Statistical Mechanics
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Lecture – 08
Many Random variables

So, good morning students, today we will talk about Many Random Variables ok.

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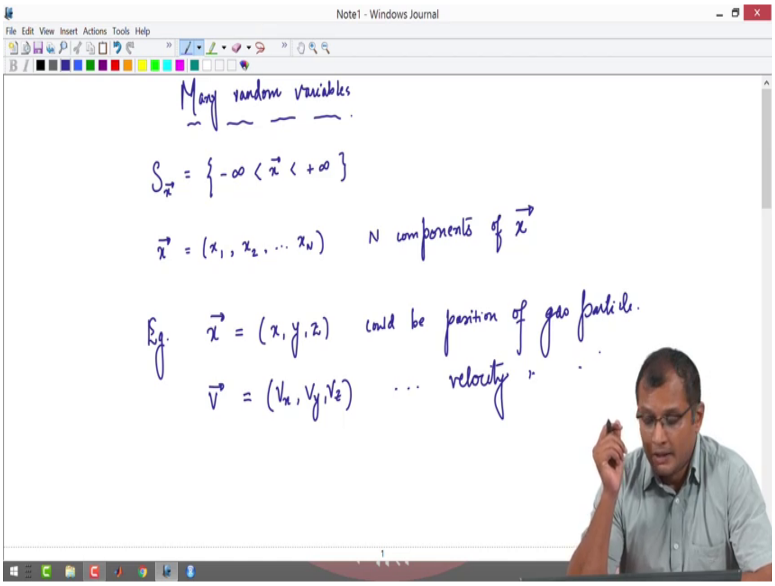

So, the subject is relevant if you have multiple random variables in the problem and you are interested in the expectation value of products of some of them. Or if you are interested in conditional probability of sets some subset of random variables in the problem and we will discuss what is meant by condition probability in the course of the lecture ok.

So, whatever we have learned so far in the context of a single random variable applies or extents to the discussion on multiple random variables. So, let me take the set of outcome for a random variable, this time I am taking it as a vector and it will become very clear why I have taken the random variable as a vector.

So, the set of outcomes let us say is between minus infinity to plus infinity ok, so that only means that I am going to write it as a vector x between minus infinity to plus infinity. Or, the meaning if this is the following. x is a random vector which means it has components and each component can take values between minus infinity to plus infinity, as the interpretation of the set of outcomes for n random variables. So, these are like N components of the vector random variable vector x ok.

Now, this is relevant for example, if you take x to be the position of a gas particle in this room, then it is nothing, but the x y z, components of the position of the gas particle, it could be the position of a for gas particle ok. You could take x as velocity, in that case you would have 3 components for the velocity for the gas particle of this room. So this could be velocity. And, hence suppose I am interested in finding the average moment or a average position of the particle in this room, I would be using probability density p of x comma v.

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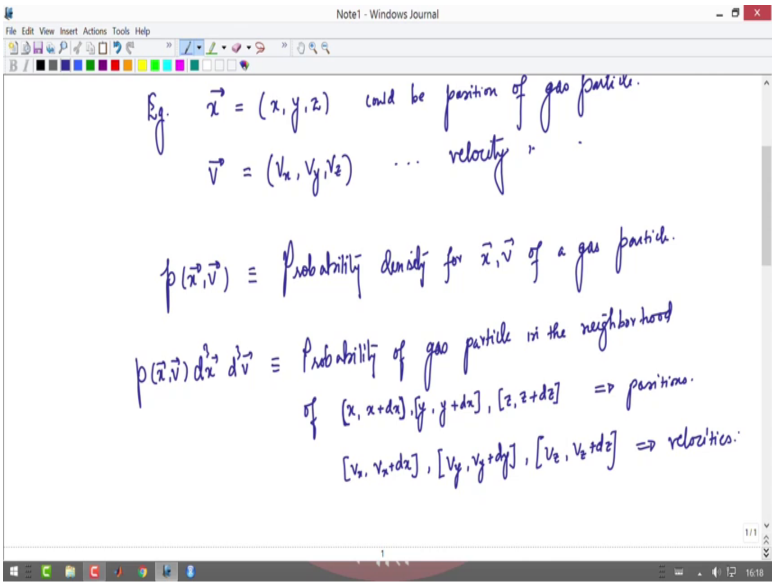

So, this could be basically giving me a probability density for x and v of a gas particle. This simply means that if I want to know the probability of finding gas particle in the neighborhood of some x v, which is position and momentum of the in this room; then I simply have to compute p of x d v x comma a v into d cube x into d cube v.

So, this will give you probability of finding gas particles of a gas particle in the neighborhood of x plus dx y y plus dx z z plus dz and the velocities ok, so these are like the values of positions ok. And the velocities are within the range v x to v x plus d x and the y component of velocity is in the range v y to v y plus d y and the z component of velocity is in the range v z plus d z ok. So, this is like 6 dimensional cube.

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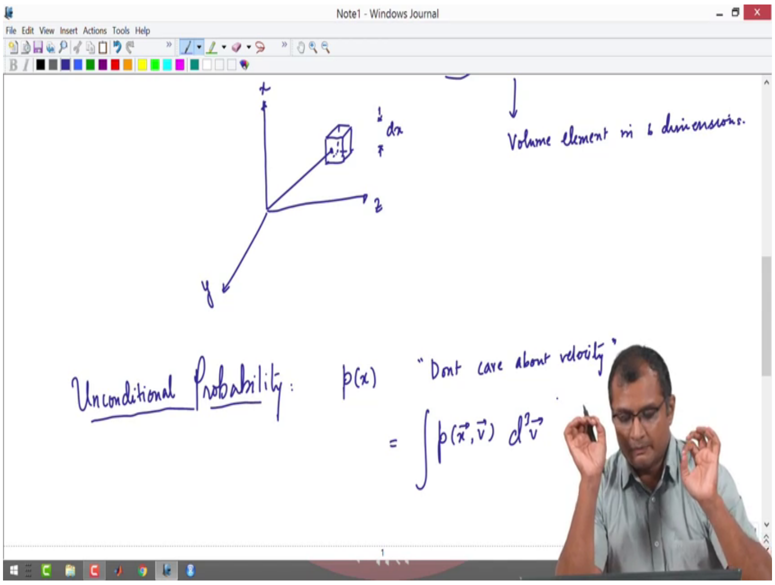

So, I am going to draw it in the 3 dimensions only taking. So, you can take x y and z and here just take a point here in this space. And, this is a 3 dimensional cube at some location the sequence of the cube is basically d x d y and d z ok.

So, this distance could be some roughly you know d of x and d x d y d z is basically volume of this cube at location x y z ok. Similarly you have take 3 axis for the momentum or velocity and then take a volume in the velocity space. And, in the 6 dimensional space of position and momentum you will see a cube of dimension 6, which is centered at point x comma a v. And, the probability of finding a particle in this cube this (Refer Time: 06:57) into cube if you want to say is the probability that are written here. So, this is the probability that is.

Student: That should be v v x plus d v x right.

Yes.

Student: That.

Right for x.

So, I have to make a small correction here. This is a typo over here they should be yeah it is d v x, d v y and d v z thanks and so this basically is a volume element in 6 dimensions ok. Now if we want to ask simple concept you know if this is the simple concept which goes as a conditional probability, we will talk about unconditional probabilities first ok.

So, if you want to know the unconditional probability of finding a particle in location x ok. So, the word unconditional probability means, suppose you want to find the probability of finding a gas particle at location x and you do not care about velocity, you say that i do not care what the velocity is all I want to know is what is the probability of finding a particle at location x.

Now, what that means, is at you, you are you are asking an unconditional probability estimate. You are saying that I will take the joint probability density, one that as all the information of positions and velocity and I will simply integrate this overall possible velocities. Because I do not care where the particle is, I do not care what is the particles moment or velocity, all I want to know what is the probability of finding a particle, probability density of finding a particle in neighborhood of x.

So, if I take this probability density and integrate overall velocities I will get what is required and that is the unconditional probability density of finding a particle at a location x ok.

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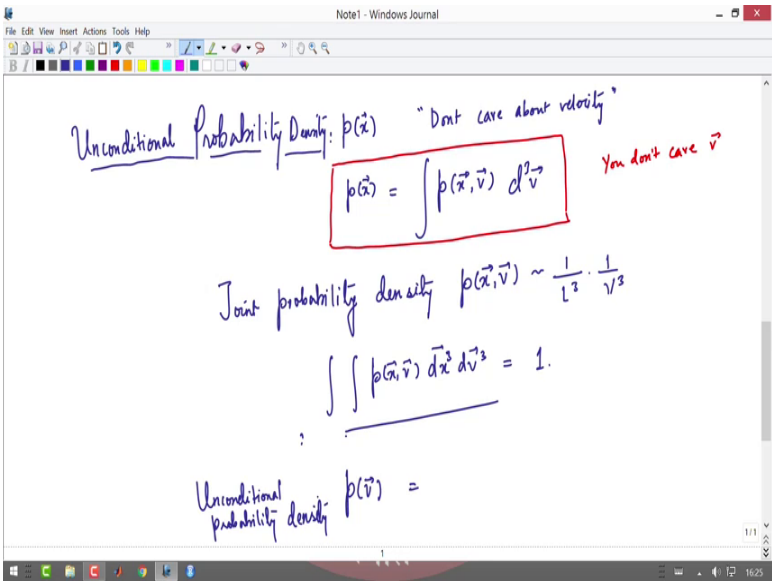

So, this is the unconditional probability density ok. So, now this means that my total probability or the joint probability density must be normalized ok. So, I am going to call this as joint probability density and it has dimension of 1 upon x cube and 2 upon v cube ok. So, it is a dimensions of 1 upon length cube into 1 upon velocity cube, some velocity cube some, velocity scale cube ok, because if you if you integrate the total or the joint probability density over entire phase space, what you get is basically the normalization which is 1 ok.

So, here the range of the integration is your minimum velocity to maximum velocity and the bounce of the volume of the box. So, that is the meaning of the joint probability density and knowing this, one can compute the unconditional probability density which has been just computed in this expression ok. Now, so this is basically concept that says that I want the information of a small subset of a variables and such that I do not care what are the values of the remaining subset of variables. So, like in this expression you do not care about the velocities because that is one subset your variable the other subset is positions you say that I want to know what is the probability of finding a particle at location x, I do not care what is the velocities are.

So, this way you just have to integrate the joint probability density over all velocities and this will give the unconditional probability density on x. Now, similarly, you could also obtained the unconditional probability density on velocities ok. So, another unconditional probability that you want to that you can construct is nothing, but what is the unconditional probability of finding? So, this is like the unconditional probability density of velocities.

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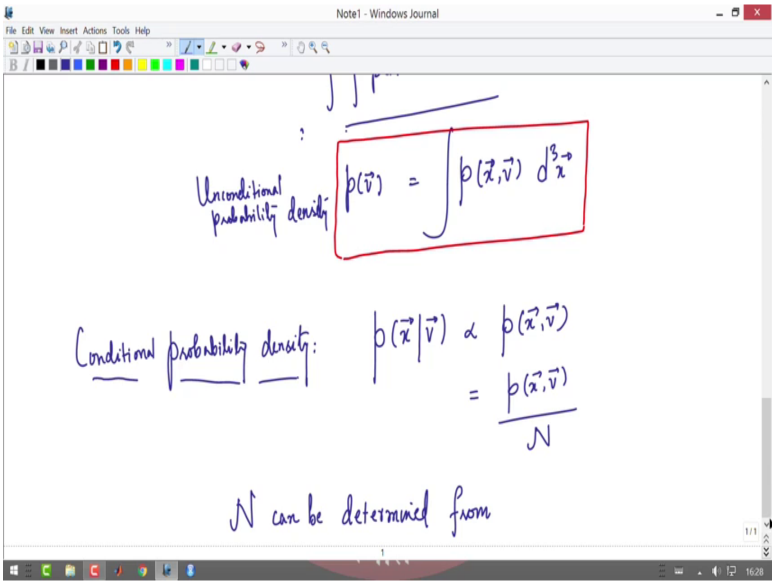

Again the procedure is same you take the joint probability density or the total probability density which are the full information and then you integrate out the position degrees of freedom. So, in to the volume integral of this and what you get is basically your unconditional probability of finding the velocity of a gas particle ok.

So, that is the answer to the problem, but you would also talk about conditional probability densities. Now this would this is more interesting this simply means that if you, if you want to know the probability of finding a gas particle at some location x provided its velocity is said to some value. Or, you may want to know the probability density of finding particle at a location right below the fan for instance provided its velocity is something.

Now, here you are not saying that I do not care about the remaining subset of the random variables if you want information on x you are also fixing the information on v and vice verse ok. So, this is defined as, so this is like asking what is the probability density of finding a measurement on x such that, the measurement on velocity is v ok. So, this called as the conditional probability density and this is definitely proportional to your joint probability density So, that is my total information and the equality can be taken by dividing this by a norm ok.

Now, how do you determine norm? The norm can be defined by demanding that the conditional probabilities are normalized. So, we integrate on both sides with respect to positions and what you get basically, so you can write down the definition for the norm from normalizing the conditional probability density.

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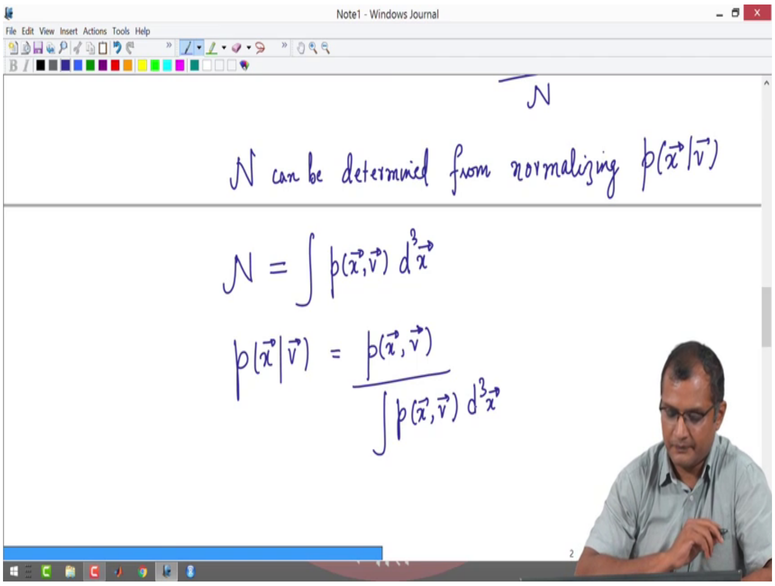

So, I can integrate with respect to v on both sides and get a definition of N. So, this could be since conditional probability density and normalize that is the requirement we are imposing this simply becomes the integral of the joint probability density with respect to the positions ok. And hence, I can write down the conditional probability density of a finding a particle at location x such that its velocity is fixed to v ok.

So, this is the condition finding a particle at location x such that, its velocity is v this is given as joint probability density divided by the norm which we have just computed as the joint probability density integrated over all space ok. So, that is the beautiful result and you could also you could also write down the result for conditional probability for position for velocities.

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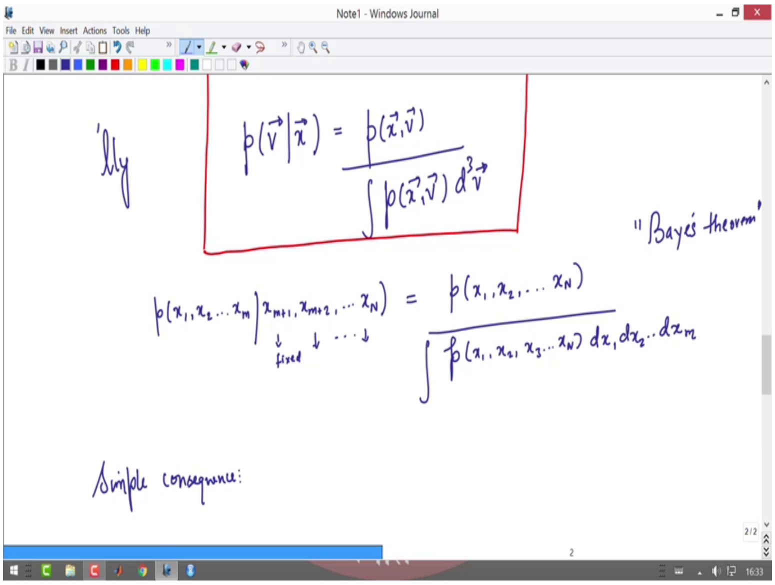

So, you can also write down similarly you can write down for the condition probability density on velocity such that, the positions are fixed to some value x. Naturally this will come out to be the joint probability density divided by the norm, which is now simply the integration of the joint probability density over the entire volume over the entire velocity ok.

So, these two relationships are very important and they give us understanding of conditional probabilities in a equilibrium systems ok. So, I am going to note this down ok. And formalizing this machinery we can only say that there is there is a theorem which goes by the name of Bayes theorem and that simply is the generalization of what we have just seen.

So, if you have, if you have if you want that out of the n random variables, you want information on a few of them let us say up to m such that, the remaining random variables I have taken up these values ok. So, you are asking: what is the probability of finding x 1 x 2 x m, such that, variables from m plus one to n have taken these values ok.

So, these are like fixed values ok. This is given by a simple Bayes theorem which says that you have the total or the joint probability density divided by simply an integration of this joint probability density over the variables ok. This is the so called Bayes theorem and simple consequence of a Bayes theorem is that if your random variables are completely uncorrelated ok. Let us take for the simply case of 2 variables ok.

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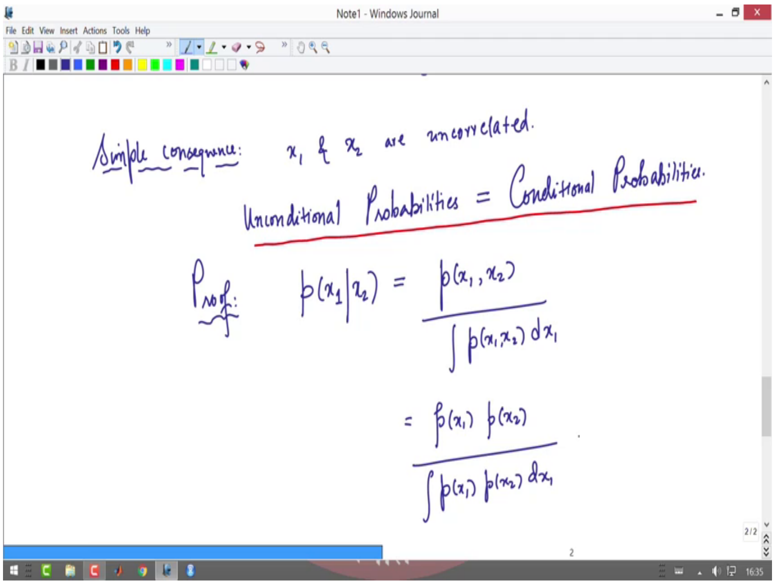

So, if x 1 and x 2 are uncorrelated, then the unconditional probabilities are the same as the conditional probability ok. So, I will first state the theorem this applies to n random variables I am just proving it for just 2 variables ok. Now, this can be very easily proved by looking at the Bayes theorem. So, that Bayes theorem just 2 variables would be the unconditional relationship between conditional probability and the unconditional probability.

So, if I write down the conditional probability from the previous example then it would simply be probability of finding x 1 when x 2 has been fixed to some value. And, this would be given as the joint probability density divided by integration of the joint probability density overall values of x 1 ok.

Now, since now since, your variables are completely uncorrelated, I can write down the numerator as product of individual probability densities ok. So, the variables are uncorrelated, the joint probability density is simply the product of individual probability densities and by the same logic I can write down the denominator as p of x 1 and p of x 2 integration over all values of x 1 ok.

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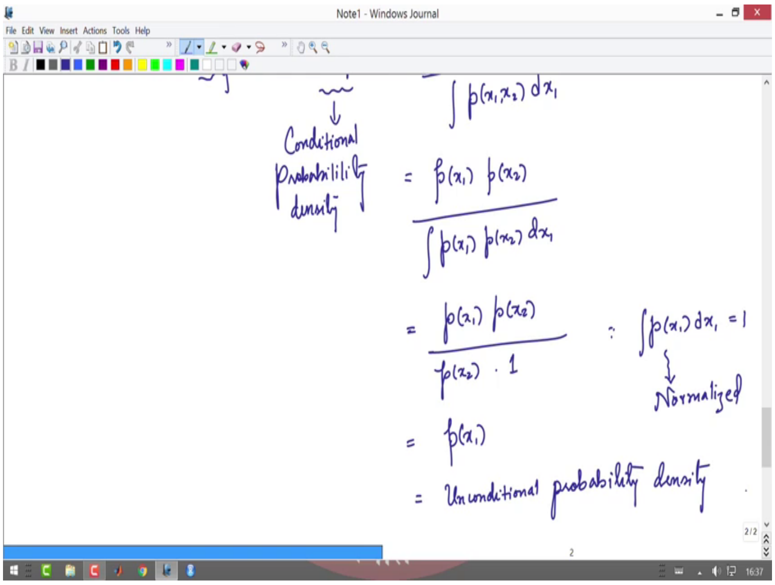

So, p of x 2 will come outside the bottom integral and integration of p x 1 d x 1 is simply 1, because p x one is normalized the distribution is normalized. Leaving as with just p of x 1 ok, this is the unconditional probability and if you look at the right hand left the left hand side, what you have is basically conditional probability.

So, if you look at this the relation what you when proving is that the random variables are uncorrelated, the conditional probability density becomes the same as the unconditional probability density ok. So, that is an important result which has to simplify problems and we shall give some problems in the upcoming assignment set, that will test your understanding of these probability concepts right.

So, now we quickly move to the next topic which is basically calculation of variable moments in variable moments in cumulants in such a multi variable distribution ok.

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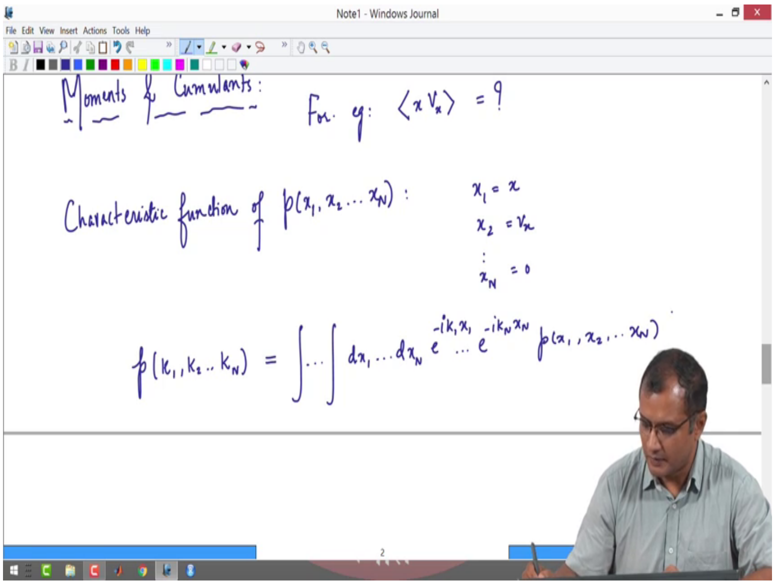

Now, what do I mean it moments and cumulants or such distributions let me give you some examples. So, I will give an example what is the probability of the product V x into x x and V x are the x component or position and velocities of a gas particle. So, this is like a joint moment of 2 random variables, how do you compute this ok. You can compute this if you know the characteristics function of multi variable distributions ok. So, let us compute these moments using the definition of characteristics functions and then we will shall proceed with an I will give another very simple bright method very simple n powerful method which is the graphical method of computing these moments ok.

So, let us compute the characteristic function of our pdf is now multi variable pdf ok. Now I could take my x 1 as x, x 2 as V x and all other values of x as 0, then this would give me the expectation value of x V x. This characteristics function would give me the expectation value of these variables. Now, so the characteristic function can be computed as the set of over it will be a function of a set of case that are nothing, but by the definition this multi dimension Fourier transform of our multi variable distribution ok.

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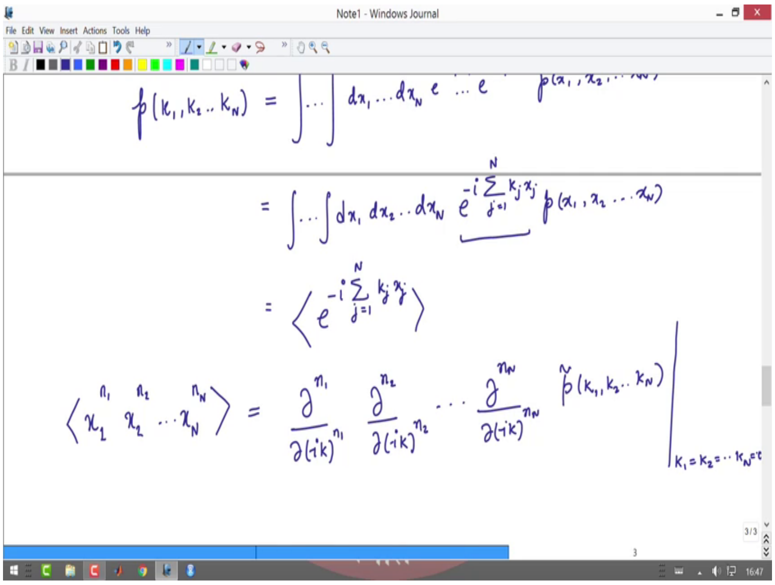

And this will reduce to the integral of this exponential, where j run from 1 to n p of x 1 x 2 all the way to x n. And this is nothing, but the expectation value of the function here, which is nothing, but e to the power minus i summation j going from one to n k j x j ok. And so now, if you want to find out the joint moment of some x 1 to the power let us say n 1 x 2 to the power n 2, all the way to x n to the power n capital then this is nothing, but d n one over d of minus i k n one the successive derivatives.

So, all the way to d n n d to the power minus i k n n of my characteristics function ok. And this entire derivative is taken at at k 1 equals to k 2 equals to all the way up to k n equals to 0.

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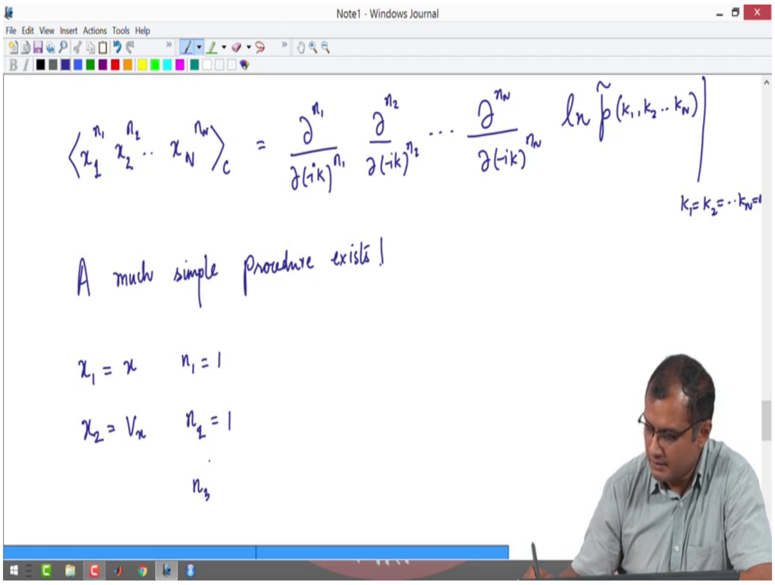

Or if you want these joint cumulants ok, so, you need the cumulant generating function and the procedure is same. So, you take these order derivatives of logarithm of the characteristics function, which is the cumulants generating function, again this is taken at the k equals 0. Now, you can see that this expression is very lengthy and p raise to (Refer Time: 31:38) is fortunately compute these joint moments and cumulants there exists a very simple diagrammatic procedure which is similar to what we have seen for single variable random distributions.

So, as much simpler procedure exists, which is diagrammatic procedure exists and that is diagrammatic in nature. For example, I take x 1 equals to x and I take n 1 equals to 1 and I take x 2 equals to V x and I take n 1 equals to n 2 equals to 1 ok. And I take all other powers as you know all other n 3 all the way to n n as 0.

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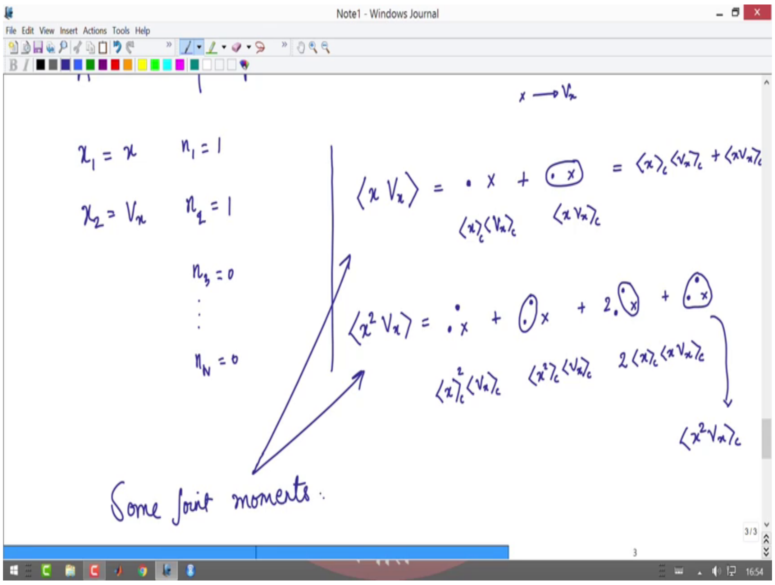

So, that means, my formula here. So, basically what I am trying to say is that I am computing the joint moment of x into V x. So, there is a gas in this room which is at equilibrium and what is the average value of the position multiplied by x component of the velocity. Now, these are 2 variables ok. So, I am going to use different symbols, I am going to use dot and a cross ok. So, dot is for a positions and a cross here represents velocity, like we did for the single random variable moments. Now I can write this in 2 ways I can leave them disconnected or I can connect them ok.

Now, when you leave something disconnected, then it is basically moment. So, this is nothing, but a moment of x into moment of V x ok, which is also there cumulants. So, we know that the first moment is also a cumulant and when you make them connected then this is simply cumulant ok. So, this is the answer to your problems, this is nothing, but cumulants of x in the x multiply plus x into V x cumulant ok. So, this is a very simple procedure, you can try and work it out for other combinations of a other powers of x and v and you will be surprised as how easy this becomes ok.

So, you would be basically interested in finding out let us say x square into V x. Now how do you do it? While there are 2 x's, so I will put the 2 dots, so dots represents x and there is one V x, which are indicates the cross 3 points. Now I can arrange this in any ways. So, one way would be to connect the 2 dots and leave the x as is there is only one way you can do it. Other way is to connect the dot and the x and leave the other dot alone right. Now, you can do it in two ways, fine and finally you can do it, you can connect all of them no more cases ok. So, when you connect all of them that closes that is the full set of Pasco possible combinations and let us now write them down. So, 2 disconnected dots is nothing, but x square ok.

So, this is nothing, but second there this is nothing, but the square the first moment and I know that the moment itself is a cumulant writing down for cross, you just have the first cumulant of the velocity. Then we have connected dots, so we will take second cumulant of the x into moment of velocity, but then moment is also cumulant and the third combination is 2 times and the 4th combination is would write it here ok. So, that is the basically value for the joint moment of x square and V x. So, these are the joint moments that we are computed today ok.

So, these are the joint moments fine. So, we close this lecture here and we will give some problems in the two because this is along the similar lines. And, we shall meet in the next lecture to discuss principal of you know maximum entropy which leads to a certain types of distribution such as, the Gaussian distribution and the exponential distribution performed in natures. So you must have wondered why a an equilibrium distribution is a Maxwell Boltzmann like. And, the reason why you get Maxwell Boltzmann distribution always is an equilibrium system is because that is the only distribution that maximizes the entropy.

So, if you leave a system isolated for a long time you know the system will always proceed towards the direction of maximum entropy and because Maxwell Boltzmann distribution is a distribution of maximum entropy, you always find equilibrium isolated systems under the under such a distribution. So, we will discuss this in the next class and do some nice problems on this ok.

Thank you.