oscillator on the right hand side and on the left hand side, I have the damped and forced oscillator. This damped and forced oscillator is what would be called non homogeneous 2nd order ODE. So, ODE stands for Ordinary Differential Equation. In contrast to an equation like the case of damped oscillator here, this is a class of what is called homogeneous 2nd order ordinary differential equation.

So, these are two distinct classes of still 2nd order ordinary differential equations. And, now to solve the case of forced oscillator, what we know from this class of differential equation is that the general solution for this class could be written like this, $x(t)$ let me write it as $x_c(t) + x_p(t)$. So, here $x_c(t)$ is let me give numbers to it. So, let me say that this is equation 1 and this is equation 2, $x_c(t)$ is simply the solution of equation 2 and $x_p(t)$ is any solution of equation 1.

But, we know the solution of equation 2 already, because we solved it and we had written down and analysed it in the previous modules. So, whatever we had already solved for the damped oscillator would be part of the general solution for the forced oscillator. So, this is called the complementary solution, which is why I had put this c here and this x_p is called the particular solution.

So, the general solution of a non-homogenous 2nd order ordinary differential equation consists of two parts; one is the solution of equation 1 with the right hand side set to 0 which is simply the solution of the corresponding homogenous part plus you somehow find out any one solution for the full forced oscillator. You add these two components together and that shall be your general solution.

In the context of the kind of oscillator problem that we are looking at it we can look at this solution in a little physical sense, x_c here is simply the complementary solution. And, this complementary solution is just the solution of the damped oscillator. And, what we know from the damped oscillator solution is that, like we just did a recap of those in all those cases whatever be the various parameters that you might choose as time tends to infinity, there is no oscillation in the system.

So, if you even for this case if you wait for long enough time the oscillations would have died down. So, in all the cases that we meet the oscillation does die down and clearly the reason is because $x_c(t)$ has this e^{-pt} term and this kills oscillation as time tends to infinity.

So, if you wait for long enough time, you would not see any oscillation. So, this kind of a behaviour is what would be called transient. The reason it is transient is because you do get oscillation, but it stays only for a short time for a particular choice of parameters. Now, what about this second part $x_p(t)$?

So, in this case again we need to see what is it that is physically happening in the system? I have an oscillatory system and it is getting damped, because of dissipative forces, but I am trying to give it some energy externally and the process of giving it an external forcing is modulated by a cosine function. So, there is a frequency ω with, which I am driving it. It is like imagine a physical situation like you go to a garden and there is a swing. And, every now and then you give the swing a push to keep it oscillating.

Now, the frequency with which your doing is the so, called driving frequency and here this ω is the driving frequency. Now, ultimately if you are doing this just imagine, what would happen to the system. The system has it is own dynamics in the sense that if you do not disturb the system, it will oscillates at it is own frequency, you can call it the natural frequency of the system, but now you are trying to interfere and forcing a different kinds of frequency on it.

So, ultimately there would be sort of come negotiation between these two opposing forces and finally, after some amount of time the system would precisely oscillate at the frequency at which you are driving it. So, what you can expect from the second part of the solution which is called the particular solution is that, maybe there would be sort of steady state established; between these two opposing tendencies or these two tendencies, which are not quite in consonant with one another .

So, you could expect steady state as a possible solution. So, just to keep this in mind let me repeat that the complementary solution is a transient which dies away after sometime. Simply, because oscillation or a small movement away from the equilibrium dies off very fast. And, after that there is no contribution from this term which is represented by $x_c(t)$. On the other hand the particular solution is a steady state solution, because it is equivalent to saying that the system finally, responds to your driving and it also beings to oscillate with the same frequency as the one with which you are driving it.

And to complete the story on the right hand side; so, the solution of the damped oscillator I will take it as $x_c(t)$ and we have seen all these properties earlier on. Now, with this understanding we are going to concentrate on the steady state solution, namely what happens to forced oscillator as time tends to infinity. So, we are not so in so much interested in the transients, which are anyway going to died down after a short amount of time; we are interested in what happens in the long time limit. So, we will be interested in working out this solution $x_p(t)$ for the case of forced oscillator ok.

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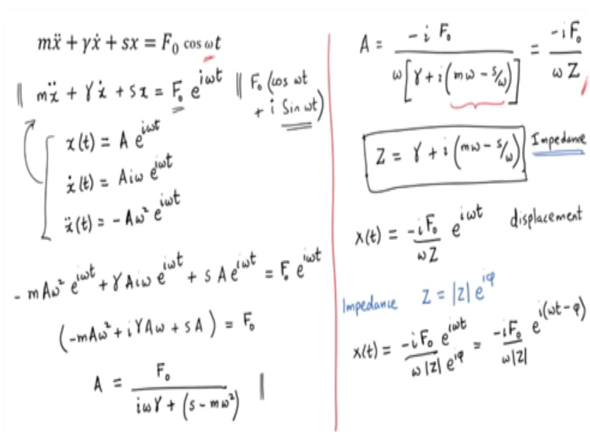

Let us now solve this equation of motion. And to solve this equation motion, I am going to do it a little more generally let me write the equation of motion here. In the way I have written down this equation, the right hand side is a complex number and I shall assume that $\ddot{x}, \dot{x}$ and x are also complex quantities. I am going to assume an ansatz for the solution $x(t)$ and as we just now discuss; if I wait for certain amount of time until the transients die down and so on and so forth.

After, that you will see that the system also will display oscillation with the angular frequency ω. In which case, it is perfectly fine for me to assume that the displacement as a function of time is some A; amplitude again remember that it will be complex because x itself is a complex number, $Ae^{i\omega t}$. Once I know this I am going to compute $\dot{x}(t)$which will be $Ai\omega e^{i\omega t}$ and $\ddot{x}$ will be i^2 will be -1, $A\omega^2 e^{i\omega t}$. Now I need to plug in these 3 quantities back in this equation and find out for what value A all this is satisfied.

So, if I do that you can cancel out $e^{i\omega t}$ on both sides I should be able to get the following equation. So, little bit of manipulation would give you this form of A. Now, I am going to manipulate it little bit I will multiply and divide by $-i$. So, you would have now understood as to why I did this, because I wanted to get this form in here. And, if you remember the quantity that we have here in the denominator looks like the definition of impedance that we saw in the previous module and it is indeed the impedance that we defined in the last module.

So, I can write this as $\frac{-iF_0}{\omega Z}$, where this quantity Z is. So, you should remember that Z being the impedance complex quantity as we see here. And, now we have the amplitude which is also a complex number in a form that is somewhat useful for us to now write the solutions. Now with this I am nearly done can write the solutions. So, $x(t)$ for example, would simply be $Ae^{i\omega t}$ which means that it is. So, I have obtained a solution which is displacement as a function of time.

And, to make this little more meaningful and explicit, I want to write Z in a slightly different form impedance which is Z remember that it is a complex number. I can write it as $|Z|e^{i\phi}$ we will define what ϕ is shortly, but if this is the complexed number which is the impedance it is easy to figure out what ϕ is we have done these manipulations again in the previous module. Now, let me insert this in my expression for $x(t)$ and this will be $\frac{-iF_0}{\omega Z}e^{i(\omega t-\phi)}$.

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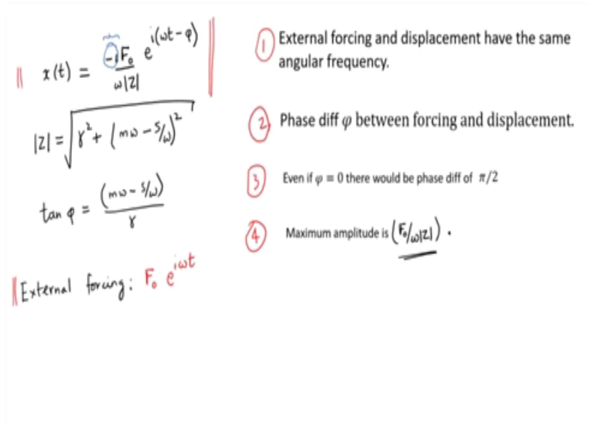

So, now I have the solution for displacement as a function of time.

So, I have the solution for displacement written here and $|Z|$ which is a magnitude of impedance is also written here, and tan ϕ from which we can actually extract ϕ would be given by $\frac{m\omega - \frac{s}{\omega}}{\gamma}$. So, we have everything in place to explicitly say something about the displacement, which is given by this formula. Now, let us look at the solution a little more carefully.

So, the first thing is that as we expected, it turns out that the displacement as a function of time is an oscillatory function and the frequency of oscillation is ω, which is actually the frequency with which you are driving the system remember that you have this ω here. So, you drive the system externally with frequency ω. And, finally, the system also responds in the same frequency displacement has the same oscillatory frequency ω. And, just to make some comparisons let me say that my external forcing is $F_0 e^{i\omega t}$. So, frequencies match, but what about the phases.

So, you would see that the displacement and the external forcing they are not quite in phase with one another. So, the first thing that you notice is that with respect to the external forcing displacement $x(t)$ has this additional phase which is ϕ attached to it and the magnitude would depend on the value of these parameters m, s and γ. So, there will always be a phase difference of ϕ between the external forcing and the displacement in a forced oscillator.

So, the first lesson is that the external forcing and displacement, they have the same angular frequency. Second lesson is that there would be a phase difference of ϕ between forcing and displacement and that you can figure out from these two equations, one for $x(t)$ and the external forcing which is $F_0 e^{i\omega t}$ so, this and this.

And, another point to note is that suppose even by chance if ϕ were equal to 0 still there would be a phase difference, because of this because of this $-i$ term which is here. That would that would ensure that still there is a phase difference of $\frac{\pi}{2}$ at least between the external forcing and the displacement.

So, even if ϕ is 0 you can always choose parameter such that ϕ is 0, but even in such a case there would be a phase difference of $\frac{\pi}{2}$ between the displacement and the external forcing. In fact, more precisely displacement would lag by a phase of $\frac{\pi}{2}$ with respect to external forcing. The maximum amplitude of displacement is $\frac{F_0}{\omega|Z|}$, which is given here.

So, with these 4 sort of basic lessons that we have learnt from the analysis that we did, now we can go and actually write down an explicit form of solution which is which more or less conveys everything that we have tried to infer here.

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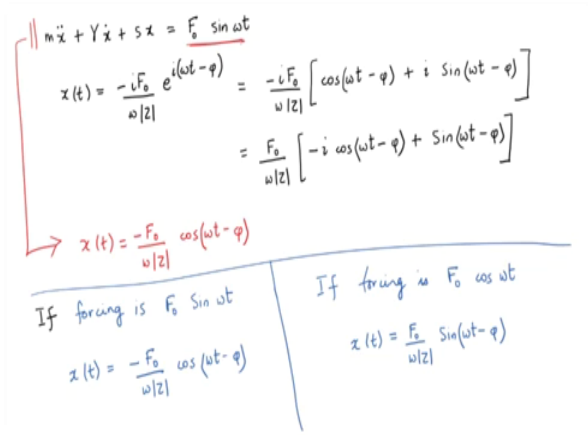

Let me say that now I am trying to solve this problem $m\ddot{x} + \gamma\dot{x} + sx = F_0 \sin\omega t$. So, I want to solve this problem specifically for the kind of drive that is written here, which is $F_0 \sin\omega t$. If, you go back and look at the problem that we actually tried to solve for the forcing there was $F_0 e^{i\omega t}$, which means that it is actually $F_0 \cos\omega t + iF_0 \sin\omega t$.

So, in some sense we have already solved for both cosine and sinusoidal driving all we need to do is to isolate the solution only for $F_0 \sin\omega t$ kind of driving. So, all I need to do is to copy the result that I have obtained for the displacement, let me write that down to begin with. So, $x(t)$ was and this is the solution that we had. So, again let me emphasize that x is the complex quantity, which means that this could have been written as let me now multiply $-i$ inside in which case I will get $\frac{F_0}{\omega|Z|}$, and $-\cos(\omega t - \phi)$; $-i$ when multiplied would give me $+1$ and that would simply boil down to this.

All, I need to now do is to recognize that this $F_0 \sin\omega t$ which is the particular choice that I made for the external driving and I need to correlate with the kind of external driving that I have put in here. So, this will be $F_0(\cos\omega t + i\sin\omega t)$ t. So, the correct solution that I expect should all be imaginary part of the solution, because the forcing here appears in the with the i here, which means that for this form of external driving. Now, the solution can be written as $x(t)$ is equal to $\frac{F_0}{\omega|Z|}$ into of course, there is a minus sign $\cos(\omega t - \phi)$.

So, now, it is very easy for me to generalise the result, if forcing is of the form $F_0 \sin \omega t$, then my solution is $-\frac{F_0}{\omega|Z|}\cos(\omega t - \phi)$. On the other hand if forcing is $F_0(\cos \omega t)$. In that case the solution $x(t)$ would be the real part of this one which would simply be $\frac{F_0}{\omega|Z|}\sin(\omega t - \phi)$.

So, as I said we have solved two different problems when we treated this as a complex quantity, having obtained the displacement as a function of time. Now, it is easy to also find the velocity as a function of time.

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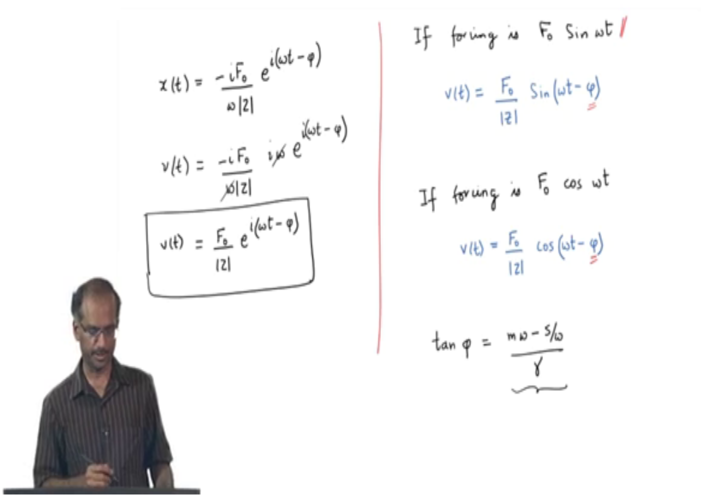

The velocity as a function of time which I am indicating by $v(t)$ would simply be the first derivative of this function, that is $\frac{iF_0}{\omega|Z|}$ will give me $i\omega e^{i\omega t - \phi}$ and ω. And ω and ω would cancel i^2 is -1 combined with the minus sign, there I will simply have $\frac{F_0}{|Z|}e^{i(\omega t - \phi)}$.

So, this is velocity as a function of time. So, once again we can ask what if my forcing is $F_0 \sin \omega t$ and $F_0 \cos \omega t$. So, if forcing is $F_0 \sin \omega t$. I just need to take the imaginary part of this quantity. And, in that case $v(t)$ would be equal to $\frac{F_0}{|Z|} e^{i(\omega t - \phi)}$.

And, similarly if external forcing is $F_0 \cos \omega t$, in that case $v(t)$ will be the real part of this result, that will be $\frac{F_0}{|Z|} \cos(\omega t - \phi)$ with this, we have obtained the solution for velocity.

So, now if I analyse these expressions for velocity a little more closely you will see that between the forcing that I have and the expression for the velocity that I have obtained there is always a phase difference of ϕ. So, whether you take your forcing to be $F_0 \sin \omega t$ or $F_0 \cos \omega t$, in either case there is going to be a phase difference of ϕ.

So, the velocity and the external forcing would have would differ in phase by ϕ, but if you choose parameters such that $\phi = 0$, in that case the forcing and velocity would be in phase with one another.

Again it does not depend on what form of forcing that you choose. And just to remind you once again ϕ is given by $\frac{m\omega - \frac{s}{\omega}}{\gamma}$. So, if you want to find ϕ, we just need to take tan inverse of this quantity. To summarize let just collect the results together, we solved the equation in complex form. So, we have results for both kinds of forcing $F_0 \sin \omega t$ and $F_0 \cos \omega t$.

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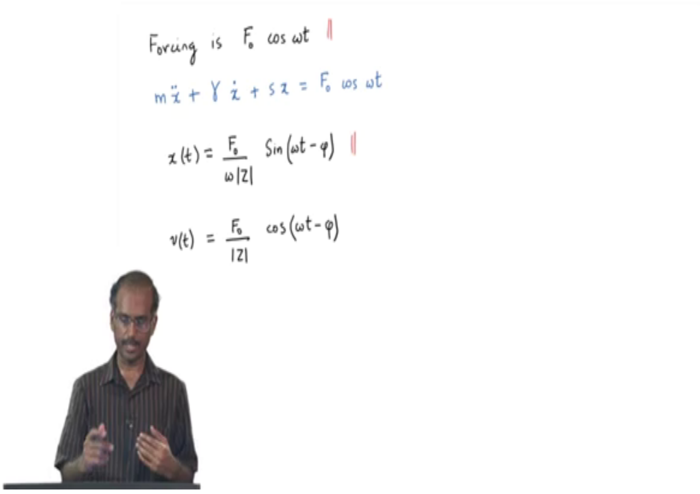

So, let me for example, assume that forcing is $F_0 \cos \omega t$, if this is the external forcing now my equation of motion would be $m\ddot{x} + \gamma\dot{x} + s$. So, this is a scalar equation all the quantities are scalar. In such a case, what we saw was displacement as a function of time would be F_0, which is a magnitude of external forcing divided by $\omega|Z|$ into $\sin(\omega t - \phi)$. And, we also saw that velocity will be $v(t)$ which will be equal to $\frac{F_0}{|Z|}\cos(\omega t - \phi)$.

And, clearly you notice that between the forcing and displacement one is a $\cos \omega t$ and other is $\sin \omega t$. So, even if ϕ were 0 there is going to be a $\frac{\pi}{2}$ phase difference, but between forcing and velocity if ϕ is 0 both would be in phase, otherwise there would be phase difference of ϕ between velocity and forcing. And, with this summary we will go and see some more properties of the forced oscillator in the subsequent modules.

Thank you.