Waves and Oscillations
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Lecture – 10
Forced Oscillator: Part 1

Welcome to the 5th module, in the last few modules we have been looking at the dissipative oscillator, the one which would oscillate and yet is subjected to dissipation. From this module onwards we are going to look at adding one more realistic effect. As you would have seen when I add dissipation to the system the oscillation finally, comes to a halt. But we want to continue the oscillation for instance in a real pendulum in a clock there is dissipation, but then it is compensated by continuously supplying it with energy. So, forced oscillator is a model for being able to do precisely that.

So, you have an oscillator which oscillates, it is subjected to dissipation and you compensate it by continuously giving it energy in some way or the other. So, that would be a forced oscillator for us and we will see what are the various properties and dynamics that a forced oscillator would show, that would be the kind of things that we would be studying in the next few modules.

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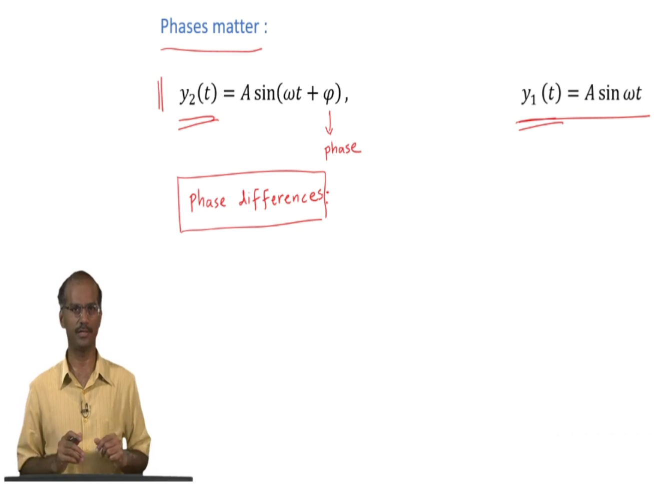

But before we get there we need to learn a few preliminaries it is mostly mathematical preliminaries ok. So, let me begin by saying that when we start looking at forced oscillator, what is going to really become important is the phases or actually the phase difference between let us say the position and acceleration or the phase difference between the natural frequency of your oscillator and the frequency with which you are forcing it for instance.

So, ultimately the important point is that phases do matter. Just to recap what phases are if I am looking at an equation of this type let us say y_1 is $A\sin\omega t$, phase is 0, it simply means that a $t = 0$ this function takes whatever the value it is and there is no initial offset in some sense ok. On the other hand if I have this function which is given by $A\sin(\omega t + \phi)$, ϕ is the phase and if I am only looking at this solution y_2 is some $A\sin(\omega t + \phi)$ all it tells me is that with respect to starting at origin there is an initial off set to the extent of about ϕ that is when I look at this oscillation in isolation.

On the other hand if I compare both these oscillations y_1 and y_2 it tells me that compared to $y_1(t)$, $y_2(t)$ has a maintains a phase difference of ϕ. So, phase difference becomes crucial in particular when you have two oscillations or more than two oscillations then you need to be worried about are they in phase or are they out of phase or do they maintain any other phase relation.

So, essentially the story of whatever is going to come in the forced oscillator is going to crucially depend on what happens to phase differences between various quantities ok. So, I am going to basically present for you some mathematical preliminaries.

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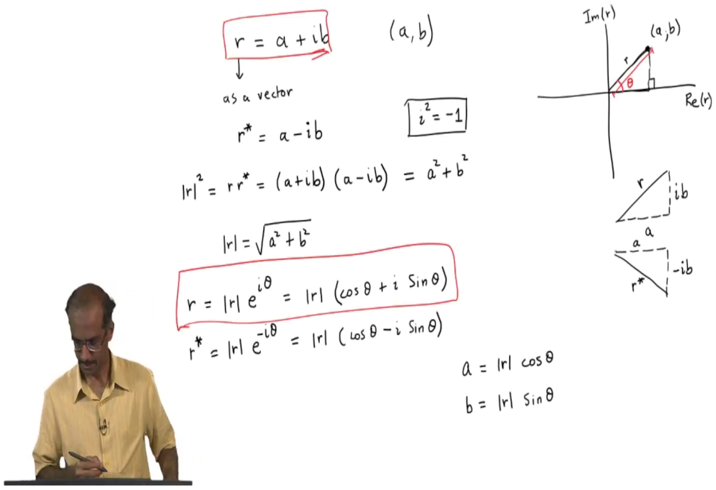

Let us start with complex number let me call it r which is some let us say $a + ib$ or you could write it as simply (a, b) ok. So, if I have to plot this an organ plane. So, this is the real part of my complex number r and y-axis is the imaginary part of the complex number r. So, let us identify this point here, this is (a, b). So, this is actually a and this will be b and this is r ok. So, if I separate it out it would look like this, this is a, this is b and this is r.

So, now, you would notice that there is actually an analogy with vectors in two dimensional spaces. So, you could think of this r as a vector imagine that it is a vector with two components. When you think of it as a complex number, it still has two components there is a real and imaginary part to it. And now given that r is this complex number $a + ib$ and if I think of it as a vector I can make the following sort of analogy. Let us say that a and b were individually vectors for instance and they were parallel to one another to begin with and by the act of adding this i or multiplying this i to be what I have done is to rotate this axis by 90 degrees. So, you could see that this corresponds to this 90 degrees here.

So, if a and b were originally parallel by multiplying by i have moved this by 90 degrees. So, that is a very important sort of picture that we should keep in mind, every time you multiply i to a certain quantity, it is going to essentially turn that vector by 90 degrees ok. Let us go back to the complex number picture itself. So, r is $a + ib$ and with that I could define a complex conjugate which is r^* which will be $a - ib$ and again it also has geometric interpretation. So, one could think of this. So, this is $-ib$, this is a and this is r^*. Now given r and r^* modulus of r^2 is simply equal to r is multiplied by r^* which would be equal to $(a + ib) \times (a - ib)$ and you can do the multiplication yourself keeping in mind that i^2 is equal to -1.

So, if you do that you will get $a^2 + b^2$. There is yet another way of representing the complex number in terms of this $|r|$, which is like the length of this vector and the angle that it makes here which I will call θ. So, in that case r could be written as $|r|e^{i\theta}$ that would be $|r|(\cos\theta + i\sin\theta)$ ok. And equivalently one can also define r^* which will be $|r|e^{-i\theta}$ this would correspond to $|r|(\cos\theta - i\sin\theta)$.

Now, if you matchup these two quantities for example, match this that r is $a + ib$ with this form of description given here, in terms of modulus of r and θ itself. So, it is very clear in that case that I can write a to be $|r|\cos\theta$ all you need to do is to simply match them and b is $|r|\sin\theta$ and from here it is also possible to define $\tan\theta$.

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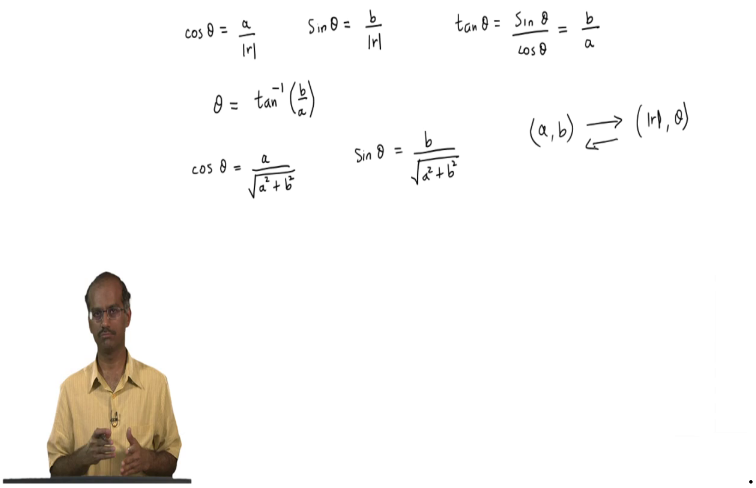

Let me first write $\cos\theta$ as $\frac{a}{|r|}$ and $\sin\theta$ is $\frac{b}{|r|}$ and $\tan\theta$ would be simply $\frac{\sin\theta}{\cos\theta}$ that would be equal to $\frac{b}{a}$. So, from this we can extract the angle θ which would be $\tan^{-1}\left(\frac{b}{a}\right)$. And given that $|r|$ is this quantity here, we can use this to rewrite $\cos\theta$ and $\sin\theta$ as; so, $\cos\theta$ would be $\frac{a}{\sqrt{a^2+b^2}}$ and sin theta would be $\frac{b}{\sqrt{a^2+b^2}}$. So, we went from describing a vector in terms of two components in the complex number representation there was a real and a imaginary part of the complex number and in the polar representation as it is called it is in terms of $|r|$ and angle θ.

So, given one we can always go to the other and we can also go back to the other. So, in other words you could go from a description in terms of (a, b) to a description in terms of $|r|$ and θ or you could go the other way around as well. So, all the transformation formulas are here. But let me remind you that the central piece of this geometrical picture that we require from here is that when you multiply vector by i it is equivalent to rotating it by 90 degrees.

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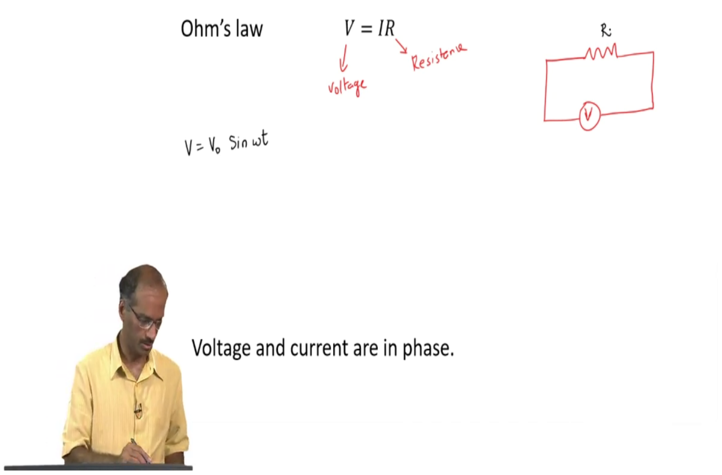

To understand this let us first look at standard way in which Ohms law is stated. So, you would have seen it stated as $V = IR$. So, V is the voltage, I is the current and R is the resistance. If I have something like a voltage being applied which is V and the only thing that is there in the circuit is a resistance and let us say current I flows through it.

So, here the statement is that current and voltage have the same phase. So, it is possible that voltage is something like $V_0 \sin \omega t$ the sinusoidally varying voltage of some frequency ω. So, in this case when the only element that is there in the circuit is resistor of resistance R. The Ohms law is stated as $V = IR$ the reason for that is that voltage and current are always in phase ok.

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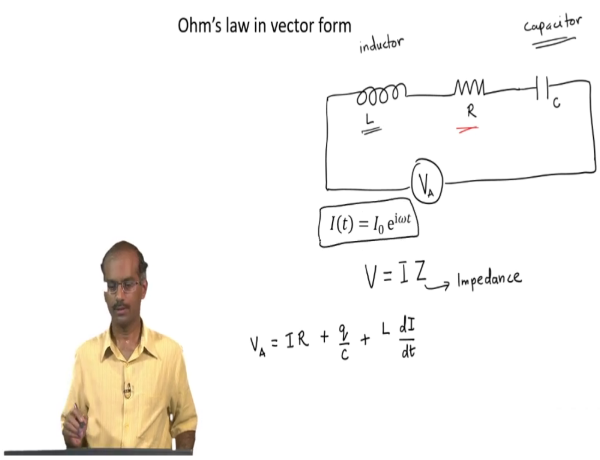

On the other hand the situation becomes a little more complicated when you add other circuit elements. So, for instance here I have a circuit which has of course, there is this resistor of resistance R, but I also have a capacitor here and an inductor. So, let me assume that it is a capacitor with capacitance of C and inductance is L and the applied voltage is V_A. So, in general in such cases the so, called vector form of Ohms law would be written as V is current times IZ where this quantity IZ is called the impedance.

So, if you compare it with the standard form it is like resistance. So, it is equivalent of resistance, but for the cases where you have inductor maybe capacitor one or both of them along with resistor in a circuit. So, this is a vector form of Ohms law the quantity is here V, I and Z are all vectors you could think of it as complex numbers.

To begin with let me write an equation equating the voltages, the voltage across the entire circuit is equal to V_A. So, that is the potential difference is equal to IR which is the potential drop across the resistor here plus $\frac{q}{C}$; q is the charge and C is the capacitance. This is the potential drop across the capacitor here plus $L \times \frac{dI}{dt}$ and this is the potential drop across the inductor.

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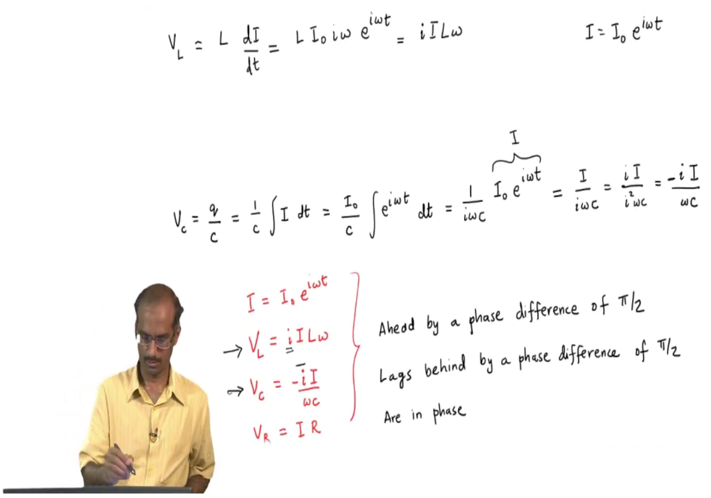

Now, let us write expressions for the potential drop across the inductor and across the capacitor and should keep in mind that we have assumed that current has this form, $I = I_0 e^{i\omega t}$. In that case the potential drop across the inductor let me indicate it by V_L would be $L\frac{dI}{dt}$ and I just need to differentiate $I_0 e^{i\omega t}$. So, that is $LI_0 i\omega e^{i\omega t}$. So, as I said we should remember that $I = I_0 e^{i\omega t}$. So, if you look at this expression here can rewrite it as $I_0 e^{i\omega t}$ I just call I. So, I will have $i \times I \times L \times \omega$.

So, let us keep that aside for a moment we will come back to that and let us also do the same kind of calculation for the voltage across the capacitor. So, if I call it let us say V_C

that will be $\frac{q}{C}$ and q itself is simply an integral of I the current over dt. So, I just need to do a simple integral involving $e^{i\omega t}$. So, this would simply become $\frac{1}{i\omega C}I_0e^{i\omega t}$ and once again this $I_0e^{i\omega t}$ is simply I and I will have $\frac{I}{i\omega C}$. Ideally I would like to have i in the numerator and it is easy to do that simply multiply and divide by small i you will get $\frac{iI}{i^2\omega C}$; $i^2 = -1$. So, I will have $\frac{-I}{\omega C}$.

Now, let us assemble three things together and see them in one place. So, now, I have current I which I wrote down as e power, $I_0e^{i\omega t}$ and then I have potential across the inductor which will be $iIL\omega$ potential across the capacitor, which is $\frac{-iI}{\omega C}$ and of course, for completeness I can also write potential across the resistor R as IR.

Now, when you compare all these equations you will see that for the case of potential drop across the inductor, there is this quantity i which multiplies capital I the current. So, as we said in the a little while ago whenever you multiply this quantity small i to any quantity it is like rotating it by 90 degrees. So, essentially what has happened is if current is $I_0e^{i\omega t}$, V_A which is the potential is rotated by 90 degrees or it has acquired an additional phase of $\frac{\pi}{2}$. So, you would say that compared to the current the potential across the inductor is ahead by a phase difference of $\frac{\pi}{2}$.

Similarly, if you compare current and the potential across the capacitor so, here the term current is multiplied to $-i$. So, in this case it is the opposite of what happens in the case of inductor. So, in this case compared to current I a voltage across the capacitor lags behind by $\frac{\pi}{2}$. So, we would say that voltage across the capacitor maintains a phase difference of $-\frac{\pi}{2}$ with respect to current.

On the other hand if you compare current and V_R which is the voltage across the resistance, I is unchanged. So, the potential across a resistor and the current are in phase.

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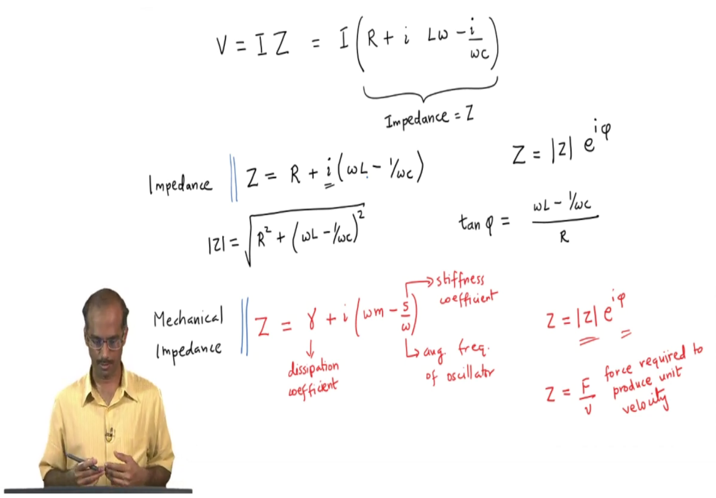

Now let us assemble all this together V in vector form, as we said $V = IZ$ and that will be equal to ok. So, now, I have the equivalent relation here and now it is very clear that the quantity that is here inside the brackets is what I would call as impedance Z. So, I can separate this and write it out the impedance, which as you can see from this equation thus the equivalent job of a resistance would simply be R plus; ωC.

And as you can see this impedance is a complex quantity there is this small i sitting here, which means it is a complex number and since it is a complex number it can also be written in that polar form. So, I could write it as $|Z|e^{i\phi}$ in general and as we did earlier, we can determined both $|Z|$ and ϕ. So, now, $|Z|$ would be $\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ and $\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}$. From this we can determine the value of ϕ ok.

The interesting part is that we obtained or tried to motivate this idea of impedance using electrical circuit, but we can carry over this idea to a mechanical problem as well. So, in this case one can think of a mechanical impedance or simply impedance in the context of say oscillatory behavior, let me for now just call it mechanical impedance or impedance in problems of mechanics.

In this case Z would be the following γ plus i into. So, here this quantity γ is the dissipation coefficient and s of course, is the stiffness coefficient m is of course, the mass. So, having define mechanical impedance in this way, you can again write it in this polar form which means here again it would be valid to say that $Z = |Z|e^{i\phi}$. So, again I can equivalently write down an expression for this $|Z|$ and the value of ϕ.

From a physical perspective for our purpose is what is important is that, this Z is simply force required to produce a unit velocity in the oscillator. Now if you compare these two equations let us say this one and this one, you will see that mass is the equivalent of inductor and capacitance is the equivalent of the inverse of stiffness and of course, resistance has it is equivalent in the dissipation coefficient. And if you go back one slide before, you would see that the effect of inductor is it adds a phase difference of $\frac{\pi}{2}$, it is ahead by $\frac{\pi}{2}$. In the case of capacitance it leads to a phase difference that lags by $\frac{\pi}{2}$, you can also make equivalent analogies with the mechanical counterpart here.

In the next module we will actually start by writing down the equation for forced oscillator and we will see how all these we will come together there to show us different effects.