Statistical Mechanics
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Lecture - 06
Poisson Distribution

Good morning to all of you. Today we will proceed the discussion and complete one very important distribution in Statistical Mechanics that is quite profound in its usage and this distribution goes by the name of Poisson Distribution.

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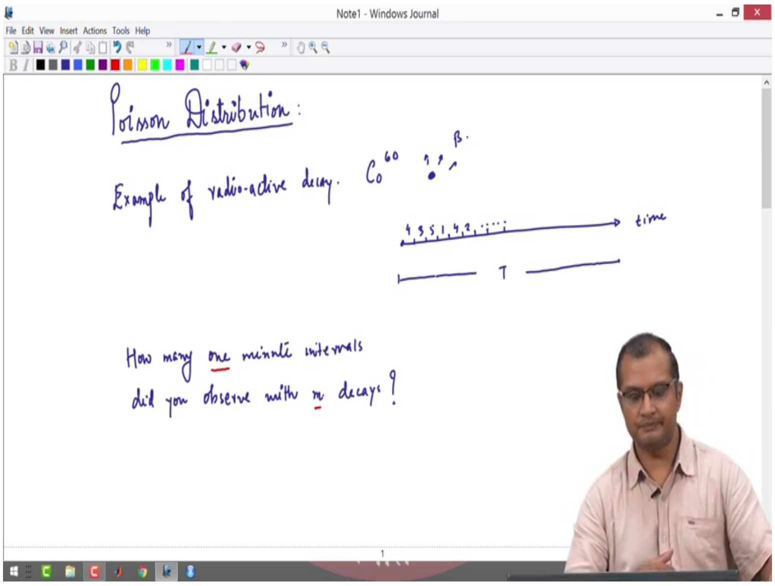

So, today we will discuss this very important distribution in statistical mechanics. So, we have already covered two distributions in the past two lectures; the Gaussian distribution and the binomial distribution. So, today this third distribution completes the set of very important distributions in statistical mechanics. And, in one of your upcoming exercise problems we will ask you to show both the binomial and Poisson distribution in the large and limit reduce to Gaussian distribution. So, that will be demonstrated in the worked out problems in the assignment sets.

So, to begin the discussion on Poisson statistics and hence the Poisson distribution I would like to take the example of beta decay or radioactive decay. So, you can think of material that is decaying such as for example, cobalt 60 ok. So, this would be basically nuclei which is giving beta particle or electrons as part of the radioactive decay process.

So, if you observe small pallet of cobalt 60 and you wait for let us say you wait for 10 minutes and this is your time axes, let us say you wait for capital T equal to 10 minutes ok.

So, this is your time axis you start here and then you ask yourself a question that in this total 10 minutes or T minute interval how many minutes did you encounter, how many 1 minute intervals did you encounter; suppose I make intervals of a 1 minute. Such that you got you know 4 decays in some interval and you got in next interval you got 3 decays and then you got 5 decays and then you got 1 decay so on and so forth then you got again a 4 decay, then you got 2 decays.

So, these are successive 1 minute intervals in the 1st 1 minute interval you got 4 decays, in 2nd 1 minute interval you got 3 decays, in the 3rd 1 minute interval you got 5 decays and so on and so forth. So, this goes on and then you ask yourself a question as to how many 1 minute intervals were observed in the total time duration which gave you 4 decays ok. So, this is a problem that can be addressed through Poisson's statistics; how many x. So, the question is: how many 1 minute intervals did you observe with x decays or let us say m decays with m decays m being 4 or 5 whatever ok.

So, you are saying that two things to keep in mind is that we have taken a 1 minute interval. So, we are asking how many 1 minute intervals are there in the total T duration which have m decays ok. So, naturally if you ask the question like how many 4 minute intervals, 4 decay intervals were there 4 decay, how many 1 minute intervals were there with 4 decays; well there were finite number of such intervals.

If you ask how many if 1 minute intervals were there with 5 decays again you may say there are some finite number of intervals, but if we ask how many 100, how many 1 minute intervals were there with 100 decays or 1000 decays. Well, the answer would be close to 0 because, you do not see such large number of decays over a span of 1 minute ok. Depend on the radioactivity of the decaying material, but then the answer is known that if I ask in my 1 minute intervals if I ask how many 0 decay events were there, the answer is will be close to 0.

Because, you will definitely see some decays; if you ask how many 1000 decay events you know were observed in you know 1 minute intervals, the answer would necessarily again be 0, because you have finite duration you cannot have 1000 decays in a finite 1 minute interval. So, definitely this is a distribution which speaks at some value and decays at extreme ends ok. So, we will develop statistics for such problems. So, let us draw this time axis again.

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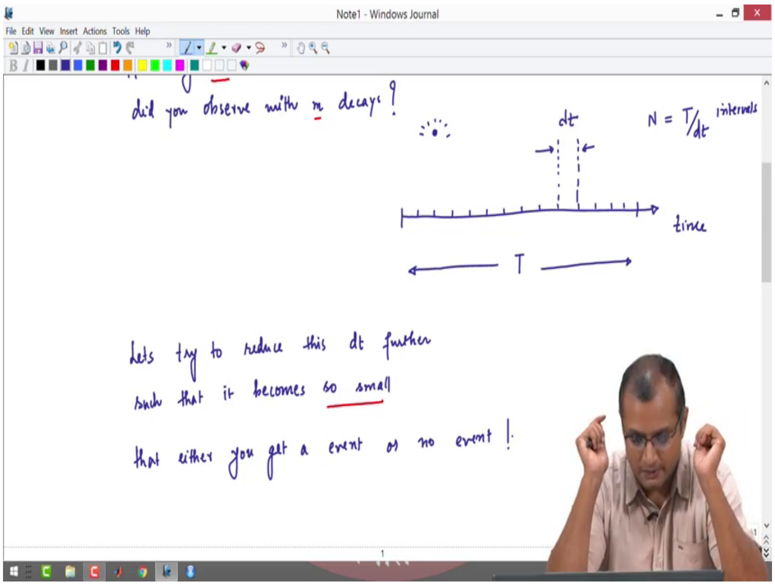

And let us say I am looking for a total time duration of T minutes or T hours. So, big T is my total time duration in some units and I am observing radioactive let us say sample which is emitting particles let us say beta particles or alpha particles and I am dividing my entire observation duration into small intervals of let us say size dt ok

So, I have divided my intervals of certain width which is dt and there are N such intervals. So, the interval the number of intervals is nothing, but the big T divided by dt ok. So, these are the intervals that I have divided my entire domain into ok; now certainly in each of these intervals there will be some decays some interval will be 5 decays and next interval there could be 2 decays in the next interval there would be 10 decays so on and so forth.

It is expected that there will be very few intervals with no event because the sample is decaying. So, there will be very few intervals with no event and there will also be very few intervals with large number of events let us say 1000's events I will not see you know a 1000 events in any one such small interval ok. So, let us try to make this picture let us try to reduce this interval delta t further such that it becomes so small that you have either an event or no event. So, small so, this is so, small that you either have a single event or no event.

So, now actually we have made the case so, aggressive we cannot talk about 5 events or 4 events or 10 events or so, on actually this dt is now so small ok, I have increase the N the number of intervals so, much that dt has shrunk to such a small width that either you get an event or no event. So, this is in my hand this is construction is in my hand I have a total time duration T and I simply decide to slice this time duration into widths such that the widths are so small that you either have one event or no event ok.

So, this is in my hand I have done that construction. Now look at this what I have ended up with this beautiful construction that I have made a Poisson problem actually a binomial problem. So, even have a event or no event there are only two possibilities now. So, this careful mathematical construction of this interval has converted my problem temporarily into a binomial problem. Now, in the binomial you know that the probabilities which are only you know for the event or no event.

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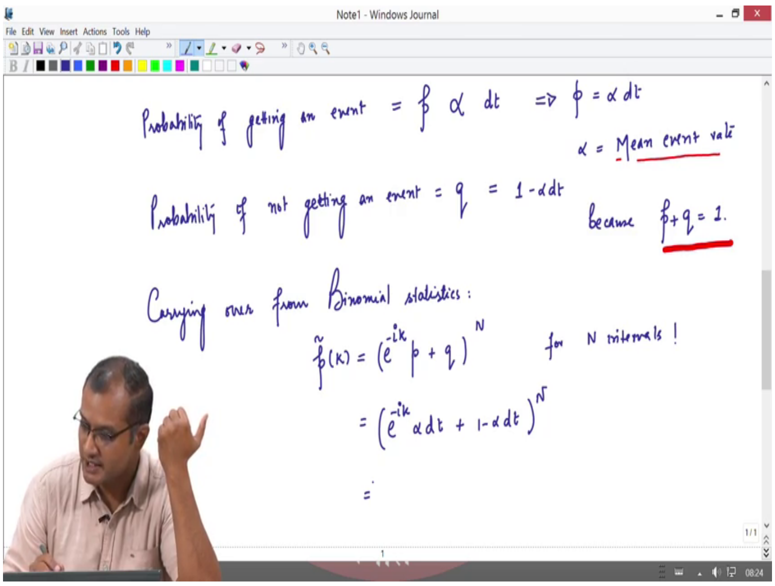

So, if I say that the probability of getting an event is p ok. So, this p would be proportional to the length of the interval because if I keep the interval dt to 0 then I will not see any event. So, the probability of seeing an event in the interval dt is 0 and if I increase this dt a little bit probability of seeing an event increases. So, I already know that this probability of seeing of an event in my small interval dt has to be proportional to the width of the interval. So, that is what I have written; which implies can write down my probability of seeing an event as some alpha times dt ok, where alpha has to be interpreted as mean event rate right.

Because, I have taken a linear interpolation of the idea that you know if there are alpha events happening per unit time what is the probability that lcn event in a time interval dt then you will simply say that its alpha times dt ok. So, this alpha is a mean event rate naturally you can ask yourself a question and it is immediately answered the probability of not seeing an event if I take it as q then q has to be automatically 1 minus alpha times dt. Because, I know that the probabilities are well behaved because p plus q has to be equal to 1 ok.

This condition has to be satisfied. So, we have this condition that probabilities are well behaved say if p is alpha dt then q must be 1 minus alpha dt because p plus q has to be 1 ok. So, then we simply carry over our concepts from binomial statistics. So, the processes been cleverly or you know aggressively transformed into a binomial problem. So, from binomial statistics I know already that the characteristic function or the generator of movements is given as e raise to minus ik into p plus q to the power N there are N intervals or for N events for N intervals. This is the result that I am simply carrying over from binomial statistics that is the result that we derived in the last class ok.

Now if you plug the values of p and q that we have just obtained this becomes e raise to minus i k the value of p that I have just obtained is alpha times dt plus the value of q which is a probability of not seeing an event is 1 minus alpha into dt. And the entire thing is raise to the power N ok, actually I know what is big N? The power N is nothing, but capital T over dt as defined here ok. So, please make note of this construction ok. So, replacing this big N by T over dt we will do we will do that in next step. So, inside the bracket you can already see a few things can be taken as common.

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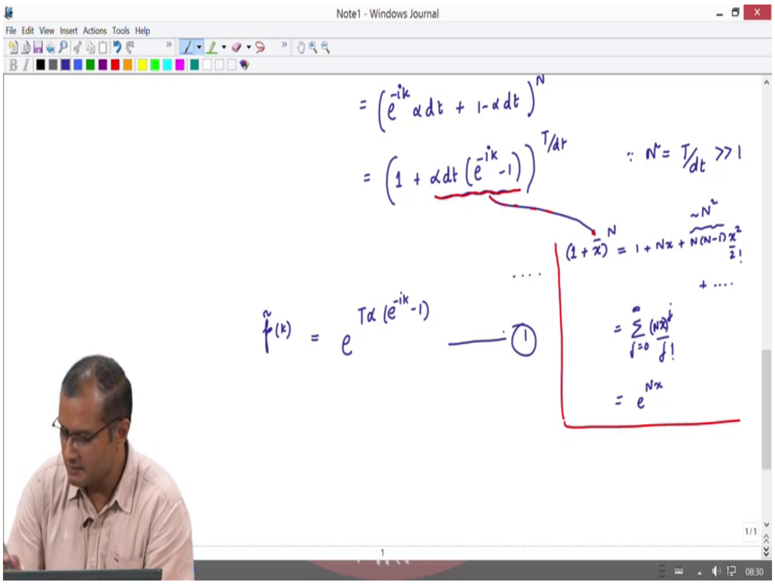

So, I am going to take 1 plus alpha into dt as a common factor and then take e raise to minus ik minus 1 entire thing has to be raised to the power p over dt because, the number of intervals is nothing, but the total interval size divided by the width of small intervals that you have ok. Now we know that this is much larger than 1, for the simple reason that I have taken dt to be very very small if I take dt to be very small such that there is either an event or no event the number of intervals has to be very large ok.

So, if your power is very large you can expand it using binomial expansion and write this as e raise to you can write the entire thing as e to the power t times alpha into e raise to minus i k minus 1 ok. So, the small interval dt cancels off what I have done here is basically the following if you if you find this slightly troublesome recall that if you have an expansion like this 1 plus x to the power N ok.

So, in this case my x here is basically this entire thing and N is t by dt. So, I can write down this 1 plus x to the power N as 1 simple binomial expansion 1 plus N x thus N into N minus 1 into x square by factorial two and so on N into N minus 1 results from the combination N see 1 ok. So, and so, on now if N is very large I can approximate this guy as N square and makes the factor as N cube and so on and so forth.

So, this is like an expansion of j going from 0 to infinity x to the power j upon j factorial ok. So in fact, this is this has to be N x to the power j because, N into N minus 1 can be approximate as N square N into N minus 1 can be approximate as N square N into N minus 1 N minus two is a approximate as N cube because N is very large it is larger than 2 it is larger than 3 and so on and so forth. Then this is nothing, but the expansion of e raise to N x. So, this is what I have done here. So, this entire thing is basically so, we can we can think of.

So, now I have the characteristic function in front of me which is generator of moments. So, if the characteristic function is in front of me I can immediately compute the e distribution function itself. So, let us keep this important result ready. So, I am going to box it because it is an important result ok. So, I am going to sort of keep it handy ok. So, let us call this some give this some equation number let us call this as equation 1 ok.

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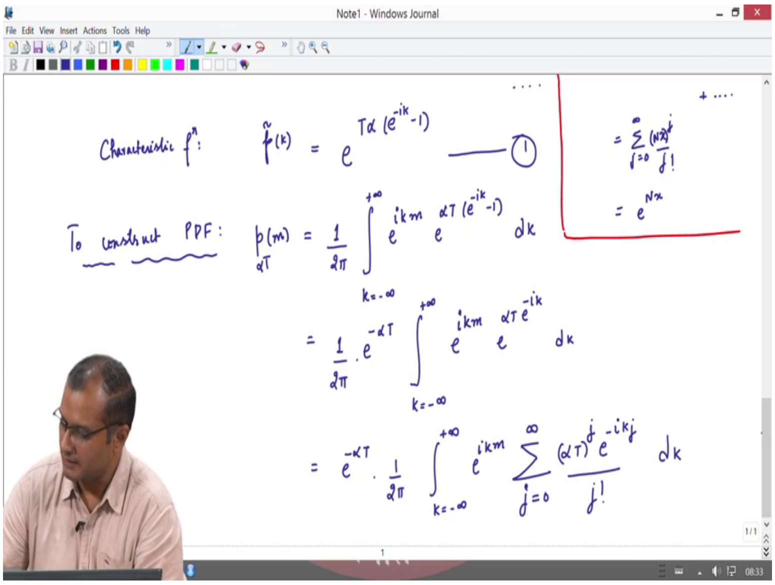

Now to construct the PDF the Poisson PDF I can take a inverse Fourier transform. So, Poisson PDF for N events in total wait you know waiting time of T interval with a null with a mean event rate is alpha is constructed by inverse Fourier transform. So, I already have a Fourier transform which is the characteristic function. So, it is my characteristic function that is just obtained and I want to obtain a PDF.

So, I must take an inverse Fourier transform. So, the inverse Fourier transform is taken as k going from minus infinity to plus infinity e to the power i km this m and that m the left hand side m are the same variable, I am taking a transform on k it is an inverse for your transform into e raise to alpha T into e raise to minus i k minus 1 ok.

So, this is my PDF this is already my Fourier transform to d of k fine. So, let us use a different colour here this is my characteristic function fine. So, if you just to give you an alert so, if you do not want to write it like this I am going to remove it. Now we can solve this further its very simple exercise you can immediately see that e raise to minus alpha T. So, I am going to take e raise to minus alpha T outside because it is not a function of k.

So, e raise to minus alpha T comes out of the integral what I have is just k going from minus infinity to plus infinity e raise to i km into e raise to alpha T into e raise to minus k fine dk and this can be written further as. So, I am going to write down the second exponential as a summation let us take a summation on j going from 0 to infinity alpha T to the power j into e raise to minus i k j divided by j factorial dk fine.

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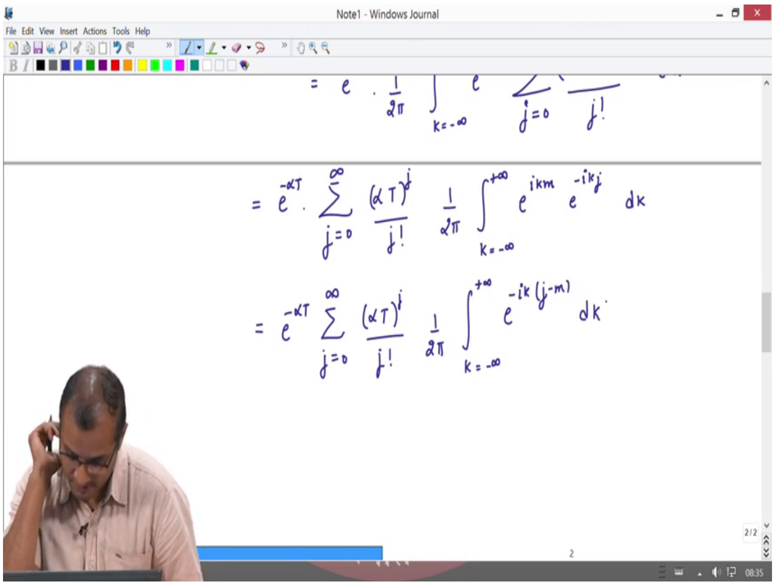

And let us simplify this further I can write this as integral k going from minus infinity to plus infinity. Now we can see that I can take out the summation so, I can take out the summation because the summation is on j and the integral is on k.

So, I can rewrite this by pulling the summation outside the integral ok. So, I am going to pull the summation outside because it's on j whereas, the integral is on k. So, everything that involves only j and not k I am going to pull outside and I will take the integral inside the sum. So, small manipulation that I have done to make things easy for me what is left over inside is e raise to minus ik j this j that is inside the integral is summed over outside let us reduce this further you can see this is summation over j alpha T to the power j upon j factorial into 1 upon 2 pi integral k going from minus k to plus minus infinity to plus infinity e to the power minus ik into j minus m d of k.

Now, this integral inside is should remind you of special function in the sense that this function is called as derived delta distribution on the argument j minus m ok.

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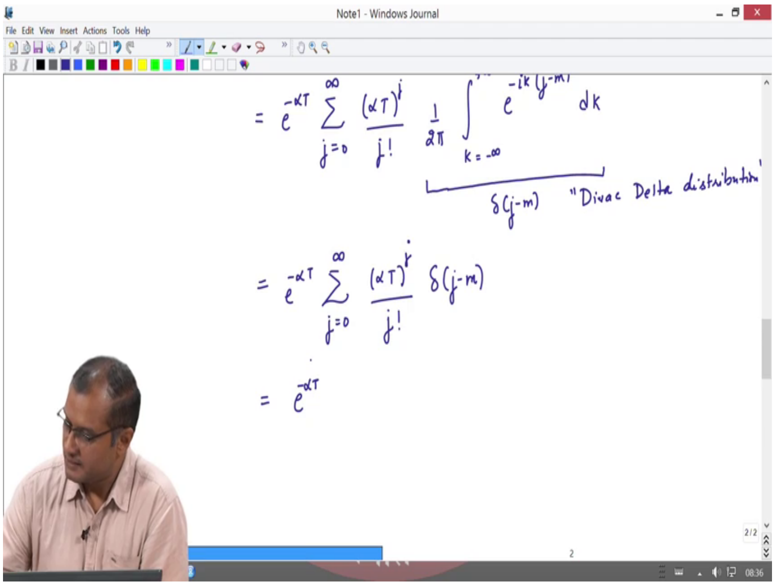

This is the derived delta distribution sorry this is derived delta distribution. So, this is now reduced to 0 to infinity alpha T alpha times T power j upon j factorial into derived delta function or derived delta distribution on j minus m. Now clearly you have infinite number of js and the pre factor alpha T to the power j upon j factorial relays as function on j that is weighed in this derived delta distribution. So, the entire summation has the enormous rate for an entire weight coming from the function evaluated j equals to m ok.

So, this integrand the sum and is evaluated at the source of the derived delta distribution and this is where it takes up its entire weight ok. So, the function alpha T to the power j upon j factorial is computed at the source of a delta function and which gives me alpha T to the power m upon m factorial.

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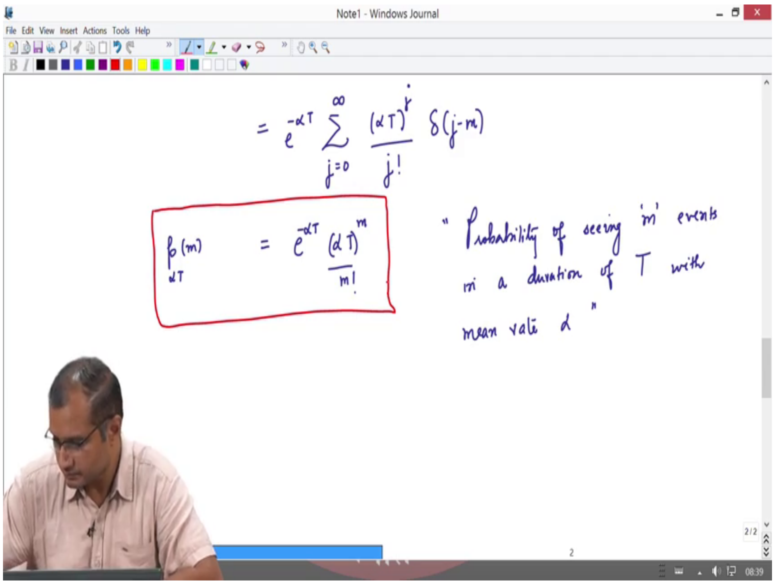

Left hand side was my Poisson distribution N probability of seeing probability of observing or seeing m events in a duration of T with mean rate alpha this is how you interpret your final answer. So, I have a duration T this T is not big T that we are it could be any T it could be 1 minute, 2 minute, 3 minute you decide it is in your hand ok. So, if you have an alpha which is the mean rate and then you ask yourself a question what is the probability of finding N events in a time duration of T well this is the answer.

So, you can you can use this result to compute your probabilities for an event that is obeying Poisson statistics how do you know a element is obeying Poisson statistics well the simple answer is that the successive events are completely un correlated or random. So, beta decay is a process which is random it is not correlated of beta decay that is happening now, is not related to a decay that happened before just moment before ok.

So, these events are completely independent of each other and the probability of observing an event is proportional to the time of observation. So, these two things will make the statistics Poisson like and you can use this Poisson distribution to get an idea is to what is the probability of seeing m events in duration that is entirely fixed by observer which is you.

So, now let us call this as equation 2 and I going to just term for the for the benefit of all of us going to pull this characteristic function right below it. So, let us pull this. So, look at this.

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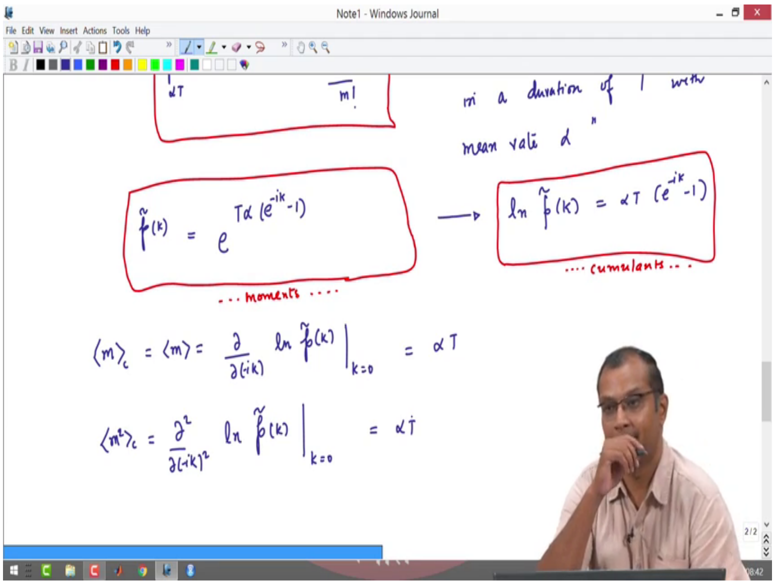

So, you have a distribution in front of you and you also have its Fourier transform which is a characteristic function and together these two expressions beautiful expressions give you everything that you require for Poisson statistics. For example, if you want to ask what is the mean of you know, what is the first cumulant of m observation. So, if you want to see m observations and you want to find out what is the first cumulant which is basically nothing, but the mean itself well then this is nothing, but you can simply compute the generator of cumulants. So, from here you can compute generator of cumulants which is nothing, but lawn of p of k and that is nothing, but alpha T into e raise to minus ik minus 1.

So, this is another important result, you have everything, you have the distribution you have the you have the distribution, you have the characteristic function which is generator of moments and you also have the logarithm of the characteristic function which is the generator of cumulants ok. So, you get here moments from here and you will get you cumulants from here. So, the first cumulant can be constructed from the procedure that we have discussed last class is nothing, but the first derivative of the cumulant generating function evaluated at origin which is nothing, but if you look at the expression of lawn p of k.

This is nothing, but alpha times T because e raise to minus ik is derivative is e raise to minus ik and taken at 0 this becomes 1, but not only that if you compute the second cumulant which is the second derivative of the cumulant generating function taken at k equals to 0. Well this also comes out to be alpha T.

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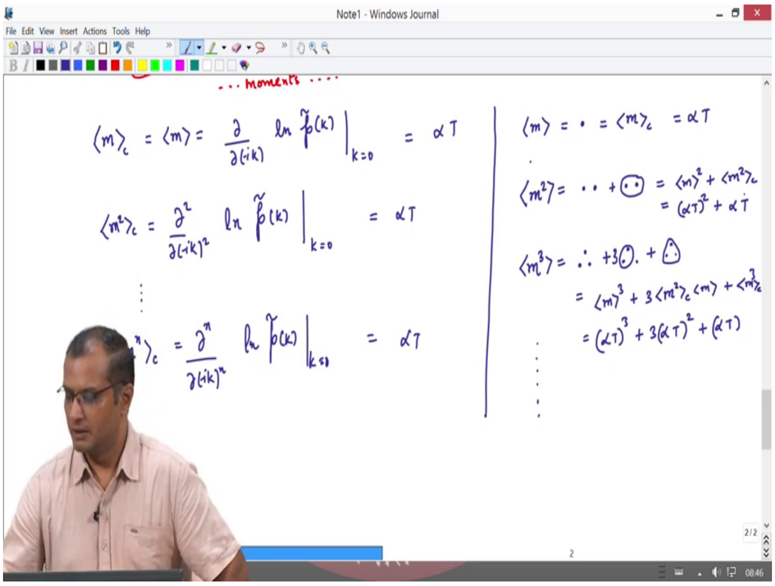

So, this tells you that all cumulants of any order suppose nth cumulant if you take which is nth derivative of the cumulant generating function comes out to be alpha T. So, this is a beautiful result that all cumulants are same for the Poisson statistics when that is the case we can construct moments straight forwardly. So, the first moment is just by my graphical computation is nothing, but the first cumulant and that is alpha T we have just obtained ok.

Let us compute the second moment well this is we need two dots this is nothing, but two dots disconnected and two dots connected no two no other possibilities are possible are available the first case is second moment ok. So, that is the first case is first moment square which is alpha T square and then you have the second cumulant which is alpha T then we have the third moment for this I will take 3 dots. But, for 3 dots I can use 3 disconnected dots plus connect 2 of them leave the third guy there are 3 such cases plus connect all 3 (Refer Time: 34:24). And what is this? Well 3 connected 3 unconnected dots is nothing, but the first moment to the power 3 plus 3 times second cumulant into first moment plus third cumulant this is nothing, but first moment as first cumulant. So, this is nothing, but alpha T to the power 3 plus 3 times second cumulant is alpha T.

So, this is alpha T square plus alpha T that is it because all cumulants are alpha T's third cumulant is also alpha T fine. So, this is how you can compute all these moments of the Poisson statistics and on the left side we have compute the cumulants. So, this is work it out you should be able to find what these moments and cumulants are. So, this is nice exercise to gain an understanding of the concepts. So, I suggest you to right out right.

So, now at this stage I will sort of discuss important a example a worked example to demonstrate how to apply Poisson distribution to solve problems ok. So, let us take a worked example.

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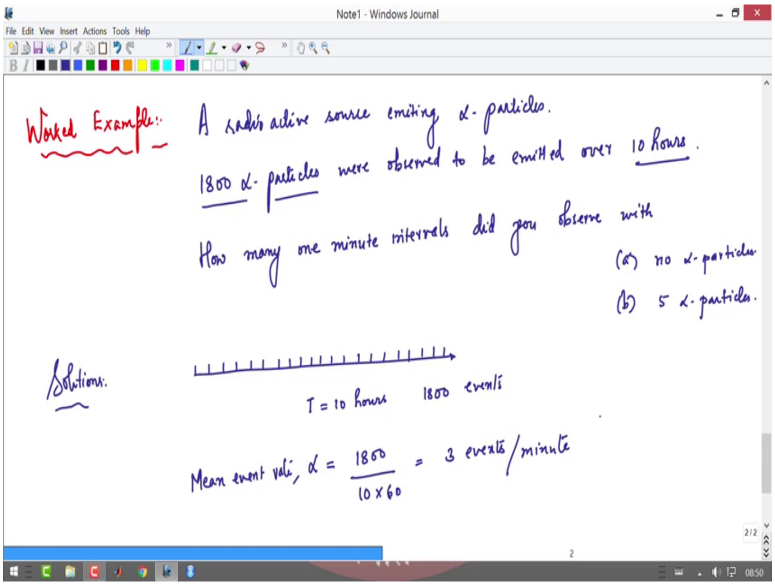

So, I am going to just give you a problem let me read out the problem. So, you have let us say you have a radioactive source emitting alpha particles let us say and you saw that 1800 alpha particles were observed to be emitted over 10 hours ok. And the question here is how many 1 minute interval is pay attention the minute in the intervals are 1 minute intervals how many 1 minute intervals did you observe with no alpha particles ok. So, that is the 1st case so, I am going to write it as a 1st case. So, how many 1 minute intervals did you observe with no alpha particles and the 2nd case is how many 1 minute intervals did you observe with 5 alpha particles that is the problem.

So, which means if you if you want to understand this question it is like you observed it for a total time of 10 hours ok. So, you have basically intervals of 1 minute and you ask yourself a question. So, you saw 1800 events. So, the alpha which is the mean event rate is nothing, but 1800 divided by 10 into 60 which is 3 events per minute ok. So, this is the mean event rate for your problem ok.

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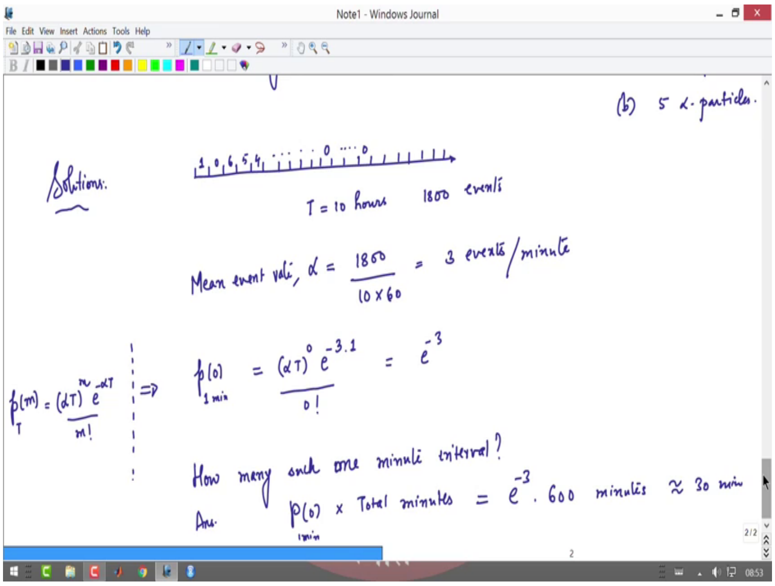

So, then suppose you saw one event here you really saw nothing here then you saw 6 events here, then you saw 5 events here, then you saw 4 events here so on and so forth; completely random now the question is how many 0 event 1 minute intervals are there? So, you already have a second interval which has no event, but there could be also 1 minute interval here with no event here further down the line you may have one more such. So, basically need to count you basically asked how many such intervals exist ok?.

So, what you have to first is compute the probability of observing no event in a 1 minute interval that the answer is simply if you if you recall your answer if you recall your distribution the probability of m events in a time interval of T is given as alpha times T to the power m e raise to minus alpha T upon m factorial ok. So, that is the result that we are going to use this gives us our m is 0 so, we get alpha T times to the power 0 which is 1 to e raise to minus alpha T.

My alpha is 3 and our interval T here is 1 minute ok. So, this is 3 into 1 divided by m factorial or 0 factorial ok. So, this comes out to be e raise to minus 3 fine is the probability of seeing a 0 event interval. Now if your question is: how many such events are there how many such intervals are there, how many such intervals such 1 minute intervals ? Now the answer is multiply the probability p 0 of 1 minute multiply by the total number of units, which is nothing, but e raise to minus 3 into you had 10 hours which means you had 600 minutes. So, basically this is this is roughly 30 minutes ok.

So, you can check this out by calculator and similarly if you want to find out how many. So, this is the answer to the first part there are roughly 30 1 minute intervals where there was no alpha disintegration ok. So, we have seen there is one such interval there is another such interval there is another such interval this way there are some 30 such 1 minute intervals where you did not see a alpha particle disintegrated the second question is straight forward. So, this was the answer to your first.

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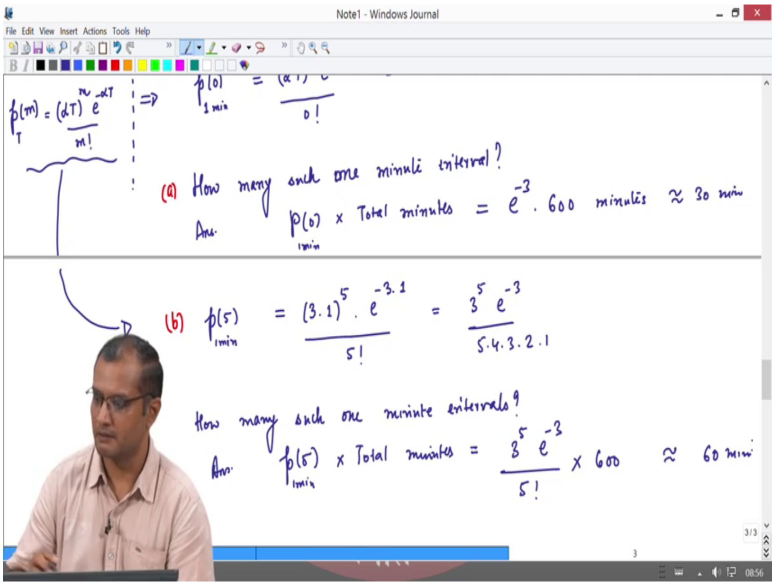

The second question it was basically how many such 1 minute intervals with 5 such disintegration happen. So, for that I must compute the probability of getting 5 disintegrations in a 1 minute interval. So, the answer would be alpha is 3 so, I am going to use this formula here. So, alpha is 3 the time is 3 time is 1 power m m is 5 into e raise to minus 3 into 1 over m which is 5 factorial ok.

So, this is 3 to the power 5 into e raise to minus 3 over 5 factorial ok. So, this will come out to be roughly 60 minutes. So, there are so, there are basically total 60 such 1 minute intervals where you see that there are 5 disintegrations of there is 1 here.

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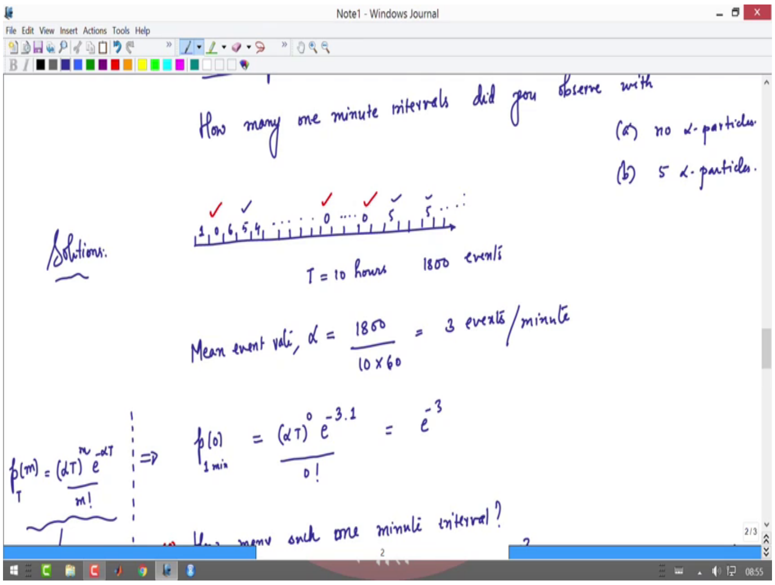

There could be 1 more here down the line there could be 1 more down the line so on and so forth and there are basically 60 such 1 minute intervals where you will see the disintegrations that are you know there are 5 disintegrations of the radioactive particle ok. So, now in fact, I have just computed the probability. So, let me just I have just made a small error here. So, this is the probability to compute the number of minute intervals I have to compute.

So, the answer how many such 1 minute intervals exist I must multiply this with total number of minutes like I did before. So, this is 3 to the power 5 into e raise to power 3 upon 5 factorial into 600 and this will be around 60 minutes ok. So, we end here and we will meet again in the next lecture and proceed with the discussion on many random variables which is again a very interesting subject.